End of the first part & Cheat sheet

A category $\mathcal{C}$ is given by

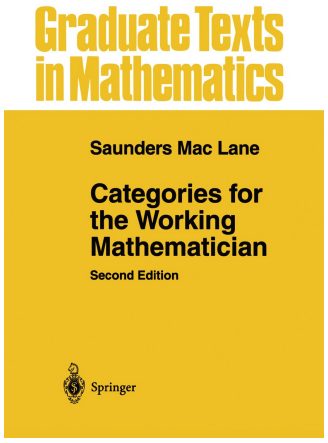
- a class of objects
- a set of $\hom(X, Y)$ of morphisms,
- plus composition, identity, associativity of composition, and neutral of identity.

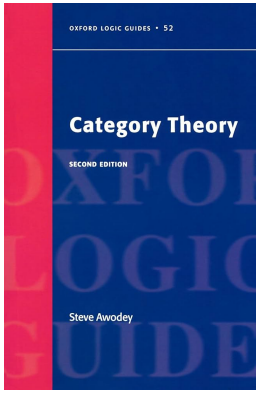
A functor $F : \mathcal{C} \to \mathcal{D}$ is given by

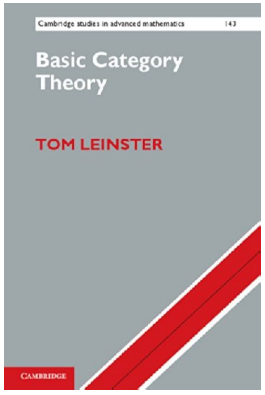
- an object FX of $\mathcal{D}$ for each $X : \mathcal{C}$,
- a function $F_{X,Y} : \hom(X, Y) \to \hom(FX, FY)$ for each pair of objects (X, Y) of $\mathcal{C}$
- plus preservation of composition and identity.

A natural transformation θ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a collection of morphisms $\theta_X : FX \to GX$ of the category $\mathcal{D}$ indexed by $X : \mathcal{C}$ such that θ_X is natural in X for all $X : \mathcal{C}$.

References







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nLab

category theory

See nLab for more references

Slides are here

Overview

How to build categories?

- Opposite category
- Products of categories
- Functor categories
- Comma categories

Limits & Colimits

How to build categories?

1. Examples, as we have already seen: theories can build categories.
2. Constructions on categories: product of categories, opposite categories, functor categories, comma categories, free categories, categories of spans (almost a category), etc
3. Using logic/type theory: syntactic category, category of models
4. There might be other ways (?).

Constructions on Categories

Opposite category

Let $\mathcal{C}$ be a category.

- $\mathcal{C}^{\mathrm{op}}$ is the category whose objects are the same as $\mathcal{C}$, and $f \in \mathcal{C}^{\mathrm{op}}(X, Y)$ if $f \in \mathcal{C}(Y, X)$.
- A *contravariant functor* F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Examples

1. $\_^{\mathrm{op}} : \mathrm{Cat} \to \mathrm{Cat}$ defines a (covariant) functor:
   - $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$, $F : \mathcal{C} \to \mathcal{D} \mapsto F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$
   - $F^{\mathrm{op}}X := FX$. Let $f \in \mathcal{C}^{\mathrm{op}}(X, Y)$, $F^{\mathrm{op}}(f) := F(f)$ where it is in $\mathcal{D}(FY, FX) = \mathcal{D}^{\mathrm{op}}(FX, FY)$
2. Let $X : \mathcal{C}$. $\hom_{\mathcal{C}}(X, \_) : \mathcal{C} \to \mathrm{Set}$ defines a functor which sends $Y : \mathcal{C}$ to the set $\mathcal{C}(X, Y)$.
3. $\hom_{\mathcal{C}}(\_, X) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ defines a contravariant functor.

Definition

A presheaf on a category $\mathcal{C}$ is a (contravariant) functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$.

- A presheaf F is *representable* if $F \cong \hom_{\mathcal{C}}(\_, X)$ for some $X : \mathcal{C}$.
- The category $\widehat{\mathcal{C}}$ of presheaves: functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, and natural transformations.
- Yoneda embedding: the functor $y : \mathcal{C} \to \widehat{\mathcal{C}}$ which sends $X : \mathcal{C}$ to $\hom_{\mathcal{C}}(\_, X)$.
  - For any $X : \mathcal{C}$, $y(X)$ is a presheaf $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$

Yoneda lemma

For any presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, there is a canonical isomorphism

$$FX \simeq \hom_{\widehat{\mathcal{C}}}(yX, F)$$

Corollary

The Yoneda embedding functor is full and faithful functor.

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Products of categories ($\times : \mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Cat}$)

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We define a new category $\mathcal{C} \times \mathcal{D}$:

- an object $\mathcal{C} \times \mathcal{D}$ is a pair (X, Y) of $X : \mathcal{C}$ and $Y : \mathcal{D}$, and
- an arrow in $(X, Y) \to (X', Y')$ is a pair (f, g) of $f : X \to X'$ and $g : Y \to Y'$.

Example (1)

1. The covariant *bifunctor* $\hom : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$ which send (X, Y) to $\hom(X, Y)$.
   - Let $(f, g) : (X, Y) \to (X', Y')$ be a morphism in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. That is, $f : X \to X'$ in $\mathcal{C}^{\mathrm{op}}$ and $g : Y \to Y'$ in $\mathcal{C}$.
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\hom(X, Y) & \xrightarrow{\hom_{\mathcal{C}}(f, Y)} & \hom(X', Y) & \xrightarrow{\hom_{\mathcal{C}}(X', g)} & \hom(X', Y') \\
\| & & & & \\
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\end{array}$$

2. The contravariant *bifunctor* $\hom : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ which send (X, Y) to $\hom(X, Y)$.
   - Let $(f, g) : (X, Y) \to (X', Y')$ be a morphism in $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$

$$\hom(X', Y') \xrightarrow{\hom(f, g)} \hom(X, Y)$$

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Right adjoint

- The covariant *bifunctor* $\hom : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$ which sends (X, Y) to $\hom(X, Y)$.
- Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *profunctor* from $\mathcal{D}$ to $\mathcal{C}$, denoted by $\mathcal{D} \nrightarrow \mathcal{C}$, is a functor of the form $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set}$.
  - The hom bifunctor is the identity profunctor $\mathcal{C} \nrightarrow \mathcal{C}$.
- Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We have the following profunctor $\mathcal{D} \nrightarrow \mathcal{C}$:

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{F^{\mathrm{op}} \times \mathcal{I}} \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\hom} \mathrm{Set}$$

  We denoted this profunctor by $\hom(F\_, \_)$.
- Let $G : \mathcal{D} \to \mathcal{C}$. Similarly, we have the profunctor $\hom(\_, G\_)$:

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\mathcal{I} \times G} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\hom} \mathrm{Set}$$

Definition

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. A *right adjoint* of F is a functor $G : \mathcal{D} \to \mathcal{C}$ such that $\hom(F\_, \_) \cong \hom(\_, G\_)$.

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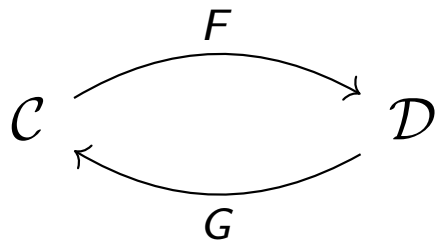

$$\frac{FX \longrightarrow Y}{X \longrightarrow GY}$$

The functor F and G are also called pair of adjoint functors.

Example

1. A pair of adjoint functors between posets is a Galois correspondence.
2. Let free : Set $\to$ Grp be the functor which gives us the free group generated by a set. The forgetful functor U : Grp $\to$ Set is the right adjoint to F.
3. "But any list is necessarily wildly incomplete." [1]

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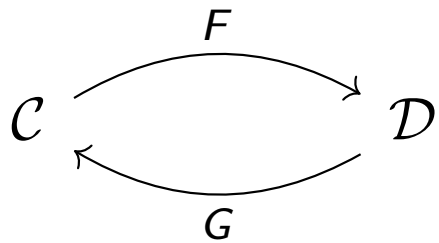

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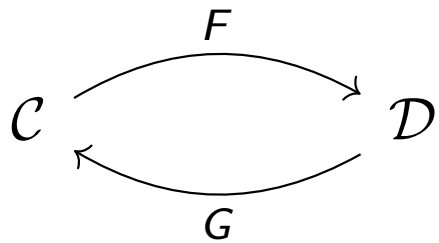

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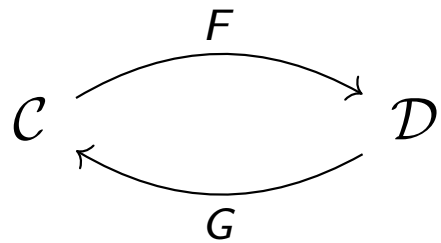

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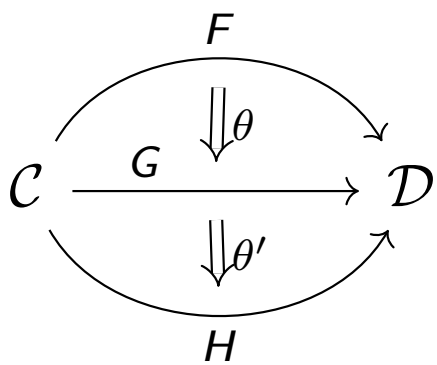
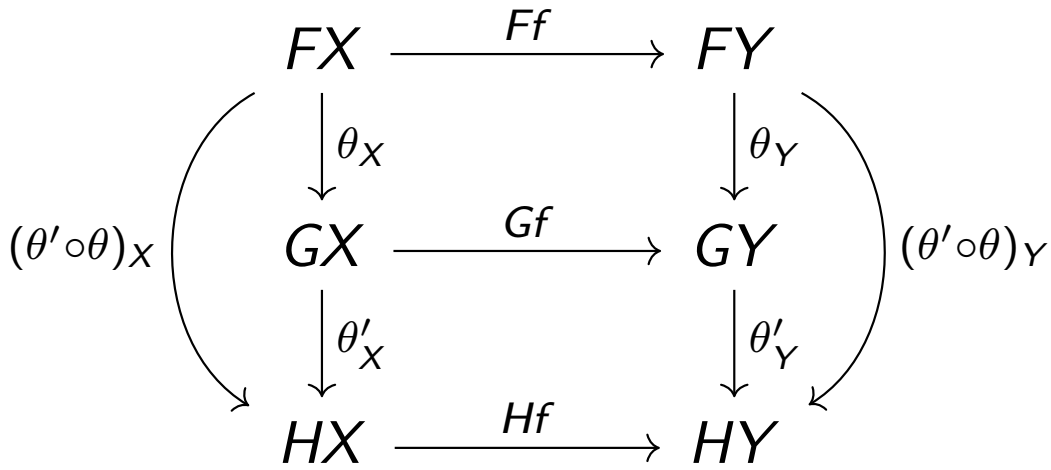
Functor categories

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We define a new category $[\mathcal{C}, \mathcal{D}]$:

- objects are functors $\mathcal{C} \to \mathcal{D}$, and
- morphisms are natural transformations $\theta : F \to G$.
- Let $\theta : F \to G$ and $\theta' : G \to H$. We define $(\theta' \circ \theta)_X$ as $\theta'_X \circ \theta_X$. Let $f : X \to Y$:



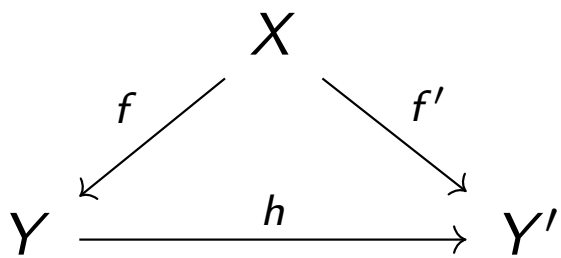


- The identity natural transformation: $\mathrm{id}_F : F \to F$ for $F : \mathcal{C} \to \mathcal{D}$ where $(\mathrm{id}_F)_X = \mathrm{id}_{FX}$.

Comma categories (1)

Let $\mathcal{C}$ be a category, and $X : \mathcal{C}$. We define a new category $X/\mathcal{C}$, called *under category*:

- objects are pairs (f, Y) where $Y : \mathcal{C}$ and $f : X \to Y$.
- Let (f, Y) and (f', Y') be two objects of $X/\mathcal{C}$. A morphism $(f, Y) \to (f', Y')$ is a morphism $h : Y \to Y$ such that



Example

Let $\star$ denote any singleton set. Then $\star/\mathrm{Set}$ is just the category of pointed sets.

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Let $\mathcal{C}$ be a category, and $X : \mathcal{C}$. We define a new category $X/\mathcal{C}$, called *under category*:

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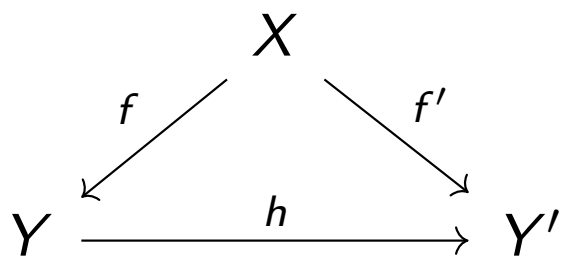

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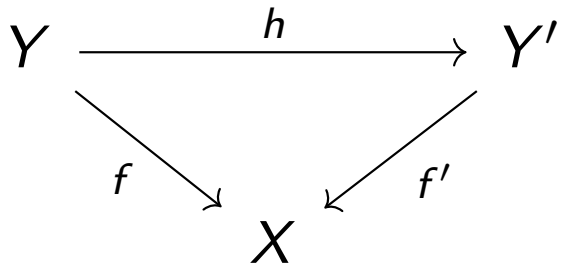

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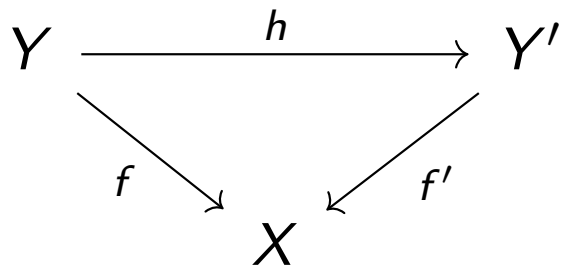

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Comma categories (3)

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $X : \mathcal{D}$. We define a new category X/F:

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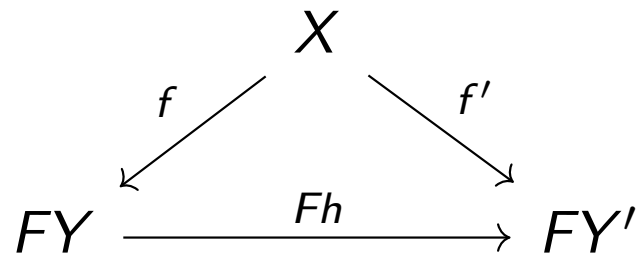

We can dually define F/X.

Example

Let $U : \mathrm{Grp} \to \mathrm{Set}$ be the forgetful functor and $X : \mathrm{Set}$. Let $\mathrm{free}(X)$ be the free group generated by the elements of X.
An example of an object of X/U can be seen as a pair $(f, \mathrm{free}(X))$ where $f : X \to UG$ is the function from X to the underlying set of $\mathrm{free}(X)$.

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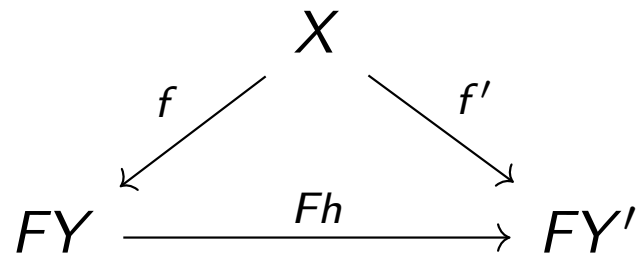

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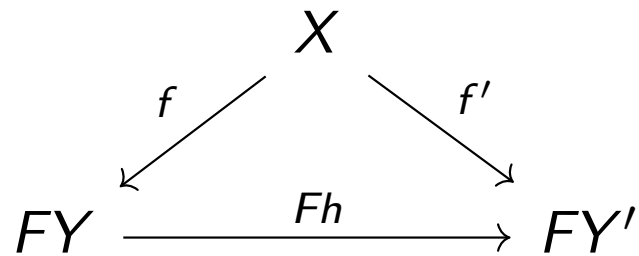

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We can find a more general framework to express the constructions of over and under categories:

Definition

Given the following categories and functors:

$$\mathcal{C} \xrightarrow{\quad F \quad} \mathcal{D} \xleftarrow{\quad G' \quad} \mathcal{E}$$

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Examples

1. An object $X : \mathcal{C}$ can be viewed as a functor $X : \mathbf{1} \to \mathcal{C}$. The under category $X/\mathcal{C}$ is then $X/\mathcal{I}$:

$$\mathbf{1} \xrightarrow{X} \mathcal{C} \xleftarrow{\mathcal{I}} \mathcal{C}$$

2. The category $\mathcal{I}_{\mathcal{C}}/\mathcal{I}_{\mathcal{C}}$ is the category of all arrows of $\mathcal{C}$.
   - $\mathcal{I}_{\mathcal{C}}/\mathcal{I}_{\mathcal{C}} = [\mathbf{2}, \mathcal{C}]$ (the functor category).
3. Consider two objects X, Y of $\mathcal{C}$ as functors:

$$\mathbf{1} \xrightarrow{X} \mathcal{C} \xleftarrow{Y} \mathbf{1}$$

   The category X/Y is the discrete category $\hom_{\mathcal{C}}(X, Y)$.
4. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\theta : F \to G$ is the same thing as a functor $\mathcal{C} \to (F/G)$.

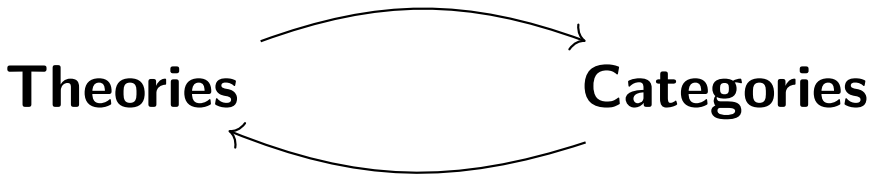
$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xleftarrow{G} \mathcal{C}$$

5. Let K be a commutative ring. The category K/CRng is just the category of commutative K-algebras (vector spaces equipped with a bilinear product).

Building categories

1. Examples, as we have already seen: theories can build categories.
2. Constructions on categories: product of categories, opposite categories, functor categories, comma categories, free categories, categories of spans (almost a category), etc
3. Using logic/type theory: syntactic category, category of models

Syntactic category & category of models



Depending on the theory, we need to assume our category have extra properties/structures: cartesian products, tensor product, closeness, (finite) limits, colimits, well-powered, has images, locally cartesian closed, etc.

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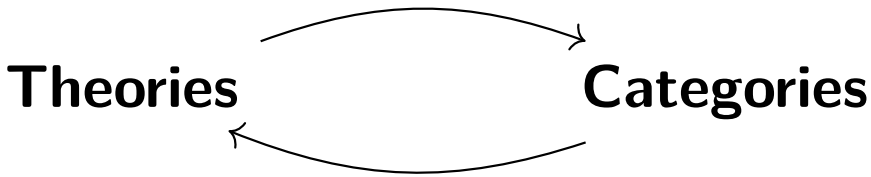


Depending on the theory, we need to assume our category have extra properties/structures: cartesian products, tensor product, closeness, (finite) limits, colimits, well-powered, has images, locally cartesian closed, etc.

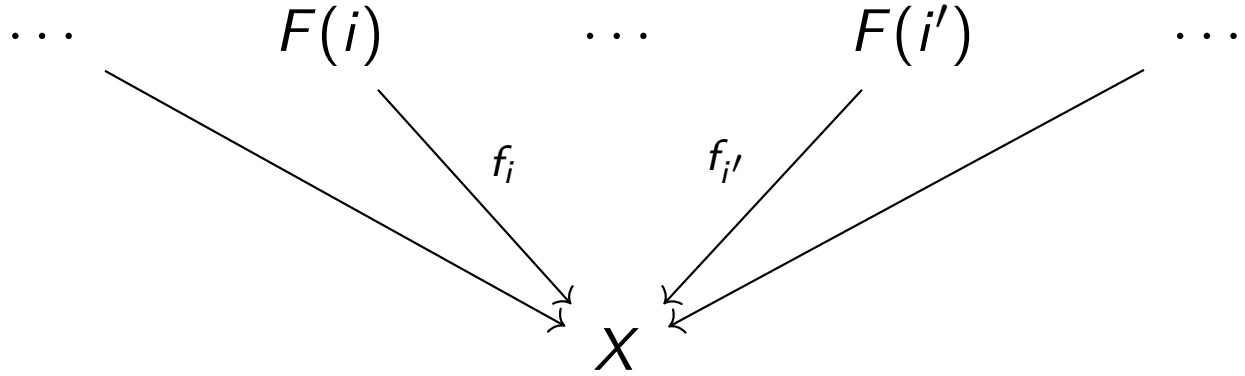
Limits & Colimits

Colimits

Let $\mathcal{C}$ and $\mathcal{J}$ be two categories and $F : \mathcal{J} \to \mathcal{C}$ be a functor [2].

Definition

A *cocone* to F is a pair $(X, (f_i)_{i \in \mathcal{J}})$ where $X : \mathcal{C}$ and $f_i : F(i) \to X$ such that



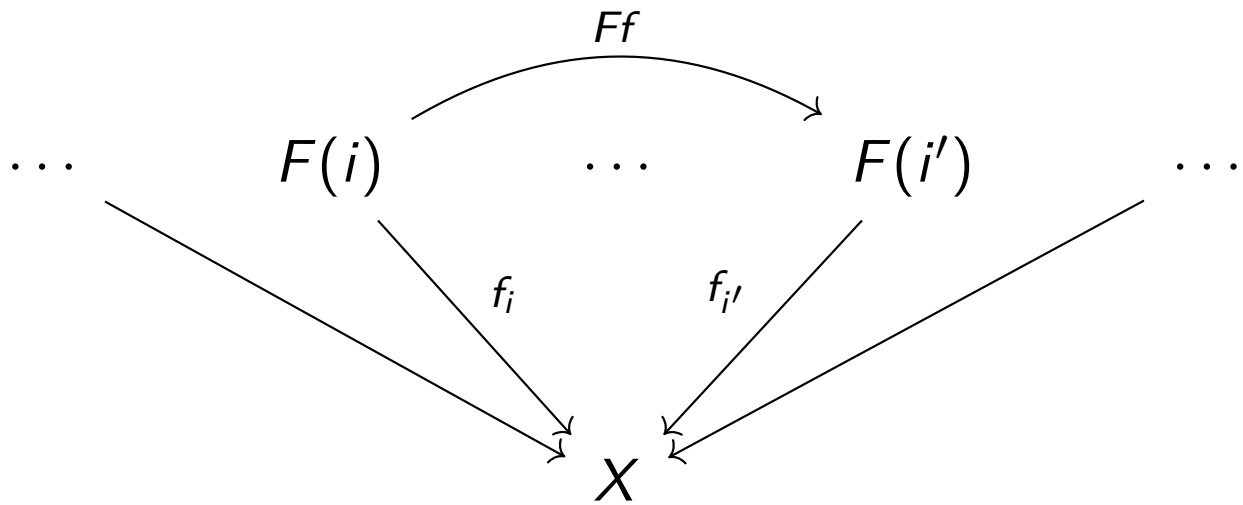
[2] It is also called diagram of shape $\mathcal{J}$ in $\mathcal{C}$

Colimits

Let $\mathcal{C}$ and $\mathcal{J}$ be two categories, and $F : \mathcal{J} \to \mathcal{C}$ be a functor.

Definition

A *cocone* to F is a pair $(X, (f_i)_{i \in \mathcal{J}})$ where $X : \mathcal{C}$ and $f_i : F(i) \to X$ such that for all $f : i \to i'$ in $\mathcal{J}$, we have $f_{i'} \circ Ff = f_i$



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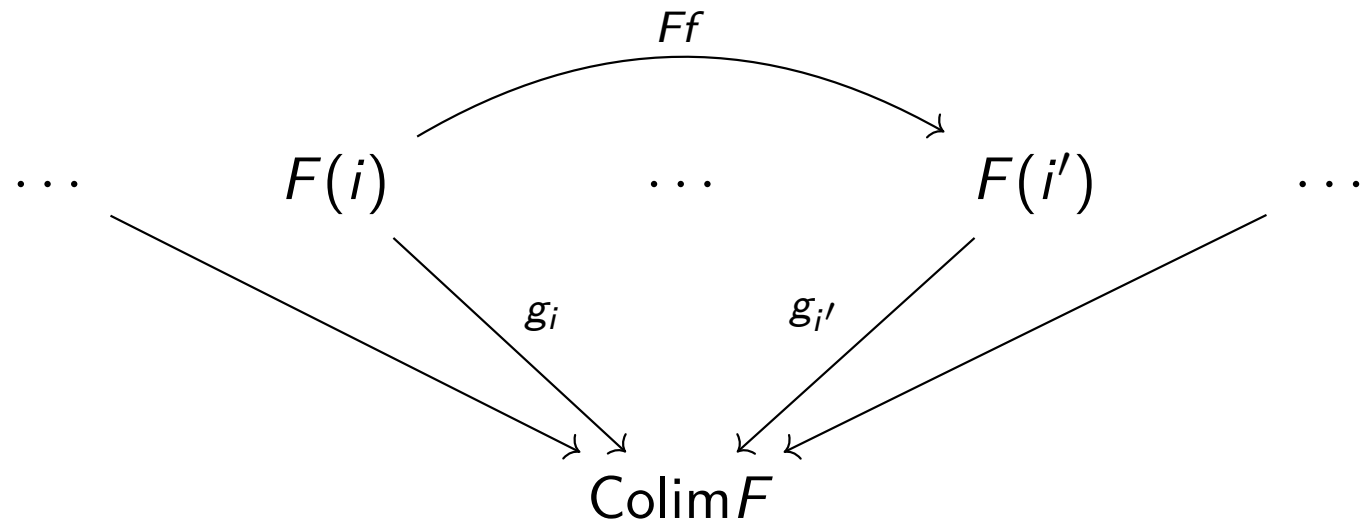
Let $\mathcal{C}$ and $\mathcal{J}$ be two categories, and $F : \mathcal{J} \to \mathcal{C}$ be a functor.

Definition

A colimit of F, denoted by $\mathrm{Colim}F$, is a cocone such that



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Let $\mathcal{C}$ and $\mathcal{J}$ be two categories, and $F : \mathcal{J} \to \mathcal{C}$ be a functor.

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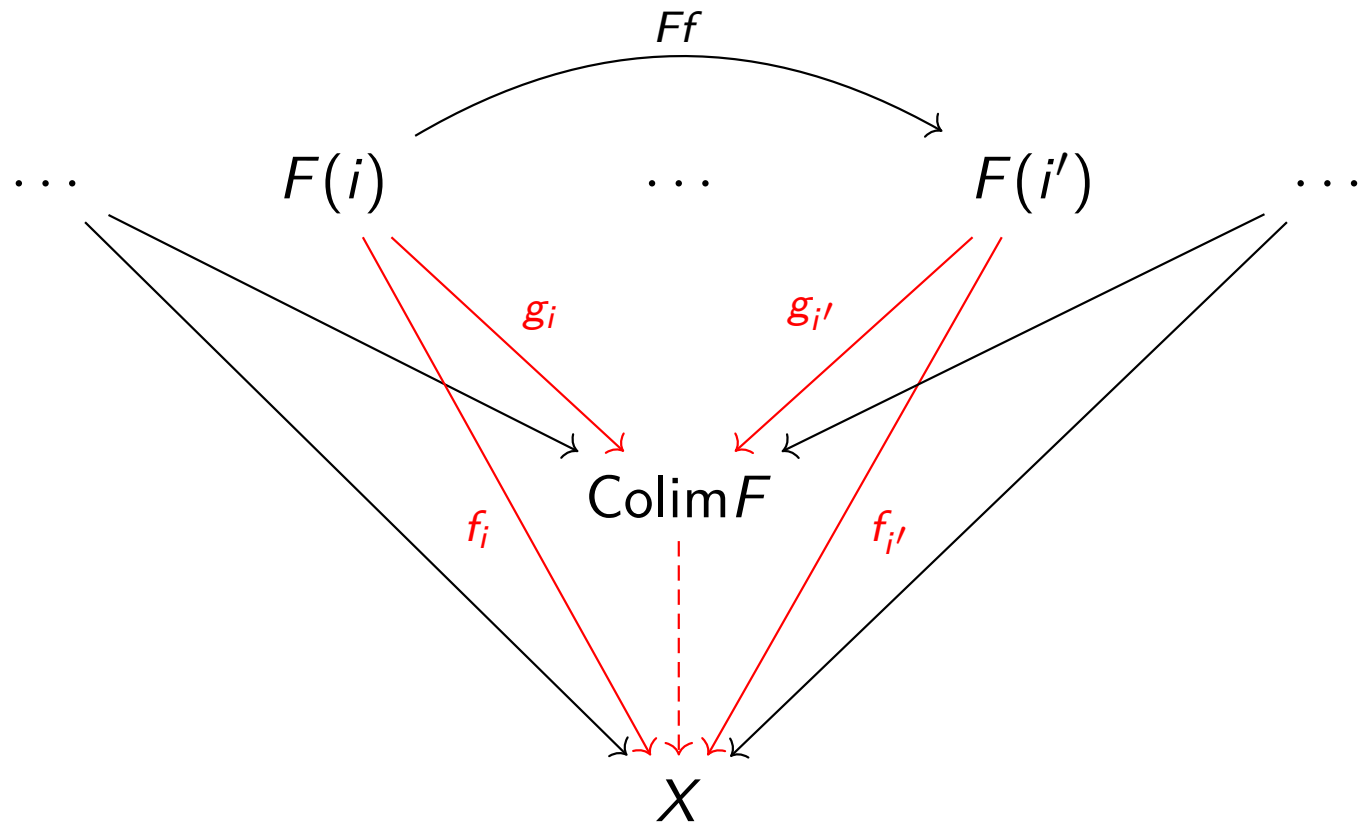
A colimit of F, denoted by $\mathrm{Colim}F$, is a cocone such that the following red diagrams commutes for any other cocone $(X, (f_i)_{i \in \mathcal{J}})$



Colimits

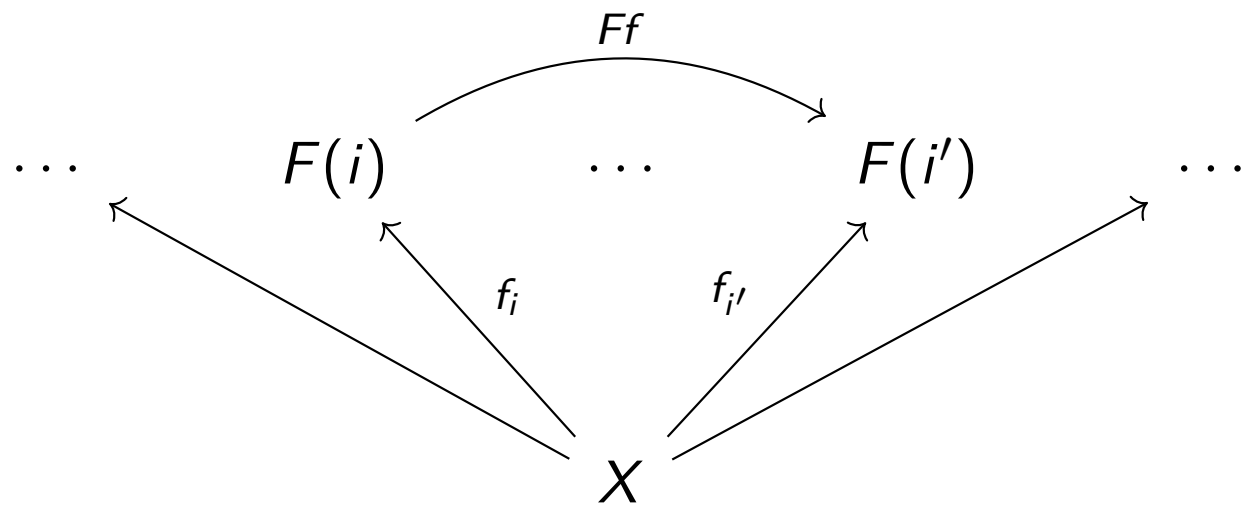
- We say $\mathcal{C}$ has *colimits of shape* $\mathcal{J}$, if $\mathrm{Colim}F$ exists for any functor $F : \mathcal{J} \to \mathcal{C}$.
- We say $\mathcal{C}$ has *all colimits* if it has colimits of every shape $\mathcal{J}$.
- A *directed colimits* is a colimit where $\mathcal{J}$ is a directed set.
- Let $\mathbf{2}_{\mathrm{dist}}$ be the discrete category with two objects. A category $\mathcal{C}$ has coproduct of two objects iff $\mathcal{C}$ has colimits of shape $\mathbf{2}_{\mathrm{dist}}$.
- One can generalize this to arbitrary coproducts.

Limits

Dually, we have the notion of cones and limits. Let $\mathcal{C}$ and $\mathcal{J}$ be two categories, and $F : \mathcal{J} \to \mathcal{C}$ be a functor.

Definition

A *cone* to F is a pair $(X, (f_i)_{i \in \mathcal{J}})$ where $X : \mathcal{C}$ and $f_i : X \to F(i)$ such that for all $f : i \to i'$ in $\mathcal{J}$, we have $Ff \circ f_i = f_{i'}$

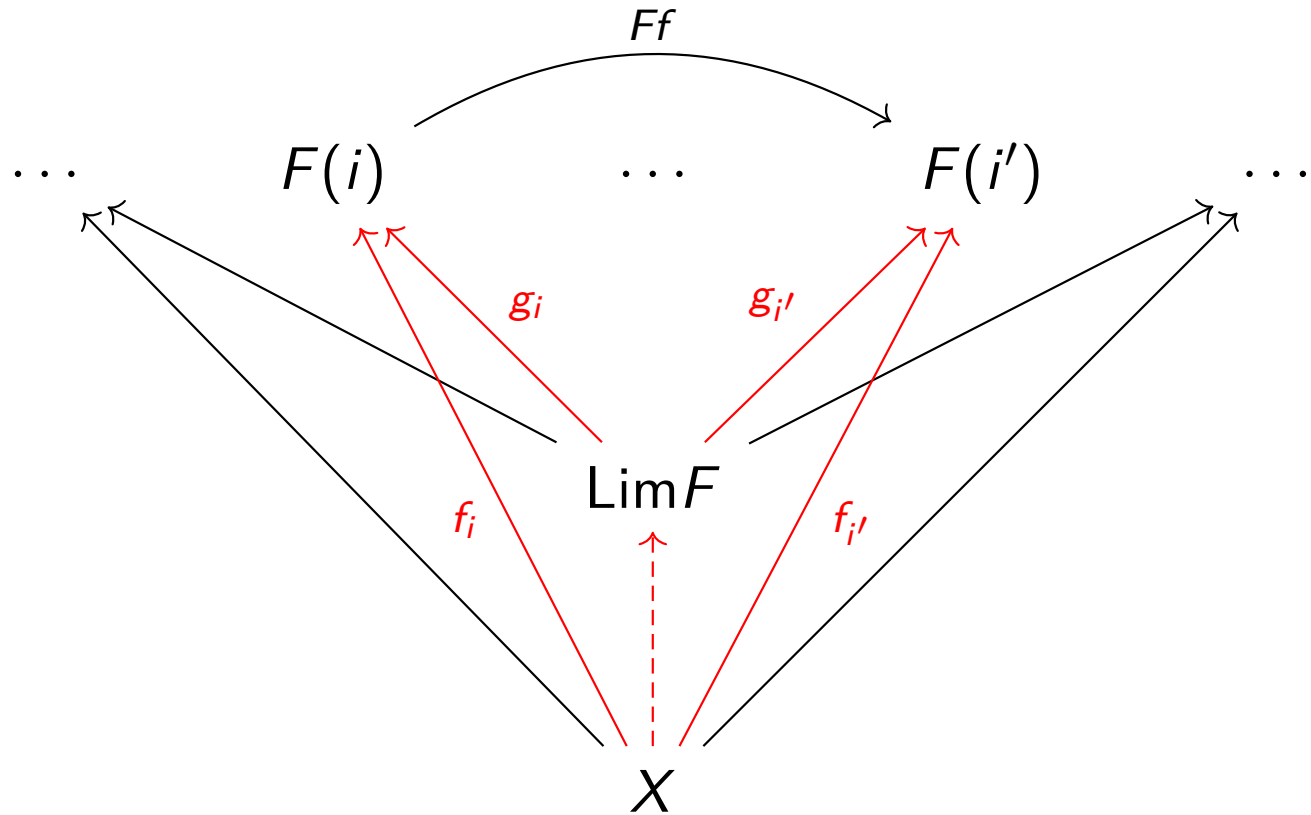


Limits

Let $\mathcal{C}$ and $\mathcal{J}$ be two categories, and $F : \mathcal{J} \to \mathcal{C}$ be a functor.

Definition

A *limit* of F, denoted by $\mathrm{Lim}F$, is a cone such that the following red diagrams commutes for any other cone $(X, (f_i)_{i \in \mathcal{J}})$



Limits

- A category $\mathcal{C}$ has product of two objects iff $\mathcal{C}$ has limits of shape $\mathbf{2}_{\text{dist}}$.
- One can generalize this to arbitrary products.
- Let $f : X \to Z$ and $g : Y \to Z$ be two morphisms in $\mathcal{C}$. The *pullback* of f and g is an object of $\mathcal{C}$, denoted by $X \times_Z Y$, with morphisms $p_1 : X \times_Z Y \to X$ and $p_2 : X \times_Z Y \to Y$ such that

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

The pullback is just limit of shape of the following category: