Automata over infinite objects

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Part I:
Finite state automata over infinite words

Words

- Finite alphabet Σ, e.g.
  $\{0,1\}, \{a,b,c,\ldots,y,z\}, \{0,1,2,3,4,5\}, \{\texttt{req, wait,grant}\}$

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Finitely bounded behaviour.

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Over finite words:

- Regular expressions
- Logics: e.g. FO, MSO
- Automata: DFA, NFA
- Monoids
- ...

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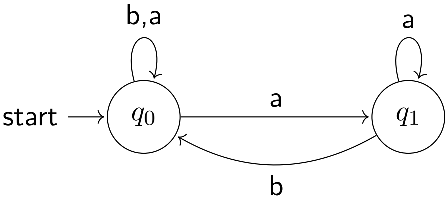
Finite automata over infinite words



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- Transitions Δ
- Deterministic or nondeterministic
- Infinite runs

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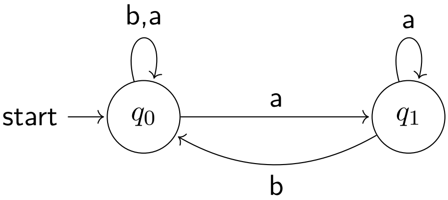

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- **Parity**: $\alpha : \Delta \to \{0, 1, \ldots, h\}$.
  Accept if the **highest priority** that occurs infinitely often is ***even***
  $\exists 2p \in [0, h]. \forall n. \exists i > n. \alpha(q_i) = 2p \wedge \exists n. \forall i > n. \alpha(q_i) \leq 2p$

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- **Müller**: Boolean combination of transitions occurring infinitely/finitely often.

Büchi and coBüchi as simple parity

- Büchi $=$ Parity $\{1, 2\}$
- coBüchi $=$ Parity $\{0, 1\}$

Exercise: automata for these languages

- Liveness: **`progress`** occurs infinitely often
- Eventually **`error`** does not occur
- {`req`,`grant`,`wait`}: Every **`request`** is eventually followed by a **`grant`**
- {`req`,`grant`,`wait`}: If **`request`** occurs infinitely often, then **`grant`** occurs infinitely often

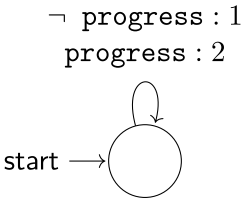


Figure: **progress** occurs infinitely often

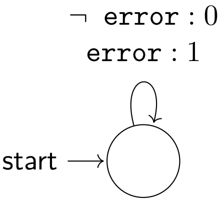
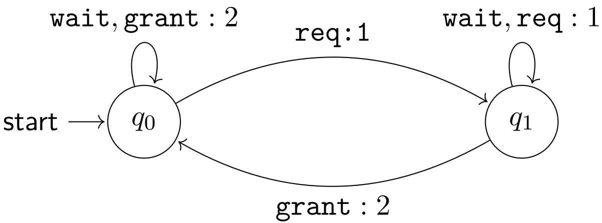
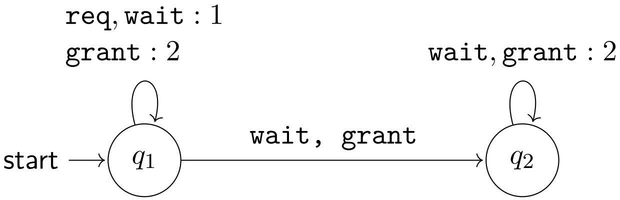


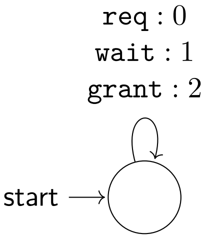
Figure: Eventually **error** does not occur

Every request is eventually followed by a grant



If infinitely many `req` then infinitely many `grants`

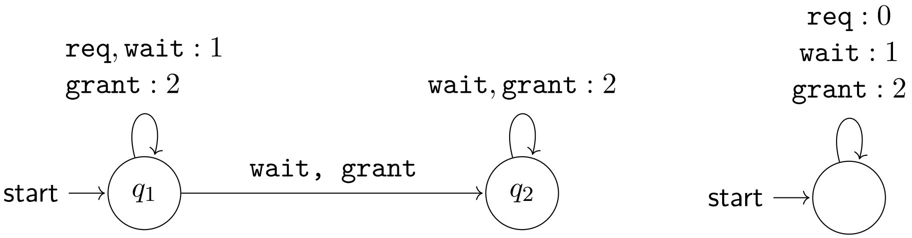




Trade-offs

- determinism vs. nondeterminism
- acceptance condition

Parity vs Büchi



Parity vs Büchi

Theorem

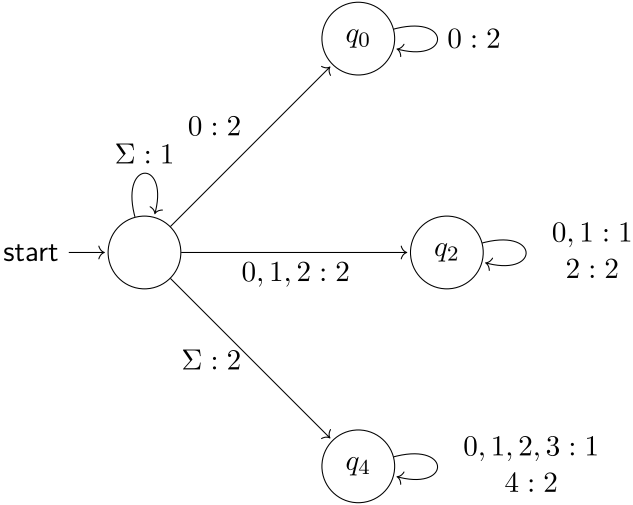
Every parity automaton can be turned into a nondeterministic Büchi automaton.

Exercise

Nondeterministic Büchi automaton for $\texttt{Parity}_{\{0,1,2,3,4,5\}}$

- Alphabet $\{0, 1, 2, 3, 4, 5\}$
- Accept if the max that occurs infinitely often is even.

Nondeterministic Büchi automaton for $\texttt{Parity}_{0,1,2,3,4}$



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Given A with priorities $0, 1, 2, 3, 4$, the composition of A with the nondeterministic Büchi automaton for $\texttt{Parity}_{\{0,1,2,3,4\}}$ is a nondeterministic Büchi automaton for A.

Parity vs. Büchi (cont.)

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Proof Idea

- Consider a deterministic automaton with priorities $\{1, \ldots, n+1\}$ over alphabet $\{0, \ldots n\}$.
- $a_0 a_1 \ldots$ where if the run over $a_0 \ldots a_i$ has as last priority p, then $a_{i+1} = p - 1$
- This word is accepted if and only if it is not in $\texttt{Parity}_{0\ldots n}$, so the automatons does not recognise this language.

Trade-offs

- For deterministic automata, the Parity-Index Hierarchy is infinite.
- For nondeterministic automata, it collapses to Büchi.

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- Non-determinism $=$ disjunctions... what about conjunctions?

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With priorities:

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- Verifier strategy $=$ run tree

Alternating automaton

Exercise

Alternating coBüchi automaton for
$\{$`wait, req, grant`$\}$: Every grant is eventually followed by a request.

Trade-offs

The following are equally expressive:

- Alternating weak automata (SCCs are accepting or rejecting)
- Nondeterministic Büchi automata
- Deterministic parity

Logic and automata

Some formulas:

- $\mu X.a \vee \Diamond X$ (eventually a)
- $\nu X.a \wedge \Diamond X$ (always a)
- $\nu X.(\Diamond X \wedge \mu Y.a \vee \Diamond Y)$ (Always eventually a)
- $\mu X.\nu Y.(a \wedge \Diamond Y) \vee (\neg a \wedge \Diamond X)$ (Eventually always a)

Part II.
Finite state automata over infinite trees

Infinite trees

- Labelled *unordered* trees
- Unfolded rooted graphs
- $(V, E, \ell : V \to \Sigma)$ where (V, E) is a tree

Examples

- a occurs finitely often on all paths.
- A subtree without any a is reachable from everywhere.
- b is reachable whenever a occurs.
- Player 1 wins the a-reachability game (alternating reachability).

Alternating parity tree automata

- $\Diamond q$: accept from q in *some* successor

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- $\Diamond q$: accept from q in *some* successor
- $\Box q$: accept from q in *all* successors
- Transitions: DNF formula over $\{\Diamond q, \Box q \mid q \in Q\}$
  $q \xrightarrow{a} (\Diamond q_1 \wedge \Diamond q_2) \vee \Box q_3$

Example

- $q_0 \xrightarrow{\Sigma:0} \Box q_0 \wedge \Diamond q_1$
- $q_1 \xrightarrow{a:1} \Diamond q_1$
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Examples

- a occurs finitely often on all paths.
- A subtree without any a is reachable from everywhere.
- b is reachable whenever a occurs.
- Player 1 wins the a-reachability game (alternating reachability).

Automata and the modal μ-calculus

Intuition:

- Formula $\longleftrightarrow$ Automaton
- subformulas $\longleftrightarrow$ states
- $\Diamond, \Box \longleftrightarrow$ transitions
- $\mu, \nu \longleftrightarrow$ priorities
- nested fixpoint semantics $\longleftrightarrow$ parity condition

Intuitive equivalence, but...

Many annoying technicalities of varying importance:

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Recall, over words, these are equally expressive:

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Over trees:

- Automata can not always be determinised
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- Parity-index hierarchy is infinite for deterministic, nondeterministic and alternating automata.

A classic for a reason

- Highly expressive
  subsumes CTL, bisimulation invariant MSO...
- Highly decidable

Some classic open problems on tree automata

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  NP$\cap$ co-NP but no known PTime algorithm...
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- Deciding the parity index:
  Given a tree language, what is the least number of priorities needed to express it with an alternating or nondeterministic automaton?

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And for cyclic proofs?

Reading

- Automata, logics, and infinite games: a guide to current research (2003) *E Grädel, W Thomas, T Wilke*
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