A categorical approach to automata learning and minimization

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Some references

T. Colcombet and D. Petrişan, *Automata minimization: a functorial approach*. Log. Methods Comput. Sci., 16(1), 2020

T. Colcombet, D. Petrisan, R. Stabile, *Learning Automata and Transducers: A Categorical Approach.* CSL 2021

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Further reading:

Q. Aristote, S. van Gool, D. Petrişan, M. Shirmohammadi, *Learning Weighted Automata over Number Rings, Concretely and Categorically* LICS 2025

`https://arxiv.org/pdf/2504.16596`

This tutorial is about …

the interplay between category theory and automata theory.

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In particular, we will see how the category-theoretic approach

- provides a unifying framework for modelling various forms of automata,
- for obtaining generic algorithms for learning algorithms,

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the interplay between category theory and automata theory.

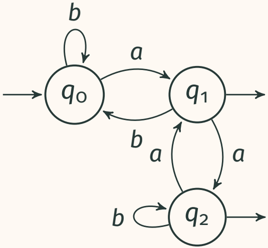
In particular, we will see how the category-theoretic approach

- provides a unifying framework for modelling various forms of automata,
- for obtaining generic algorithms for learning algorithms,
- highlights the link between automata learning and minimization.

Automata with effects

Complete Deterministic Finite Automata

Let’s rewrite the definition of a complete DFA…



$$\begin{tikzcd}
1 \arrow[r, "q_0"] & \{q_0, q_1, q_2\} \arrow[loop above, "\delta_a"] \arrow[loop below, "\delta_b"] \arrow[r, "\chi_{q_1,q_2}"] & 2
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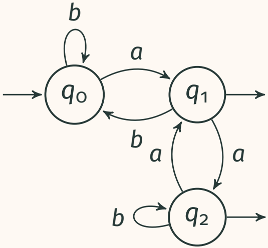

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To give a complete DFA over A amounts to give a set Q and the functions depicted below.

$$\begin{tikzcd}
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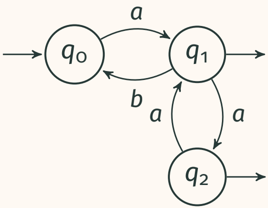

$$\begin{tikzcd} 1 \arrow[r, "q_0", "\shortmid" marking] & \{q_0, q_1, q_2\} \arrow[loop above, "\delta_a", "\shortmid" marking] \arrow[loop below, "\delta_b", "\shortmid" marking] \arrow[r, "\chi_{q_1,q_2}", "\shortmid" marking] & 1 \end{tikzcd}$$

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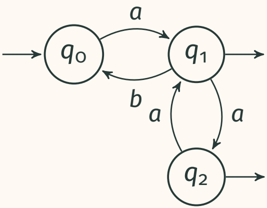

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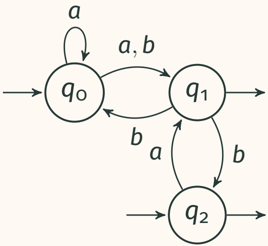
\end{tikzcd}$$

To give a DFA over A amounts to give a set Q and the **partial functions** depicted below.

$$\begin{tikzcd}
1 \arrow[r, "q_0", "/" marking] & Q \arrow[loop above, "\delta_\sigma (\sigma \in A)", "/" marking] \arrow[r, "\chi_F", "/" marking] & 1
\end{tikzcd}$$

Nondeterministic Finite Automata

Let's rewrite the definition of an NFA...



$$\begin{tikzcd}
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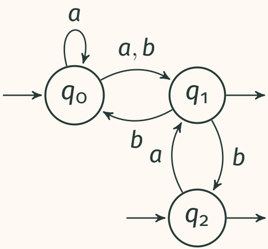

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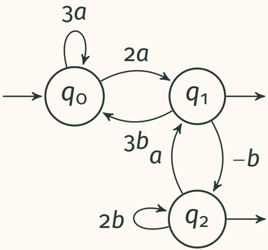
\end{tikzcd}$$

To give an NFA amounts to give a set Q and the **relations** depicted below.

$$\begin{tikzcd}
1 \arrow[r, "I", "\bullet"{marking}] & Q \arrow[loop above, "\delta_\sigma\,(\sigma \in A)", "\bullet"{marking}] \arrow[r, "\chi_F", "\bullet"{marking}] & 1
\end{tikzcd}$$

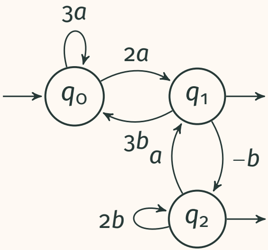
Weighted automata over a semiring



$$\begin{tikzcd} R \arrow[r, "i"] & R^{\{q_0,q_1,q_2\}} \arrow[r, "fin"] \arrow[loop above, "\delta_a"] \arrow[loop below, "\delta_b"'] & R \end{tikzcd}$$

e.g., $\delta_a(q_0) = 3q_0 + 2q_1$,
$fin(q_1) = fin(q_2) = 1$, etc...

Weighted automata over a semiring



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e.g., $\delta_a(q_0) = 3q_0 + 2q_1$,
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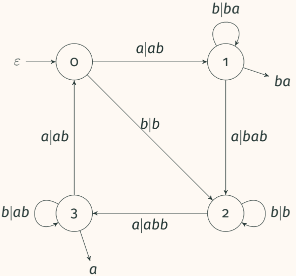
To give a WA over R amounts to give a free module R^Q and the **linear maps** depicted below.

$$\begin{tikzcd} R \arrow[r, "i"] & R^Q \arrow[loop above, "\delta_\sigma (\sigma \in A)"] \arrow[r, "f"] & R \end{tikzcd}$$

Sequential transducers

A sequential transducer with input alphabet A and output alphabet B consists of:

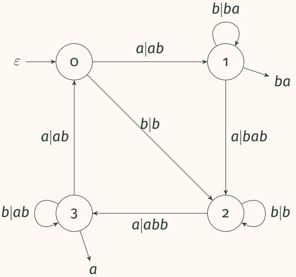
- a finite set of states Q
- an initial state with an initial output in B^*, or an undefined initial state



Sequential transducers

A sequential transducer with input alphabet A and output alphabet B consists of:

- a finite set of states Q
- an initial state with an initial output in B^*, or an undefined initial state
- for each $a \in A$ a transition function $Q \to B^* \times Q + 1$



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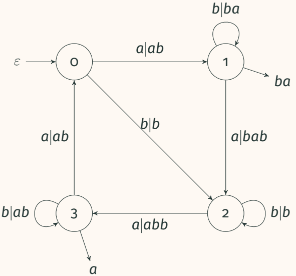
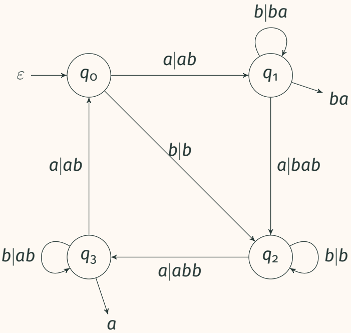
A sequential transducer with input alphabet A and output alphabet B consists of:

- a finite set of states Q
- an initial state with an initial output in B^*, or an undefined initial state
- for each $a \in A$ a transition function $Q \to B^* \times Q + 1$
- for each state in Q, either an output word in B^* or undefined.



Sequential Transducers

Let's rewrite the definition of a sequential transducer with input alphabet $\{a, b\}$ and output alphabet $B = \{a, b\}$.



$$\begin{tikzcd}
1 \arrow[r, "q_0", "\blacklozenge" marking] & \{q_0, q_1, q_2, q_3\} \arrow[loop above, "\delta_a", "\blacklozenge" marking] \arrow[loop below, "\delta_b"', "\blacklozenge" marking] \arrow[r, "\chi_{q_1,q_2}", "\blacklozenge" marking] & 1
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Note that an arrow $X \xrightarrow{\blacklozenge} Y$ is actually a function $X \to 1 + B^* \times Y$.

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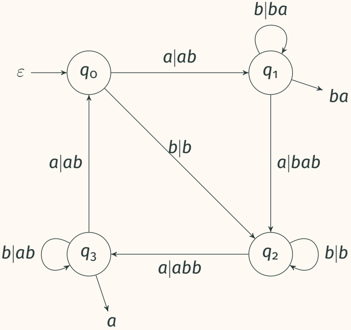

$$\begin{tikzcd} 1 \arrow[r, "q_0"] & \{q_0, q_1, q_2, q_3\} \arrow[r, "\chi_{q_1,q_2}"] \arrow[loop above, "\delta_a"] \arrow[loop below, "\delta_b"] & 1 \end{tikzcd}$$

Note that an arrow $X \longrightarrow Y$ is actually a function $X \to 1 + B^* \times Y$.

To give a seq. transducer amounts to give a set Q and arrows:

$$\begin{tikzcd} 1 \arrow[r, "q_0"] & Q \arrow[r, "\chi_F"] \arrow[loop above, "\delta_\sigma (\sigma \in A)"] & 1 \end{tikzcd}$$

Word automata

$$\begin{tikzcd}
1 \arrow[r, "q_0"] & Q \arrow[r, "\chi_F"] \arrow[loop above, "\delta_\sigma\,(\sigma\in A)"] & 2
\end{tikzcd}$$

complete DFAs — Set (sets and functions)

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DFAs — Set$_\bullet$ (sets and partial functions)

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$$\begin{tikzcd}
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NFA — Rel (sets and relations)

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$$\begin{tikzcd}
R \arrow[r, "i", "\circ" marking] & R^Q \arrow[loop above, "\delta_\sigma\,(\sigma\in A)", "\circ" marking] \arrow[r, "f", "\circ" marking] & R
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WAs over R — $\mathsf{FreeMod}_R$ (R-modules and linear maps)

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WAs over R — FreeMod_R (R-modules and linear maps)

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Sequential transducers — ?

The output category for subsequential transducers

We consider partial actions for the free monoid B^*.

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We consider a category $\mathcal{T}$ with

- objects: sets $X, Y, Z, \ldots$
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How to compose $f\colon X \relbar\joinrel\blacklozenge\joinrel\rightarrow Y$ and $g\colon Y \relbar\joinrel\blacklozenge\joinrel\rightarrow Z$?

$g \circ f\colon X \relbar\joinrel\blacklozenge\joinrel\rightarrow Z$ (i.e. $g \circ f\colon X \to B^* \times Z + 1$) is given by

$$g \circ f(x) = \begin{cases} (uv, z) & \text{if } f(x) = (u, y) \text{ and } g(y) = (v, z) \\ \perp & \text{otherwise.} \end{cases}$$

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This is the Kleisli category for the monad $T\colon \mathrm{Set} \to \mathrm{Set}$ given by $T(X) = B^* \times X + 1$, which associates to each set X the free partial action of B^* on X.

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WAs over R — FreeMod$_R$ (R-modules and linear maps)

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\end{tikzcd}$$

Sequential transducers — $\mathcal{T}$

$$\begin{tikzcd}
I \arrow[r, "i"] & Q \arrow[loop above, "\delta_\sigma\,(\sigma\in A)"] \arrow[r, "f"] & O
\end{tikzcd}$$

$(\mathcal{C}, I, O)$-automata — $\mathcal{C}$

Word automata as functors

Word automata on A^* are **functors** $\mathcal{A}\colon\mathcal{I}\to\mathcal{C}$, where the input category $\mathcal{I}$ is freely generated by

$$\begin{tikzcd}
\text{in} \arrow[r, "\triangleright"] & \text{states} \arrow[r, "\triangleleft"] \arrow[loop above, "a\ (a\in A)"] & \text{out}
\end{tikzcd}$$

The data given by the functor $\mathcal{A}$ is a tuple $\langle Q, i, f, (\delta_a)_{a\in A}\rangle$, where

- Q is an object of $\mathcal{C}$.
- $i\colon I\to Q$ is the «initial» arrow, for some object I of $\mathcal{C}$
- $f\colon Q\to F$ is the «final» arrow, for some object F of $\mathcal{C}$
- $\delta_a\colon Q\to Q$ is the «transition» arrow for each $a\in A$

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The language accepted by $\mathcal{A}$ is a map $L_{\mathcal{A}}\colon A^*\to\mathcal{C}(I,F)$ that associates to a word $w=a_1\dots a_n$ the composite morphism

$$I\xrightarrow{i}Q\xrightarrow{\delta_{a_1}}Q\xrightarrow{\delta_{a_2}}\dots\xrightarrow{\delta_{a_n}}Q\xrightarrow{f}F$$

Automata and languages as functors

An automaton $\mathcal{A}$ accepts a language $\mathcal{L}$ when the next diagram commutes

$$\begin{tikzcd}
\text{in} \arrow[rr, "\triangleright w \triangleleft \,:\, w \in A^*"] & & \text{out} \\
\text{in} \arrow[r, "\triangleright"] & \text{states} \arrow[r, "\triangleleft"] \arrow[loop above, "a"] & \text{out}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O} \arrow[r, "\mathcal{L}"] \arrow[d, hook] & \mathcal{C} \\
\mathcal{I} \arrow[ur, "\mathcal{A}"'] &
\end{tikzcd}$$

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\text{in} \arrow[r, "\triangleright"] & \text{states} \arrow[r, "\triangleleft"] \arrow[loop above, "a"] & \text{out}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O} \arrow[r, "\mathcal{L}"] \arrow[d, hook] & \mathcal{C} \\
\mathcal{I} \arrow[ur, "\mathcal{A}"'] &
\end{tikzcd}$$

For every language $\mathcal{L} \colon \mathcal{O} \to \mathcal{C}$ we consider a category $\mathsf{Auto}_{\mathcal{L}}$ of automata accepting $\mathcal{L}$.

$\mathcal{O}$ can be seen as an “observation” subcategory of $\mathcal{I}$.

Much of the ensuing theory can be developed independently on the precise shape of $\mathcal{I}$.

The output categories we have seen so far

What do these categories have in common ?

- Set – the category of sets and functions
- $\mathbf{Set}_\bullet$ – the category of sets and partial functions
- Rel – the category of sets and relations
- Vec – the category of vector spaces and linear transformations
- $\mathcal{T}$ – the category of free partial actions of some free monoid B^* and their morphisms

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Answer. They are categories of free algebras (aka Kleisli categories) for monads specifying some effect:

- the identity monad
- the Maybe monad (aka option)
- the powerset monad – non-determinism
- the monad of partial free actions of B^*.

Changing output categories

Adjunctions – Recap

Having an adjunction

$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & D \arrow[l, bend left, "U"] \end{tikzcd}$$

means we have isomorphisms $C(X, UY) \cong D(FX, Y)$ natural in both X and Y.

$f\colon FX \to Y$ yields $f_\flat\colon X \to UY$

$g\colon X \to UY$ yields $g^\sharp\colon FX \to Y$

Adjunctions – example 1

Exercise. Describe an adjunction between Set and $\mathrm{Set}_\bullet$.

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Set}_\bullet \arrow[l, bend left, "U"]
\end{tikzcd}$$

means we have isomorphisms $Set(X, UY) \cong \mathrm{Set}_\bullet(FX, Y)$ natural in both X and Y.

$$f\colon FX \not\longrightarrow Y \quad \text{in } \mathrm{Set}_\bullet \text{ yields } f_\flat\colon X \to UY \text{ in Set}$$

$$g\colon X \to UY \text{ in Set yields } \quad g^\sharp\colon FX \not\longrightarrow Y \quad \text{in } \mathrm{Set}_\bullet$$

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Exercise. Describe an adjunction between Set and $\mathsf{Set}_\bullet$.

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means we have isomorphisms $Set(X, UY) \cong \mathsf{Set}_\bullet(FX, Y)$ natural in both X and Y.

Answer. $FX = X$, $UX = 1 + X$...

$$f{:}X \mathrel{\ooalign{$\longrightarrow$\cr\hfil$/$\hfil}} Y \quad \text{in } \mathsf{Set}_\bullet \text{ yields } f_\flat{:}X \to 1 + Y \text{ in Set}$$

$$g{:}X \to 1 + Y \text{ in Set yields } \quad g^\sharp{:}X \mathrel{\ooalign{$\longrightarrow$\cr\hfil$/$\hfil}} Y \quad \text{in } \mathsf{Set}_\bullet$$

Adjunctions – example 2

Exercise. Describe an adjunction between Set and Rel.

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Rel} \arrow[l, bend left, "U"]
\end{tikzcd}$$

means we have isomorphisms $Set(X, UY) \cong \mathrm{Rel}(FX, Y)$ natural in both X and Y.

$$f\colon FX \longrightarrow\!\!\bullet\!\!\longrightarrow Y \quad \text{in Rel yields } f_\flat\colon X \to UY \text{ in Set}$$

$$g\colon X \to UY \text{ in Set yields } \quad g^\sharp\colon FX \longrightarrow\!\!\bullet\!\!\longrightarrow Y \quad \text{in Rel}$$

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Answer. $FX = X$, $UX = \mathcal{P}X$...

$$f : X \longrightarrow\!\!\bullet\!\!\longrightarrow Y \quad \text{in Rel yields } f_\flat : X \to \mathcal{P}Y \text{ in Set}$$

$$g : X \to \mathcal{P}Y \text{ in Set yields} \quad g^\sharp : X \longrightarrow\!\!\bullet\!\!\longrightarrow Y \quad \text{in Rel}$$

Adjunctions – example 3

The above adjunctions are all standard in category theory:
They are adjunctions between Set and the Kleisli category for a monad.

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Kl}(T) \arrow[l, bend left, "U"]
\end{tikzcd}$$

with F identity on objects and $UX = TX$.

Getting rid of effects – or lifting adjunctions

Suppose we have the ‘same’ language interpreted in two different categories related by an adjunction $F \dashv U$:

$$L_C\colon A^* \to C(X, UY) \text{ and } L_D\colon A^* \to D(FX, Y).$$

$$\begin{tikzcd}
Auto(L_C) \arrow[rr, bend left] \arrow[d, "States"] \arrow[rr, phantom, "\perp"] & & Auto(L_D) \arrow[ll, bend left] \arrow[d, "States"] \\
C \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & D \arrow[ll, bend left, "U"]
\end{tikzcd}$$

Lifting adjunctions – completing DFAs

Suppose we have the ‘same’ regular language interpretted in two different categories (Set and Rel) related by an adjunction $F \dashv U$:

$$L_{\mathrm{Set}} : A^* \to \mathrm{Set}(1,2) \text{ and } L_{\mathrm{Set}_\bullet} : A^* \to \mathrm{Set}_\bullet(1,1).$$

$$\begin{tikzcd}
\mathit{Auto}(L_{\mathrm{Set}}) \arrow[r, bend left, "\mathit{inclusion}"] \arrow[r, phantom, "\perp"] \arrow[d, "\mathit{States}" description] & \mathit{Auto}(L_{\mathrm{Set}_\bullet}) \arrow[l, bend left, "\mathit{completion}"] \arrow[d, "\mathit{States}" description] \\
\mathrm{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Set}_\bullet \arrow[l, bend left, "U"]
\end{tikzcd}$$

Corollary 1. The completion of a DFA is a right adjoint to inclusions of complete DFA in DFA.

Lifting adjunctions – determinization

Suppose we have the ‘same’ regular language interpretted in two different categories (Set and Rel) related by an adjunction $F \dashv U$:

$$L_{\mathsf{Set}}\colon A^* \to \mathsf{Set}(1, U1) \text{ and } L_{\mathsf{Rel}}\colon A^* \to \mathsf{Rel}(F1, 1).$$

$$\begin{tikzcd}
\mathit{Auto}(L_{\mathsf{Set}}) \arrow[r, bend left] \arrow[r, phantom, "\perp"] \arrow[d, "\mathit{States}" description] & \mathit{Auto}(L_{\mathsf{Rel}}) \arrow[l, bend left] \arrow[d, "\mathit{States}" description] \\
\mathsf{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{Rel} \arrow[l, bend left, "U"]
\end{tikzcd}$$

Corollary 1. The determinization of NFA is a right adjoint to inclusions of DFA in NFA.

Lifting adjunctions – determinization

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\end{tikzcd}$$

Corollary 1. The determinization of NFA is a right adjoint to inclusions of DFA in NFA.

Corollary 2. Initial automata for free in Kleisli valued automata.

Automata in a category: minimization

DFA Minimization

Given a language $L \subseteq A^*$ and a word $u \in A^*$ the left quotient $u^{-1}L$ is the set

$$\{v \in A^* \mid uv \in L\}$$

The Myhill-Nerode equivalence is defined by

$$u \cong_L v \text{ iff } u^{-1}L = v^{-1}L$$

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Theorem (Myhill-Nerode). A language L is regular iff it has only finitely many left quotients iff $\cong_L$ has finite index.

Proof. $\Rightarrow$ If an automaton $\mathcal{A} = (Q, q_0, F, (\delta_a)_{a \in A})$ accepts a language L, then the automaton $(Q, \delta_u(q_0), F, (\delta_a)_{a \in A})$ accepts $u^{-1}L$.

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$\Leftarrow$ Consider the Nerode automaton of L, that is $(Q, q_0, F, (\delta_a)_{a \in A})$, where

- $Q = \{u^{-1}L \mid u \in A^*\}$,
- $q_0 = L$
- $F = \{u^{-1}L \mid u \in L\}$ and
- $\delta_a(u^{-1}L) = (ua)^{-1}L$.

DFA Minimization

How do we minimize an automaton $\mathcal{A}$?

- remove all states that are not accessible from the initial state. We obtain the reachable sub-automaton Reach$(\mathcal{A})$.
- Merge all states that accept the same language, we obtain the observable quotient Obs(Reach$(\mathcal{A})$).

DFA Minimization

How do we minimize an automaton $\mathcal{A}$?

- remove all states that are not accessible from the initial state. We obtain the reachable sub-automaton $\mathrm{Reach}(\mathcal{A})$.
- Merge all states that accept the same language, we obtain the observable quotient $\mathrm{Obs}(\mathrm{Reach}(\mathcal{A}))$.

There are several algorithms for minimizing automata:
Moore, Hopcroft, Brzozowski.

Minimization of $\mathcal{C}$-automata

- What does it mean for a $\mathcal{C}$-automaton to be minimal?
- What are sufficient conditions on $\mathcal{C}$ so that a minimal automaton for a language exists?
- Can we compute the minimal automaton effectively?

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Thus we need a notion of «quotient» (surjection for sets) and «sub-object» (injection for sets), i.e. a factorization system.

Three more category-theoretic notions

- An initial object in a category $\mathcal{C}$ is an object X such that for any object A of $\mathcal{C}$ there is a unique morphism $! \colon X \to A$.
  Question: what is the initial object in Set*? And in* Rel*?*

Three more category-theoretic notions

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  Question: what is the final object in Set*? And in* Rel*?*
- A factorization system provides the category-theoretic generalizations for the notions of “quotients” and “subobjects”, definition on next slide...

The three ingredients for minimization

When does a ‘minimal’ automaton accepting a language $\mathcal{L}$ exist?

$$\begin{tikzcd}
\mathcal{O} \arrow[r, "\mathcal{L}"] \arrow[d, hook] & \mathcal{C} \\
\mathcal{I} \arrow[ur, dashed, "\mathtt{Min}(\mathcal{L})?"] &
\end{tikzcd}$$

The three ingredients for minimization

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left Kan ext?

$$\begin{tikzcd}
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\mathcal{I} \arrow[uuur] \arrow[uur, dashed] \arrow[ur] & &
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- a factorization system

then $\texttt{Min}(\mathcal{L})$ is obtained as the factorization

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$$\begin{tikzcd}
\mathcal{O} \arrow[r, "\mathcal{L}"] \arrow[d, hook] & \mathcal{C} \\
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\end{tikzcd}$$

If the category of automata accepting $\mathcal{L}$ has

- an initial object $\mathcal{A}_{\texttt{init}}(\mathcal{L})$, ✓when $\mathcal{C}$ has copowers
- a final object $\mathcal{A}_{\texttt{final}}(\mathcal{L})$, and, ✓when $\mathcal{C}$ has powers
- a factorization system ✓when $\mathcal{C}$ has one

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Factorization systems

Factorization systems are a generalization of the next situation: Every function $f\colon X \to Y$ can we written as a composite

$$\begin{tikzcd} X \arrow[r, two heads, "e"] & Z \arrow[r, tail, "m"] & Y \end{tikzcd}$$

with e a surjection and m and injection, and, moreover, such a decomposition is unique up-to isomorphism.

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In a category $\mathcal{C}$, a factorization system consists of two classes of morphisms, E and M, so that:

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- every morphism $f : X \to Y$ can we written as a composite $m \circ e$ with $e \in E$ and $m \in M$;
- the decomposition is functorial, i.e. any two decompositions are isomorphic

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Trivial example: minimizing DFAs

The initial automaton $\mathcal{A}_{\mathtt{init}}$ for Set-automata accepting a language L is the following :

$$\begin{tikzcd}
1 \arrow[r, "\varepsilon"] & A^{*} \arrow[r, "L?"] \arrow[loop above, "w \mapsto wa"] & 2
\end{tikzcd}$$

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The final automaton $\mathcal{A}_{\mathtt{final}}$ for Set-automata accepting a language L is the following :

$$\begin{tikzcd}
1 \arrow[r, "L"] & 2^{A^*} \arrow[r, "\varepsilon?"] \arrow[loop above, "K \mapsto a^{-1}K"] & 2
\end{tikzcd}$$

Trivial example: minimizing DFAs accepting L

The unique map from the initial to the final automaton is given by $!\colon A^* \to 2^{A^*}$, defined by $w \mapsto w^{-1}L$.

$$\begin{tikzcd}
 & A^* \arrow[d, "\text{reachedState}" description] \arrow[dr, bend left, "\mathcal{L}?"] & \\
1 \arrow[ur, bend left, "\varepsilon"] \arrow[r, "i"] \arrow[dr, bend right, "\mathcal{L}"'] & Q \arrow[r, "f"] \arrow[d, "\text{acceptedLanguage}" description] & 2 \\
 & 2^{A^*} \arrow[ur, bend right, "\varepsilon?"'] &
\end{tikzcd}$$

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Another trivial example

$\mathbb{R}$-weighted automata, i.e. $(\mathsf{Vec}, \mathbb{R}, \mathbb{R})$-automata
accepting a $(\mathsf{Vec}, \mathbb{R}, \mathbb{R})$-language

$$\begin{tikzcd}
 & \bigoplus\limits_{u\in A^*} \mathbb{R} \arrow[dr, bend left, "L?"] \arrow[d, "\mathsf{reachedState}" description] & \\
\mathbb{R} \arrow[ur, bend left, "\varepsilon"] \arrow[r, "i"] \arrow[dr, bend right, "L"'] & Q \arrow[r, "F"] \arrow[d, "\mathsf{acceptedLanguage}" description] & \mathbb{R} \\
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 & \prod_{u\in A^*}\mathbb{R} \arrow[ur, bend right, "\varepsilon?"'] &
\end{tikzcd}$$

The minimal transducer in a picture

We obtain $\mathtt{Min}(\mathcal{L})$ – the minimal subsequential transducer as obtained by Choffrut!

$$\begin{tikzcd}
 & A^* \arrow[d, two heads] \arrow[dr, bend left, "\mathcal{L}?"] & \\
1 \arrow[ur, bend left, "\varepsilon"] \arrow[r, "i"] \arrow[dr, bend right, "\mathcal{L}"'] & \mathtt{Min}(\mathcal{L}) \arrow[r, "f"] \arrow[d, tail] & 1 \\
 & \mathsf{Irr}(A^*, B^*) \arrow[ur, bend right, "\varepsilon?"'] &
\end{tikzcd}$$

The minimal transducer in a picture

We obtain $\texttt{Min}(\mathcal{L})$ – the minimal subsequential transducer as obtained by Choffrut! In fact it also works if we replace B^* by a trace monoid.

$$\begin{tikzcd}
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Minimal Automaton $\mathtt{Min}(\mathcal{L})$ for a Language

The automaton $\mathtt{Min}(\mathcal{L})$ divides any other automaton accepting $\mathcal{L}$.

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dr, bend left] & \\
\mathcal{A}_{\mathtt{init}}(L) \arrow[ur, bend left] & & \mathcal{A}_{\mathtt{final}}(L)
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$$\begin{tikzcd}
 & & \mathcal{A} \arrow[rd, bend left] & \\
\mathcal{A}_{\mathtt{init}}(L) \arrow[rru, bend left] \arrow[r, two heads] & \mathtt{reach}(\mathcal{A}) \arrow[ru, tail] & & \mathcal{A}_{\mathtt{final}}(L)
\end{tikzcd}$$

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\mathcal{A}_{\texttt{init}}(L) \arrow[urr, bend left] \arrow[r, two heads] & \texttt{reach}(\mathcal{A}) \arrow[ur, tail] \arrow[rr, two heads] & & \texttt{obs}(\texttt{reach}(\mathcal{A})) \arrow[r, tail] & \mathcal{A}_{\texttt{final}}(L)
\end{tikzcd}$$

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 & & \mathcal{A} \arrow[rrd, bend left] & & \\
\mathcal{A}_{\texttt{init}}(L) \arrow[rru, bend left] \arrow[r, two heads] \arrow[rrd, two heads, bend right] & \texttt{reach}(\mathcal{A}) \arrow[ru, tail] \arrow[rr, two heads] & & \texttt{obs}(\texttt{reach}(\mathcal{A})) \arrow[r, tail] & \mathcal{A}_{\texttt{final}}(L) \\
 & & \texttt{Min}(L) \arrow[rru, tail, bend right] & &
\end{tikzcd}$$

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 & & \mathcal{A} \arrow[dr, bend left] & \\
\mathcal{A}_{\mathtt{init}}(L) \arrow[urr, bend left] \arrow[r, two heads] \arrow[drr, bend right, two heads] & \mathtt{reach}(\mathcal{A}) \arrow[ur, tail] \arrow[r, two heads] & \mathtt{obs}(\mathtt{reach}(\mathcal{A})) \arrow[r, tail] \arrow[d, dash, dashed] & \mathcal{A}_{\mathtt{final}}(L) \\
 & & \mathtt{Min}(L) \arrow[ur, tail, bend right] &
\end{tikzcd}$$

Thus far we identified simple sufficient conditions on $\mathcal{C}$ so that minimization of $\mathcal{C}$-automata is guaranteed!

Learning

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- The algorithm stops when the teacher agrees that the hypothesis automaton accepts the language L.

The L^*-algorithm: some definitions

- At each step, we maintain a pair of sets of words (Q, T), starting with $(\{\epsilon\}, \{\epsilon\})$.
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- consistency : $\forall q, q' \in Q\,.\forall a \in A\,.\;\; q \sim_T q' \Rightarrow qa \sim_T q'a$
- When (Q, T) is closed and consistent it is possible to build a hypothesis automaton $\mathcal{H}(Q, T)$

L^*-revisited

- At the (Q,T) stage of the algorithm the learner only has access to a fragment of the language:

$$L_{Q,T} : QAT \cup QT \rightarrowtail A^* \xrightarrow{L} 2$$

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- This can be represented by a notion of (Q, T)-biautomaton

$$\begin{tikzcd}
1 \arrow[r, "\triangleright q", "(q \in Q)"'] & Q_1 \arrow[r, shift left, "a \ (a \in A)"] \arrow[r, shift right, "\varepsilon"'] & Q_2 \arrow[r, "t \triangleleft", "(t \in T)"'] & 2
\end{tikzcd}$$

such that the following coherence diagrams commute

$$\begin{tikzcd}
 & Q_1 \arrow[dr, "a"] & \\
1 \arrow[ur, "\triangleright q"] \arrow[dr, "\triangleright qa"'] & & Q_2 \\
 & Q_1 \arrow[ur, "\varepsilon"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Q_2 \arrow[dr, "t \triangleleft"] & \\
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 & Q_2 \arrow[ur, "at \triangleleft"'] &
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Minimal (Q, T)-biautomaton and the hypothesis automaton

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We obtain a generic FunL* algorithm that instantiates to

- Angluin's original algorithm,
- the weighted automata over fields variant of L^*
- the sequential transducer variant

Further details: Thomas Colcombet, Daniela Petrisan, Riccardo Stabile: Learning Automata and Transducers: A Categorical Approach. CSL 2021

Perspectives

Minimization/learning for free ?

Are there some conditions on a monad T so that $\mathrm{Kl}(T)$ has all the required properties required for the existence of minimization/learning of $\mathrm{Kl}(T)$-automata ?

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Learning nominal automata, building on Victor Iwaniack's work on automata in toposes.

More details for learning

Minimal (Q, T)-biautomaton and the hypothesis automaton

We can compute the minimal (Q, T)-biautomaton in an arbitrary category* using off-the-shelf results from (Colcombet,P., 2017).

$$\begin{tikzcd}
1 \arrow[r, "\triangleright q_{min}"] & Q/{\sim_{T \cup AT}} \arrow[r, shift left, "a_{min}"] \arrow[r, shift right, "\varepsilon_{min}"'] & (Q \cup QA)/{\sim_T} \arrow[r, "t\triangleleft_{min}"] & 2
\end{tikzcd}$$

Recall $w \sim_T v$ iff $\forall u \in T.\quad wu \in L \Leftrightarrow vu \in L$

* under mild assumptions

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- ε_{min} is surjective iff (Q,T) is closed
- ε_{min} is injective iff (Q,T) is consistent

```
$Q = T := \{\varepsilon\}$
repeat
  while $(Q,T)$ not closed and consistent
    if $(Q,T)$ is not closed enlarge $Q$
      $(\forall q \in Q.\forall a \in A.\exists p \in Q.\ p \sim_T qa)$
    if $(Q,T)$ is not consistent enlarge $T$
      $(\forall q,q' \in Q.\forall a \in A.\ \ q \sim_T q' \Rightarrow qa \sim_T q'a)$
  ask an equivalence query for $\mathcal{H}(Q,T)$
  if the answer is no then
    add the counterexample and its
    prefixes to $Q$
until the answer is yes
return $\mathcal{H}(Q,T)$
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- ε_{min} is surjective iff (Q,T) is closed
- ε_{min} is injective iff (Q,T) is consistent
- If ε_{min} is an isomorphism we merge the two states of the (Q,T)-biautomaton and obtain $\mathcal{H}(Q,T)$.

```
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```

The $FunL^*$-algorithm

```
input: teacher of the target language $L$                       in a catgeory $\mathcal{C}$
output: $Min(L)$
$Q := T := \{\varepsilon\}$
repeat
   while $\varepsilon_{min}$ is not an isomorphism do           Iso = $E \cap M$
      if $\varepsilon_{min} \notin E$ then                      $(E, M)$ fact. system
         add $QA$ to $Q$
      if $\varepsilon_{min} \notin M$ then
         add $AT$ to $T$
   ask an equivalence query for the hypothesis automaton $\mathcal{H}(Q, T)$
   if the answer is no then
      add the counterexample and all its prefixes to $Q$
until the answer is yes
return $\mathcal{H}(Q, T)$
```