# Basics of (co)algebras

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## Coalgebra – a brief history

- dual of algebra in the work of Arbib and Manes, in their category theoretic approach to dynamical systems and automata
- Park and Milner invent bisimulation as a notion of behavioural equivalence for concurrent processes
- Aczel's theory of non-well-founded sets : Aczel and Mendler generalise Park and Milner's notion of bisimulation to the level of arbitrary coalgebras
- abstract theory of systems
- disclaimer: nothing new in this tutorial !
- excellent references on the theory of coalgebra by Rutten, Jacobs, Kurz, etc...

# Algebras

Algebras are classically presented as sets with operations.

Consider the set of natural numbers $\mathbb{N}$ equipped with the operations

$$\begin{aligned} \mathrm{zero}\colon 1 &\to \mathbb{N} & \qquad \mathrm{succ}\colon \mathbb{N} &\to \mathbb{N} \\ \mathrm{zero}(*) &= 0 & \mathrm{succ}(n) &= n + 1 \end{aligned}$$

These form an algebra $[\mathrm{zero}, \mathrm{succ}]\colon 1 + \mathbb{N} \to \mathbb{N}$.

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Consider the set $\{0, 1\}$ equipped with the operations

$$\mathrm{zero}\colon 1 \to \{0, 1\} \qquad \mathrm{flip}\colon \{0, 1\} \to \{0, 1\}$$
$$\mathrm{zero}(*) = 0 \qquad \mathrm{flip}(x) = 1 - x$$

These form an algebra $[\mathrm{zero}, \mathrm{flip}]\colon 1 + \{0, 1\} \to \{0, 1\}$.

## Algebras for a functor

**Definition.** Let $F: \mathcal{C} \to \mathcal{C}$ be a functor. An $F$-algebra is a pair $(X, \alpha)$ consisting of an object $X$ of $\mathcal{C}$ and a morphism

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**Example.** $M$-algebras, where $M\colon \mathrm{Set} \to \mathrm{Set}$ is defined by $MX = 1 + X$.

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**Exercise.** Define a morphims of $M$-algebras $(\mathbb{N}, [\mathrm{zero}, \mathrm{succ}]) \to (\{0, 1\}, [\mathrm{zero}, \mathrm{flip}])$.

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An $F_\Sigma$-algebra is then given by a map $\alpha\colon \coprod_{n \in \mathbb{N}} \Sigma_n \times A^n \to A$.

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To give such an $\alpha$ means that for all $n \in \mathbb{N}$ and for all $\mathtt{op} \in \Sigma_n$ we given an interpretation of $\mathtt{op}_A\colon A^n \to A$.

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What if we want to add equations ? We will need to speak about monads ...

# Coalgebras – the dual of algebras

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**Example.** Let $\mathbb{N}^\omega$ denote the set of infinite streams of natural numbers, that is, $\mathbb{N}^\omega = \{\sigma \mid \sigma\colon \mathbb{N} \to \mathbb{N}\}$. Consider the operations

$$\begin{aligned} \mathtt{head}\colon \mathbb{N}^\omega &\to \mathbb{N} \\ \sigma &\mapsto \sigma(0) \end{aligned} \qquad\qquad \begin{aligned} \mathtt{tail}\colon \mathbb{N}^\omega &\to \mathbb{N}^\omega \\ \sigma &\mapsto \sigma' = \lambda n.\sigma(n+1) \end{aligned}$$

Their pairing

$$\langle \mathtt{head}, \mathtt{tail} \rangle\colon \mathbb{N}^\omega \to \mathbb{N} \times \mathbb{N}^\omega$$

equips $\mathbb{N}^\omega$ with a coalgebra structure for the functor $\mathsf{Str}\colon \mathsf{Set} \to \mathsf{Set}$ defined by $\mathsf{Str}(X) = \mathbb{N} \times X$.

# Morphisms of coalgebras

**Definition.** A morphism between two $F$-coalgebras $(X, \xi)$ and $(Y, \zeta)$ is a $\mathcal{C}$-morphism $f\colon X \to Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\zeta} \\ FX & \xrightarrow{Ff} & FY \end{array}$$

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**Exercise.** Consider the Str-coalgebra $(\mathbb{N}^\omega \times \mathbb{N}^\omega, \langle \mathtt{h}, \mathtt{t} \rangle)$, where

$$\begin{aligned} \mathtt{h}: \mathbb{N}^\omega \times \mathbb{N}^\omega &\to \mathbb{N} \\ (\sigma, \tau) &\mapsto \sigma(0) \end{aligned} \qquad \begin{aligned} \mathtt{t}: \mathbb{N}^\omega \times \mathbb{N}^\omega &\to \mathbb{N}^\omega \times \mathbb{N}^\omega \\ (\sigma, \tau) &\mapsto (\tau, \mathsf{tail}(\sigma)) \end{aligned}$$

Define a coalgebra morphism $(\mathbb{N}^\omega \times \mathbb{N}^\omega, \langle \mathtt{h}, \mathtt{t} \rangle) \to (\mathbb{N}^\omega, \langle \mathtt{head}, \mathtt{tail} \rangle)$. How should we call it ?

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Answer: zip$\colon \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega$

# Induction and coinduction

# Initial algebras

$F$-algebras and their morphisms form a category $\mathrm{Alg}(F)$.

**Definition.** Let $F\colon \mathcal{C} \to \mathcal{C}$ be a functor. An initial $F$-algebra is an initial object in the category $\mathrm{Alg}(F)$.

That is, an $F$-algbera $\alpha\colon FA \to A$ is initial if for every $F$-algebra $\beta\colon FB \to B$ there exists a unique $F$-algebra morphism $u\colon A \to B$.

$$\begin{tikzcd}
FA \arrow[r, dashed, "Fu"] \arrow[d, "\alpha"] & FB \arrow[d, "\beta"] \\
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**Exercise.** What is the initial algebra for the functor $F_\Sigma$, induced by a signature $\Sigma$ ?

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$$\text{initial algebra} \xrightarrow{\exists!\ \text{morphism}} \text{arbitrary algebra}$$

- existence —> definition by induction
- uniqueness —> proof by induction

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**Example.** $(List(A), [nil, cons])$ is the initial algebra for the functor $TX = 1 + A \times X$. To define a map $f\colon List(A) \to B$ to some other $T$-algebra $B$, we need to define $f$ inductively on the constructors $nil$ and $cons$.

# Final coalgebra

$F$-coalgebras and their morphisms form a category $\mathrm{CoAlg}(F)$.

**Definition.** Let $F\colon \mathcal{C} \to \mathcal{C}$ be a functor. A final $F$-coalgebra is an final object in the category $\mathrm{CoAlg}(F)$.

That is, an $F$-coalgbera $\xi\colon X \to FX$ is final if for every $F$-coalgebra $\zeta\colon Z \to FZ$ there exists a unique $F$-coalgebra morphism $u\colon Z \to X$.

$$\begin{tikzcd} Z \arrow[r, dashed, "\exists!\, u"] \arrow[d, "\zeta"] & X \arrow[d, "\xi"] \\ FZ \arrow[r, dashed, "Fu"] & FX \end{tikzcd}$$

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**Example.** $(\mathbb{N}^\omega, \langle \mathtt{head}, \mathtt{tail} \rangle)$ is the final coalgebra for the functor $\mathrm{Str}$ defined by $\mathrm{Str}(X) = \mathbb{N} \times X$.

**Exercise.** Define a coalgebra structure on $\mathbb{N}^\omega$ that induces by coinduction the function even. Same question for odd.

## More examples of (final) coalgebras

- deterministic automata over alphabet $A$ can be seen as coalgebras

  $$\langle o, tr\rangle : X \to 2 \times X^A$$

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- deterministic automata over alphabet $A$ can be seen as coalgebras
$$\langle o, tr\rangle : X \to 2 \times X^A$$
(plus a point $1 \to X$)
- The final coalgebra for the functor $FX = 2 \times X^A$ is the automaton of all languages
$$\langle eps, der\rangle : 2^{A^*} \to 2 \times (2^{A^*})^A\,,$$
where
  - $eps(L) = 1$ iff $\epsilon \in L$
  - $der(L)(a) = a^{-1}L$

# Part II

# Recap

$F$-algebra for a functor $F\colon \mathcal{C} \to \mathcal{C}$

$$\alpha\colon FX \to X$$

$F$-coalgebra for a functor $F\colon \mathcal{C} \to \mathcal{C}$

$$\xi\colon X \to FX$$

# Recap

$F$-algebra for a functor $F: \mathcal{C} \to \mathcal{C}$

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$F$-algebra morphism

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initial $F$-algebra $(I, \alpha)$

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final $F$-coalgebra $(\Omega, \zeta)$

$$\begin{tikzcd} X \arrow[r, dotted, "\exists! f"] \arrow[d, "\xi"] & \Omega \arrow[d, "\zeta"] \\ FX \arrow[r, dotted, "Ff"] & F\Omega \end{tikzcd}$$

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| $FX = 2 \times \mathcal{P}(X)^A$ (nondeterminisitic automata) | |

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| $FX = 2 \times \mathcal{P}(X)^A$<br>(nondeterminisitic automata) | |
| $FX = (O \times X)^I$<br>(Mealy machines) | causal functions $I^* \to O^*$ |

## Behaviours of systems

**A final coalgebra $\nu.F$ — when it exists — captures the possible behaviours of the systems modelled as $F$-coalgebras.**

# Computing initial algebras as colimits

**Theorem.** Let $\mathcal{C}$ be a category with an initial object $0$ and colimits of $\omega$-chains. Assume $F$ preserves such colimits. Then the colimit of the chain

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2 !} \cdots$$

carries a structure of an initial $F$-algebra.

# Computing initial algebras as colimits

**Theorem.** Let $\mathcal{C}$ be a category with an initial object $0$ and colimits of $\omega$-chains. Assume $F$ preserves such colimits. Then the colimit of the chain

$$0 \xrightarrow{\;!\;} F0 \xrightarrow{\;F!\;} F^2 0 \xrightarrow{\;F^2!\;} \ldots$$

carries a structure of an initial $F$-algebra.

**Exercise.** Take $\Sigma = \{e : 0, m : 2\}$ be a signature. Compute the colimit of the initial chain of the associated functor $F_\Sigma$.

# Computing initial algebras as colimits: example

**Exercise.** Take $\Sigma = \{e : 0, m : 2\}$ be a signature. Compute the colimit of the initial chain of the associated functor $F_\Sigma$.

Recall $F_\Sigma \colon \mathrm{Set} \to \mathrm{Set}$ is defined by $FX = \{e\} + \{m\} \times X \times X$.

The initial chain $\quad 0 \xrightarrow{\ !\ } F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2!} \dots \quad$ is then:

$$\varnothing \xrightarrow{\ !\ } \{e\} \xrightarrow{F!} \{e, m(e,e)\} \xrightarrow{F^2!} \dots$$

where the maps $F^n!$ are just inclusions. The colimit of this chain is the set of terms over the signature $\Sigma$ and is the carrier of the initial $F_\Sigma$-algebra.

# Computing final coalgebras as limits

**Theorem.** Let $\mathcal{C}$ be a category with an final object 1 and limits of $\omega^{op}$-chains. Assume $F$ preserves such limits. Then the limit of the chain

$$1 \xleftarrow[!]{} F1 \xleftarrow[F!]{} F^2 1 \longleftarrow \dots$$

carries a structure of an final $F$-coalgebra.

# Congruences and bisimulations

# $F$-congruence

Let $F\colon \mathrm{Set} \to \mathrm{Set}$ be a functor. Let $(A, \alpha)$ and $(B, \beta)$ be two $F$-algebras. A relation $R \subseteq A \times B$ is called an $F$-congruence if there exists an $F$-algebra structure $\gamma\colon F(R) \to R$ such that the projections $\pi_1\colon R \to A$ and $\pi_2\colon R \to B$ are $F$-algebra morphisms.

$$
\begin{tikzcd}
 & FR \arrow[dl, "F\pi_1"'] \arrow[d, dotted, "\gamma"] \arrow[dr, "F\pi_2"] & \\
FA \arrow[d, "\alpha"'] & R \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & FB \arrow[d, "\beta"] \\
A & & B
\end{tikzcd}
$$

**Exercise.** Show that a congruence on $(\mathbb{N}, [\mathit{zero}, \mathit{succ}])$ is a relation $R \subseteq \mathbb{N} \times \mathbb{N}$ such that $(0, 0) \in R$ and $(n, m) \in R$ implies $(\mathit{succ}(n), \mathit{succ}(m)) \in R$.

## $F$-bisimulation

Let $F\colon \mathrm{Set} \to \mathrm{Set}$ be a functor. Let $(X, \xi)$ and $(Y, \zeta)$ be two $F$-coalgebras. A relation $R \subseteq X \times Y$ is called an $F$-bisimulation if there exists an $F$-coalgebra structure $\gamma\colon R \to FR$ such that the projections $\pi_1\colon R \to X$ and $\pi_2\colon R \to Y$ are $F$-coalgebra morphisms.

$$\begin{tikzcd}
 & R \arrow[dl, "\pi_1"'] \arrow[d, dotted, "\gamma"] \arrow[dr, "\pi_2"] & \\
X \arrow[d, "\xi"'] & FR \arrow[dl, "F\pi_1"'] \arrow[dr, "F\pi_2"] & Y \arrow[d, "\zeta"] \\
FX & & FY
\end{tikzcd}$$

**Coinduction proof principle.** Every bisimulation on a final $F$-coalgebra $X \to F(X)$ is contained in the diagonal on $X$.

# Example: bisimulation for labelled transition systems

A labelled transition system is a coalgebra $\xi\colon X \to \mathcal{P}(L \times X)$.

Note that $(l, x') \in \xi(x)$ means $x \xrightarrow{\;l\;} x'$ .

# Example: bisimulation for labelled transition systems

A labelled transition system is a coalgebra $\xi : X \to \mathcal{P}(L \times X)$.

Note that $(l, x') \in \xi(x)$ means $\quad x \xrightarrow{\;l\;} x' \quad$.

$$\begin{tikzcd}
 & R \arrow[dl, "\pi_1"'] \arrow[d, dotted, "\gamma"] \arrow[dr, "\pi_2"] & \\
X \arrow[d, "\xi"'] & \mathcal{P}(L \times R) \arrow[dl, "\mathcal{P}(L\times\pi_1)"'] \arrow[dr, "\mathcal{P}(L\times\pi_2)"] & Y \arrow[d, "\zeta"] \\
\mathcal{P}(L \times X) & & \mathcal{P}(L \times Y)
\end{tikzcd}$$

**Exercise.** Prove that this is equivalent to the standard notion of bisimulation. Recall that $R \subseteq X \times Y$ is a bisimulation iff $\forall (x, y) \in R$

- if $\quad x \xrightarrow{\;l\;} x' \quad$ then $\exists\, y' \in Y$ s.t. $\quad y \xrightarrow{\;l\;} y' \quad$ and $(y, y') \in R$.
- if $\quad y \xrightarrow{\;l\;} y' \quad$ then $\exists\, x' \in X$ s.t. $\quad x \xrightarrow{\;l\;} x' \quad$ and $(x, x') \in R$.

# Example: bisimulation of stream systems

$$\begin{tikzcd}
 & R \arrow[dl, "\pi_1"'] \arrow[d, dotted, "\gamma"] \arrow[dr, "\pi_2"] & \\
X \arrow[d, "\langle o_X, tr_X \rangle"'] & A \times R \arrow[dl, "1_A \times \pi_1"'] \arrow[dr, "1_A \times \pi_2"] & Y \arrow[d, "\langle o_Y, tr_Y \rangle"] \\
A \times X & & A \times Y
\end{tikzcd}$$

A relation $R \subseteq X \times Y$ is a bisimulation iff for all $(x, y) \in R$ we have $o_X(x) = o_Y(y)$ and $(tr_X(x), tr_Y(y)) \in R$.

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- A relation that is compatible for a suitable extension of $F$ to the category of sets and relation – sometimes called Hermida-Jacobs bisimulations

S. Staton, *Relating coalgebraic notions of bisimulation*, 2011.
`https://arxiv.org/abs/1101.4223`

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- A relation satisfying a 'congruence' condition, proposed by Aczel and Mendler
- A relation which is the kernel of a common compatible refinement of the two coalgebras

S. Staton, *Relating coalgebraic notions of bisimulation*, 2011.
`https://arxiv.org/abs/1101.4223`

# Further reading

In this tutorial we just scratched the surface...

- J. Rutten, *The Method of Coalgebra: exercises in coinduction*, 2019
  `https://ir.cwi.nl/pub/28550/rutten.pdf`
- B. Jacobs and J. Rutten, *A tutorial on (co)algebras and (co)induction*, 1997
- A. Kurz, *Coalgebras and Modal Logic*, 2001,
  `https://alexhkurz.github.io/papers/cml.pdf`