The μ-calculus and its proof theory

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Aussois (France)

Outline

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Section 1

Introduction

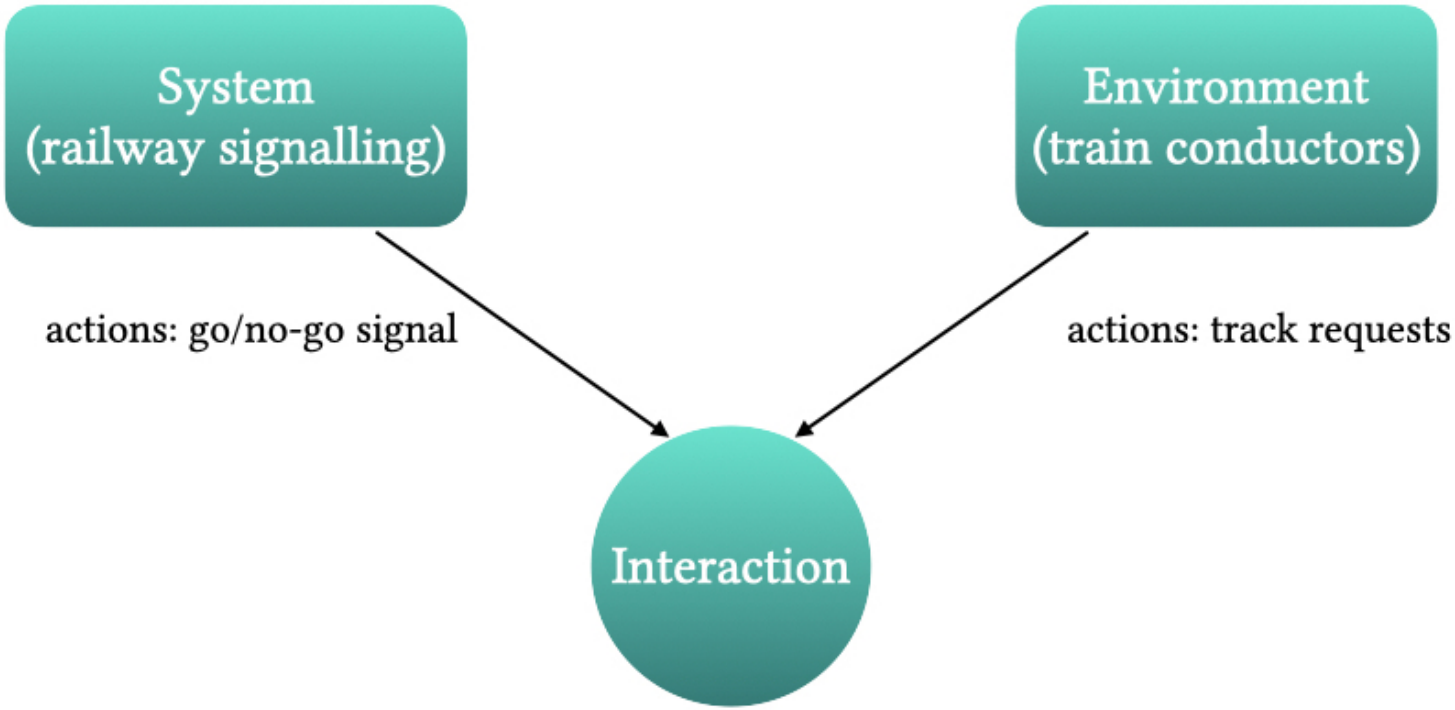
Modal μ-calculus

- *Blue Book (Chapter 8)* Temporal Logics in Computer Science: Finite-State Systems. Demri, Goranko & Lange, CUP (2016)
- *The μ-calculus and Model Checking.* Bradfield & Walukiewicz. Springer International Publishing, Cham (2018)
- *Green Book (Part V)* Automata, Logics, and Infinite Games. Grädel, Thomas & Wilke (editors). LNCS (2002)
- *Yellow Book (Chapter 6)* Rudiments of μ-calculus. Arnold & Niwinski. Vol. 146. Elsevier (2001)

Proof Theory

- **Games for the μ-calculus.** Niwinski & Walukiewicz. TCS 163(1&2):99-116 (1996)
- **On the Proof Theory of the Modal mu-Calculus**. Studer. Studia Logica 89, 343-363 (2008)
- **Finitary Proof Systems for Kozen's μ.** Afshari and Leigh Oberwolfach Preprints. DOI 10.14760/OWP-2016-26 (2016)
- **Demystifying μ.** Afshari et al. CoRR abs/2401.01096 (2024)

Railway signalling



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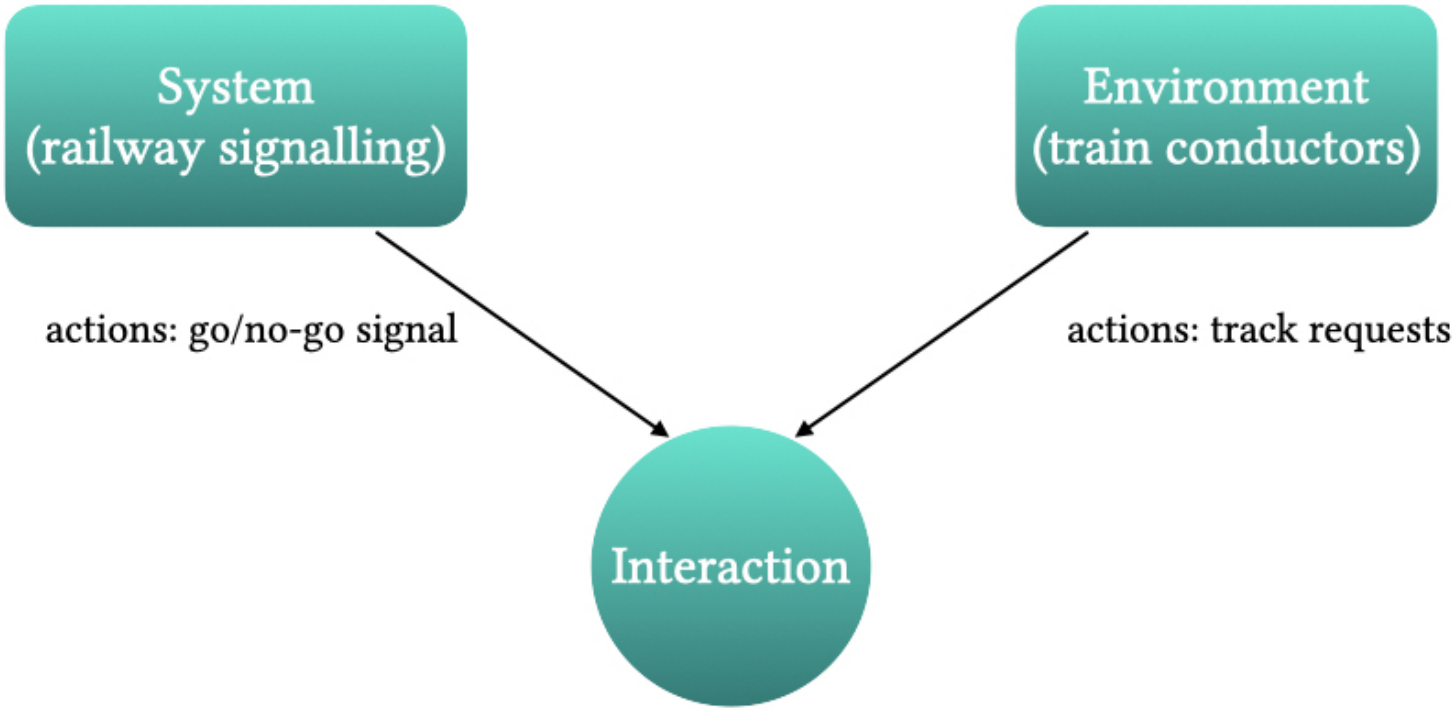

Questions Is the system operating correctly? Are there any faults?

Reactive systems

Main characteristics:

- interact with the environment
- safety-critical
- run indefinitely

Other examples: *air traffic control systems* or *automated stock trading*

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Want to verify

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– **Safety Bad things never happen**

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- $3 \equiv \text{not}(p_1 \text{ and } p_2)$ and **Next**(3)
- $\mathsf{x} \equiv \varphi(\mathsf{x})$ where φ is a statement using finite vocabulary such as **Next** and logical connectives.

Questions of interest

1. Are there logics suitable to express such temporal properties?
2. What is the cost of checking liveness, safety and other fairness constraints?
3. Can practical algorithms be developed for real-world examples?

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Section 2

Modal Logic

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- Examples: $\Box(p \vee \Diamond q)$, $\Box\Diamond\neg p$

A model of modal logic, called a **transition system** (or Kripke structure), is a triple $T = (S, \to, \lambda)$ with a

- set of states S;
- transition relation $\to \subseteq S \times S$, writing also $s \to t$ for $(s, t) \in \to$
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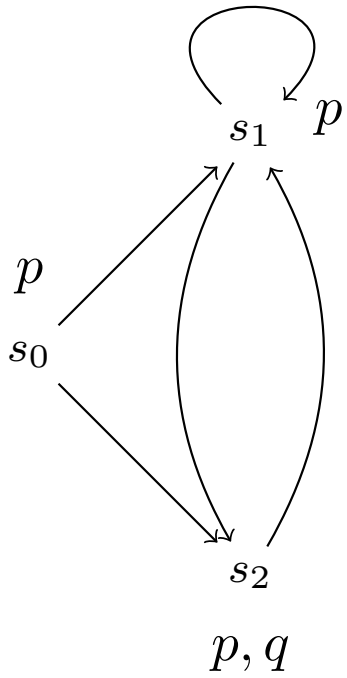
Note that a transition system is simply a **directed graph** whose nodes are labelled by **finite sets of propositions**.

Example



Transition system with states $\{s_0, s_1, s_2\}$

We define what it means for a transition system T to model/satisfy a formula φ at a world s, written $T, s \models \varphi$, by structural induction:

- $T, s \models p$ iff $p \in \lambda(s)$
- $T, s \models \neg\varphi$ iff $T, s \not\models \varphi$
- $T, s \models \varphi \wedge \psi$ iff $T, s \models \varphi$ and $T, s \models \psi$
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We write $\|\varphi\|^T$ to denote the set of all states s of transition system T that satisfy the formula φ. In other words,

$$s \in \|\varphi\|^T \text{ if and only if } T, s \models \varphi.$$

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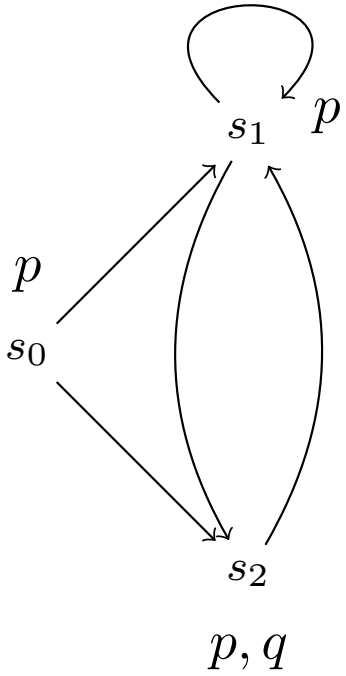

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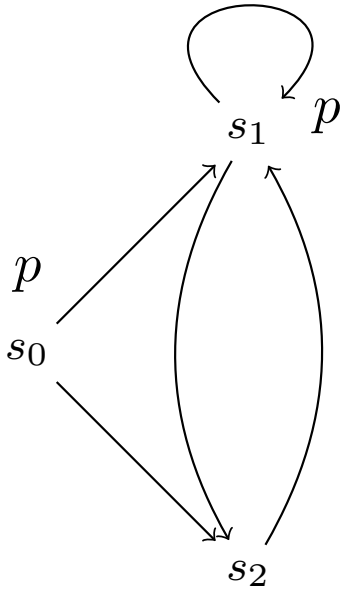

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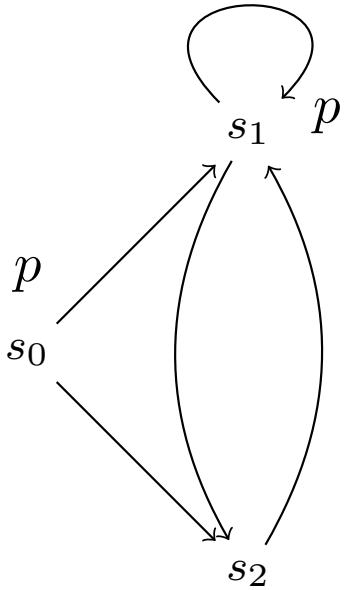

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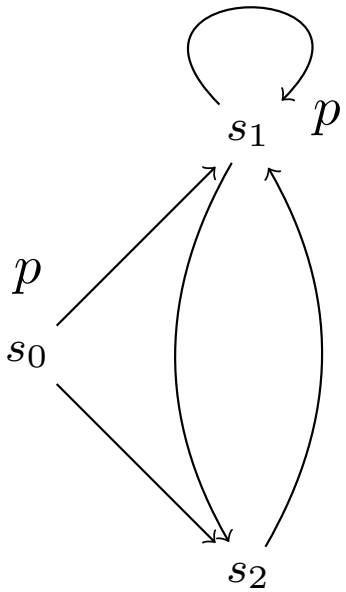

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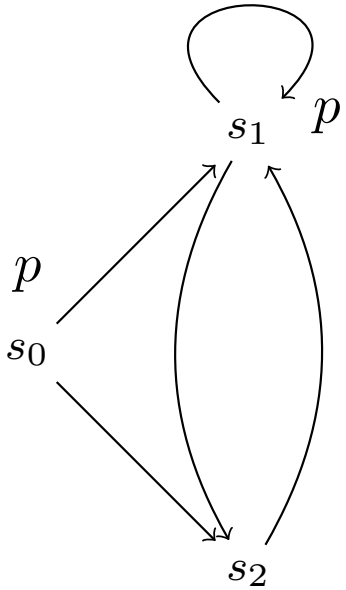

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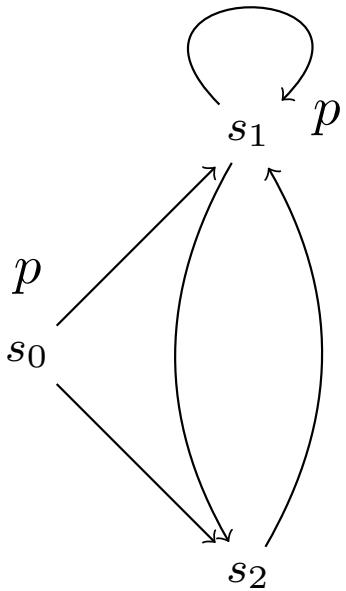

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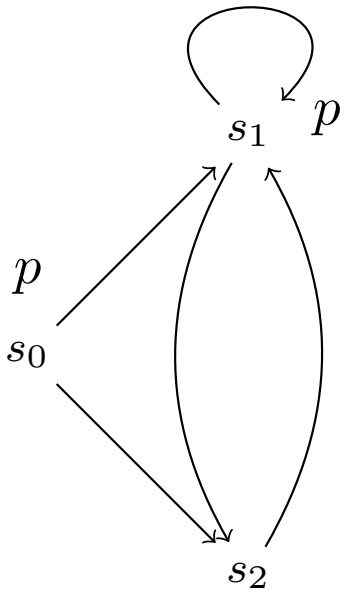

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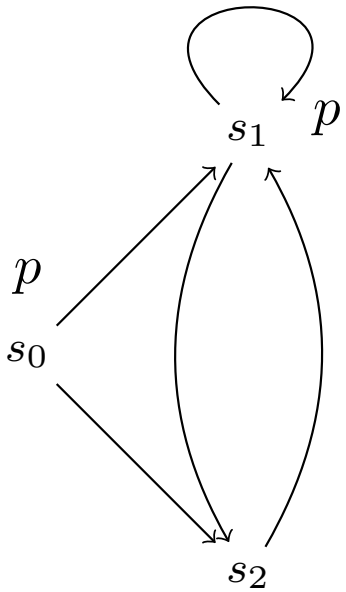

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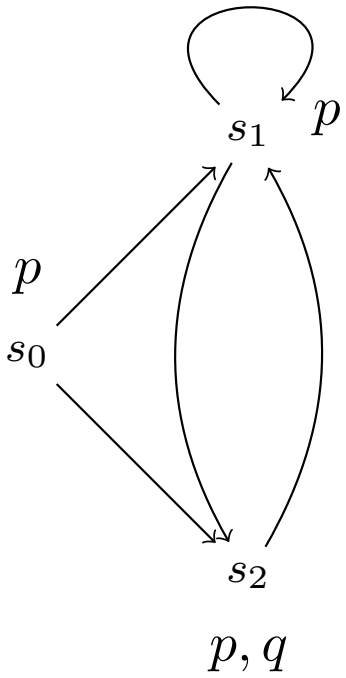

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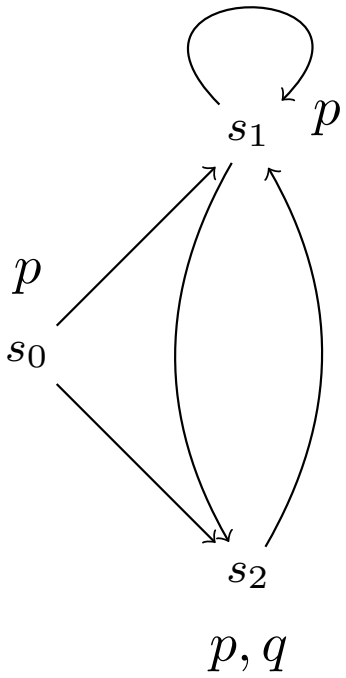

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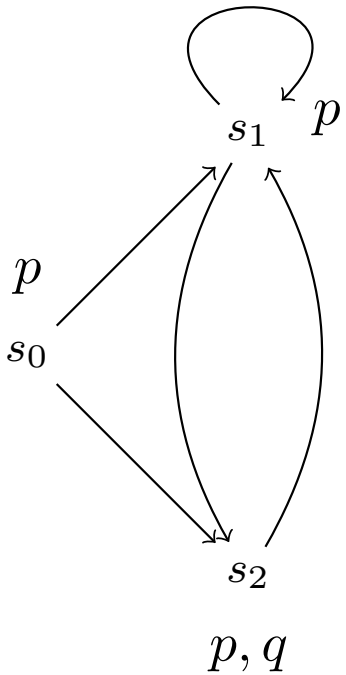

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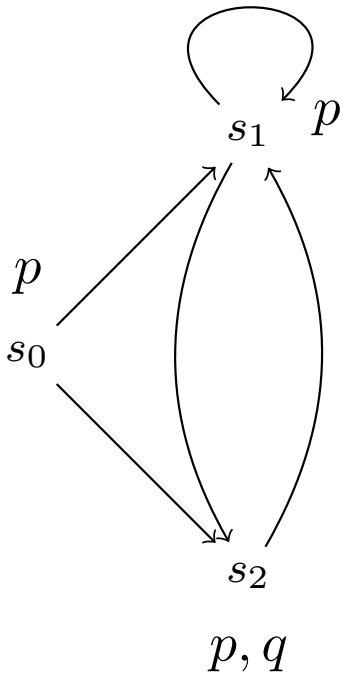

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Syntactic methods:

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Remark Tableaux are often designed for satisfiability but one can also talk about tableaux for validity. As the two notions are dual if you have a tableau system for one you can define an analogous one for the other.

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4. When should we use tableaux? Seeking computational properties.

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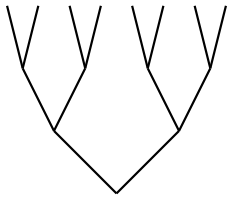
Our tableaux are going to look like (upwards growing) trees:



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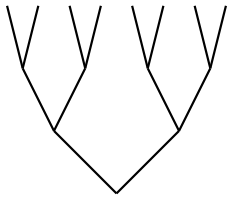

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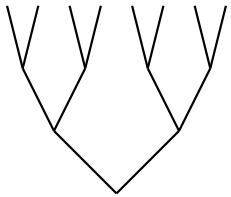

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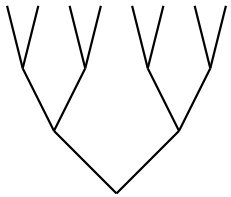

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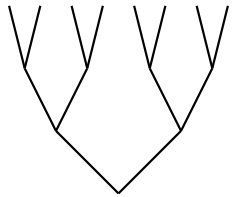

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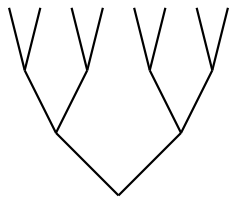

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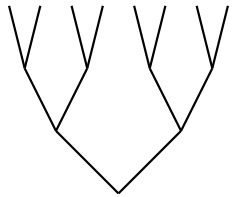

Given a formula φ we want to check if it satisfiable. The idea is to systematically build a model from **using the logical structure** of φ:

- Start the tree by putting φ at the **root**;
- From the root make new **branches** (unary, binary or more);
- The idea of these new nodes is to reduce checking satisfiability of φ to checking **satisfiability of its constituents**;
- To check satisfiability of $\varphi_1 \wedge \varphi_2$ we check both φ_1 and φ_2;
- Continue breaking down φ until only **literals** remain.

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- $\neg\Diamond(\neg p \wedge q)$ is not in NNF but $\Box(p \vee \neg q)$ is.
- What about $\Box\neg(p \wedge q)$?

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NNF is not a restriction on expressibility:

Lemma (Exercise)

Every formula of modal logic is equivalent to one in NNF. The equivalence is provable in **K**.

Motivating example

Consider the formula $\varphi = (\Box\Box p \wedge \Diamond q) \vee (\neg p \wedge \Box\neg q)$. We aim to find a model of φ, namely some $M = (W, \to, V)$ and $w_0 \in W$ such that $M, w_0 \models \varphi$.

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This search for satisfiability (or soundness) can be formalised using tableaux, a **tree** where each node is labelled by a **subset of** $Sub(\varphi)$ (the set of all subformulas of φ).

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A **tree** (over Σ) is a triple $t = \langle S, \to, \lambda \rangle$ with a distinguished node $\rho \in S$ which satisfies the following conditions.

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A **path** through tree t is a function $\mathbb{P} \colon \mathbb{N} \to S$ such that

- $\mathbb{P}(0) = \rho$,
- For every n, if $\mathbb{P}(n)$ is not a leaf, then $\mathbb{P}(n) \to \mathbb{P}(n+1)$.

Notation

- From now on, all formulas considered are in **NNF**.
- We use $\varphi, \psi, \delta, \gamma, \ldots$ (also with indices) for denoting formulas.
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- Γ, φ means $\Gamma \cup \{\varphi\}$, and Γ, Δ denotes $\Gamma \cup \Delta$.

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$$\wedge\ \frac{\Delta, \varphi_0, \varphi_1}{\Delta, \varphi_0 \wedge \varphi_1} \qquad \vee_0\ \frac{\Delta, \varphi_0}{\Delta, \varphi_0 \vee \varphi_1} \qquad \vee_1\ \frac{\Delta, \varphi_1}{\Delta, \varphi_0 \vee \varphi_1}$$

$$\text{mod}\ \frac{\Delta, \delta_1 \qquad \Delta, \delta_2 \qquad \cdots \qquad \Delta, \delta_n}{\Box\Delta, \Diamond\{\delta_1, \cdots, \delta_n\}, \Theta}\ \Theta \subseteq \textsc{Lit} \text{ is consistent}$$

Rules for **K** pre-tableaux

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- A pre-tableau is a finitely branching tree.
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- The mod-rule can be applied only if no other rule is applicable.
- For modal logic **K** tableaux are well-founded trees. In other words, every path is finite. **Why?**

Examples of pre-tableaux

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Question How many pre-tableaux does $(\neg p \vee \Diamond\Diamond q) \wedge (p \vee \Box\Box\neg q)$ have?

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Question Which of these are tableaux?

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Lemma (Exercise)

Let $t = \langle S, \to, \lambda \rangle$ be a tableau and suppose $s \in S$. Then for all $p \in \textsc{Prop}$ we have $\{p, \neg p\} \not\subseteq \lambda(s)$.

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3. $w \in V(p)$ iff there exists $s \in S$ such that $\tau(s) = w$ and $p \in \lambda(s)$.

Exercise Complete the proof by first proving

- Show that for every $s \in S$ and $\psi \in \lambda(s)$ we have

$$\psi \in \lambda(s) \qquad \text{iff} \qquad M, \tau(s) \models \psi$$

Hint: Double induction i) complexity of ψ & ii) distance of s from leaves.

How to extract the tree-model?

Both of the following are tableaux.

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The models they give us are respectively:

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Corollary

Modal logic has the finite model property.

Exercises

Consider the formulas φ and ψ given by

$$\varphi := \Box p \to \Box\Box p$$
$$\psi := \Diamond(p \to q) \to (\Box p \to \Diamond q)$$

1. Give an equivalent NNF formula for each
2. Using tableaux check if they are valid/satisfiable

Notes

- If you are claiming it is satisfiable then you should provide a tableau and describe the model it gives rise to.
- If you are claiming it is not satisfiable then you must argue that it has no tableaux; you can do this by listing all pre-tableaux and showing none satisfy the tableaux condition.
- Validity of φ can be deduced from satisfiability of $\neg\varphi$.

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We can try (and fail) using the modal syntax: $p \vee \Diamond p \vee \Diamond\Diamond p \vee \cdots$

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Expressive limitation of modal logic

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Temporal Logics

Temporal logics provide us with a language that can express path quantification:

- "along every path ..."
- "there exists a path along which ..."

and much more complex properties.

Recap

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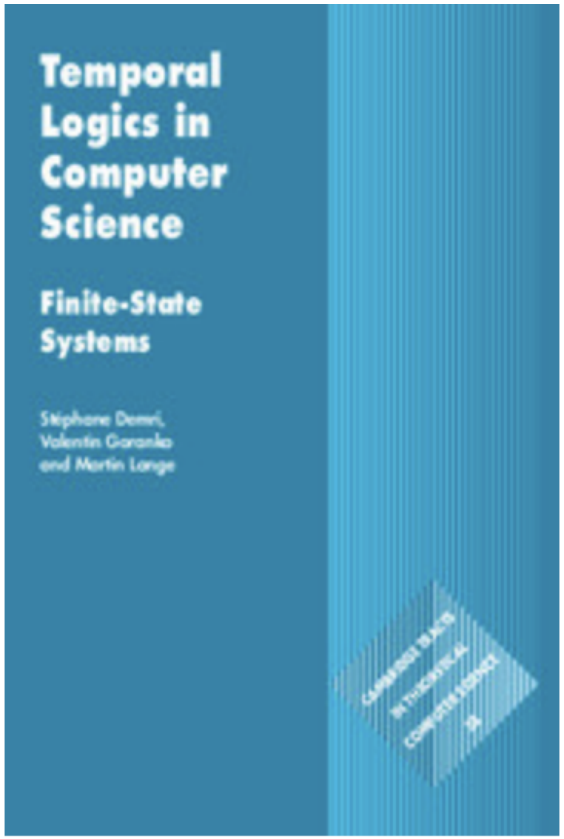
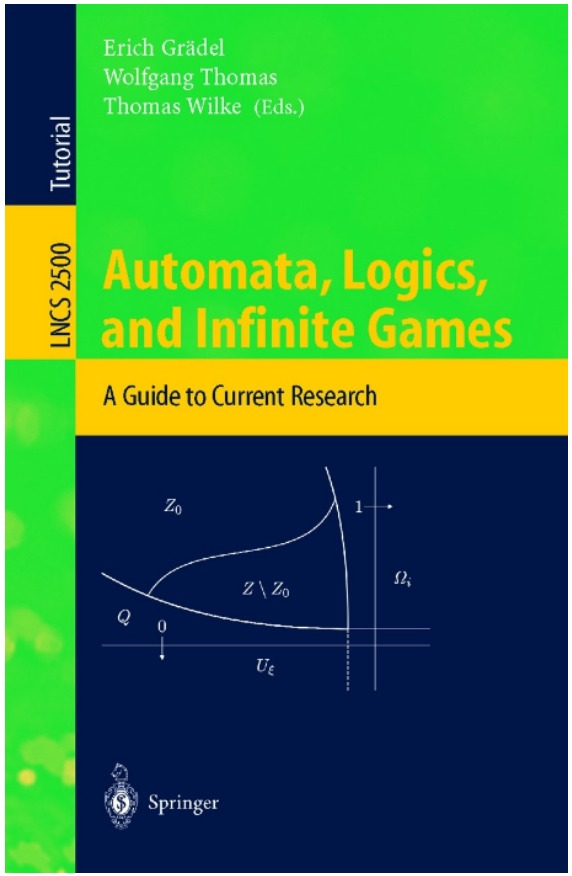
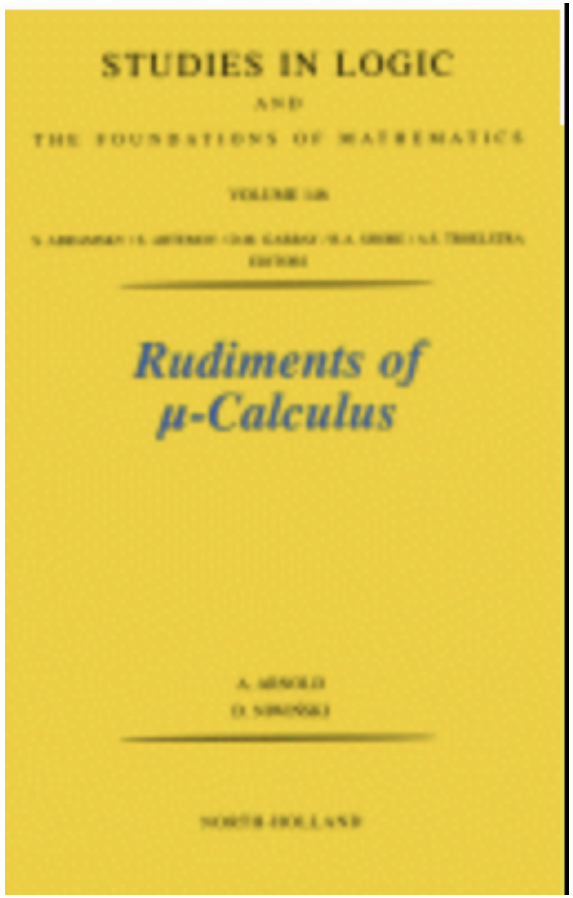

Temporal Logics in Computer Science
Finite-State Systems
Stéphane Demri, Valentin Goranko and Martin Lange


Tutorial
LNCS 2500
Erich Grädel
Wolfgang Thomas
Thomas Wilke (Eds.)
Automata, Logics, and Infinite Games
A Guide to Current Research
Springer


STUDIES IN LOGIC
Rudiments of μ-Calculus

LTL-formulas, over atomic propositions $p_1, \cdots, p_n$, are defined

$$\begin{array}{rcll} \varphi & ::= & p_i & \text{atomic proposition} \\ & | & \neg\varphi & \text{negation} \\ & | & \varphi \wedge \psi & \text{conjunction} \\ & | & \varphi \vee \psi & \text{disjunction} \\ & | & \mathbf{X}\,\varphi & \text{next} \\ & | & \varphi\,\mathbf{U}\,\psi & \text{until} \end{array}$$

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Intended meaning

$\mathbf{X}\,\varphi$ "φ is true at the **next** time-step"

$\varphi\,\mathbf{U}\,\psi$ "φ is true **until** ψ is true (and ψ holds eventually)"

Two additional constructs

$\mathbf{F}\,\varphi$ "φ is **eventually** true"
φ is true at **some** point in the future

$\mathbf{G}\,\varphi$ "φ is **always** true"
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They are expressible in LTL by

$$\begin{aligned} \mathbf{F}\,\varphi &:= \mathsf{true}\,\mathbf{U}\,\varphi \\ \mathbf{G}\,\varphi &:= \neg(\mathbf{F}\,\neg\varphi) \end{aligned}$$

Examples of properties expressible in LTL

- **Recurrence**: "p_1 holds infinitely often"

$$\mathbf{G}\,(\mathbf{F}\,p_1)$$

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$$\mathbf{G}\,(p_1 \to \mathbf{X}\,\mathbf{F}\,p_2)$$

- **Fairness**: "If p_1 is true infinitely often, then so is p_2"

$$\mathbf{G}\,\mathbf{F}\,p_1 \to \mathbf{G}\,\mathbf{F}\,p_2$$

Kripke structures revisited

A **(pointed) Kripke structure** over a set of atomic propositions $\{ p_1, \ldots, p_n \}$ is a quadruple (S, R, λ, s_0) with

- a finite state-set S
- an initial/distinguished state $s_0 \in S$
- a transition relation $R \subseteq S \times S$, and
- a labelling function $\lambda \colon S \longrightarrow \mathcal{P}(\{ p_1, \cdots, p_n \})$, associating to each $s \in S$ the set $\lambda(s)$ of those p_i that are satisfied at s.

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Notation

We write $\lambda(s)$ as a bit vector $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{B}^n$ such that $b_i = 1$ iff $p_i \in \lambda(s)$.

Paths and label sequences

An infinite **path** $s_0 s_1 s_2 \cdots$ through a Kripke structure (S, R, λ, s_0) induces an infinite sequencee of "labels" over the alphabet $\mathbb{B}^n$

$$\lambda(s_0)\,\lambda(s_1)\,\lambda(s_2)\cdots$$

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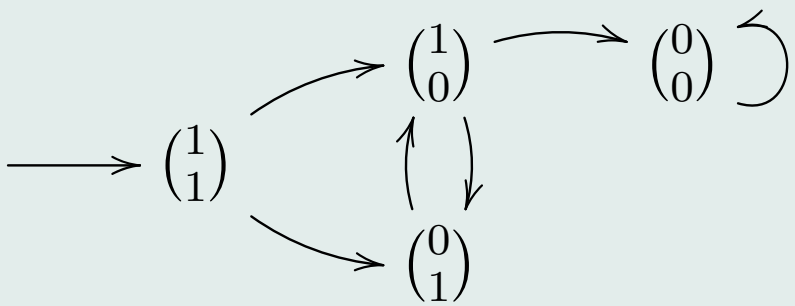
Example

A Kripke structure over $\{p_1, p_2\}$.



$\binom{1}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{0}{0}\binom{0}{0}\cdots$
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Notation. Let $\alpha = \alpha(0)\ \alpha(1)\ \cdots \in (\mathbb{B}^n)^\omega$:

1. α^i stands for $\alpha(i)\alpha(i+1)\cdots$, the ***i*-suffix** of α; so $\alpha = \alpha^0$.
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Satisfaction relation. Let $i \geq 0$. Define $\alpha^i \vDash \varphi$ by recursion over φ:

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\alpha^i \vDash \varphi \wedge \psi & \text{iff} & \alpha^i \vDash \varphi \ \text{ and } \ \alpha^i \vDash \psi \\
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We say α **satisfies** φ, denoted $\alpha \vDash \varphi$, if $\alpha^0 \vDash \varphi$.

Linear vs Branching

- **Linear-time properties** set the same conditions on every infinite path through a system modelled by a Kripke structure.
- **Branching-time properties** are conditions on the structure of the tree formed by all paths through a Kripke structure

See e.g.
Branching vs. linear time: Final showdown, M.Y. Vardi, TACAS 2001.

- We work in **two-sided** sequent calculus so we can handle negation.
- Rules for the propositional part is as usual with left and right **intro/outro rules** for each connective.
- The main task is to deal with $\varphi \,\mathbf{U}\, \psi$.
- We treat this new connective by exploiting the following equivalence

$$\varphi \,\mathbf{U}\, \psi \equiv \psi \vee (\varphi \wedge \mathbf{X}\,(\varphi \,\mathbf{U}\, \psi))$$

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$$X\ \frac{\Gamma \Rightarrow \Delta}{\Sigma, \mathbf{X}\,\Gamma \Rightarrow \mathbf{X}\,\Delta, \Lambda}\ \text{with } \Sigma, \Lambda \text{ arbitrary sequents}$$

$$U_R\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi \qquad \Gamma \Rightarrow \psi, \mathbf{X}\,(\varphi \,\mathbf{U}\, \psi)}{\Gamma \Rightarrow \Delta, \varphi \,\mathbf{U}\, \psi}$$

$$U_L\ \frac{\Gamma, \psi \Rightarrow \Delta, \qquad \Gamma, \varphi, \mathbf{X}\,(\varphi \,\mathbf{U}\, \psi) \Rightarrow \Delta}{\Gamma, \varphi \,\mathbf{U}\, \psi \Rightarrow \Delta}$$

Non-wellfounded proofs

Pre-proofs are finite branching but can have infinitely long branches

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Proofhood condition every infinite path carries an infinite trace that passes through U_L with the until formula principal.

Example

$$R\,\frac{\varphi\,\mathbf{U}\,\psi \Rightarrow \top, \psi \qquad L\,\dfrac{\psi \Rightarrow \psi, \mathbf{X}\,(\top\,\mathbf{U}\,\psi) \qquad X\,\dfrac{\varphi\,\mathbf{U}\,\psi \Rightarrow \top\,\mathbf{U}\,\psi}{\varphi, \mathbf{X}\,(\varphi\,\mathbf{U}\,\psi) \Rightarrow \psi, \mathbf{X}\,(\top\,\mathbf{U}\,\psi)}}{\varphi\,\mathbf{U}\,\psi \Rightarrow \psi, \mathbf{X}\,(\top\,\mathbf{U}\,\psi)}}{\varphi\,\mathbf{U}\,\psi \Rightarrow \top\,\mathbf{U}\,\psi}$$

LTL finitary axiomatisation

We need to add an **induction** axiom to handle the until operator:

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Or equivalently,

$$\text{ind}\ \frac{\psi \to \delta \qquad \varphi \wedge \mathbf{X}\,\delta \to \delta}{\varphi\,\mathbf{U}\,\psi \to \delta}$$

Outline

Section 3

Cyclic and Ill-founded Proofs

Proof Systems à la Hilbert

A proof of a sequent Γ in a given system $\mathscr{S}$ is a **finite tree** in which leaves are labelled by axioms, the labelling of inner nodes respect the inference rules of $\mathscr{S}$, and the root is labelled by Γ.

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But these 'concrete' proofs are hard to produce because of

- non-determinism
- induction 'axioms'

Cyclic Proofs: the core idea

Geared towards *proof search*, a cyclic proof formalises **an infinite decent argument**.

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$$\sqrt{2} = p/q = p'/q' = p''/q'' = \cdots \qquad \text{with } q > q' > q'' > \cdots$$

Cyclic proofs for modal logic K4

Formulas $p \mid \overline{p} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi$

Sequents $\Gamma, \Delta, \ldots$ finite sets of formulas treated as disjunction

Axioms $\Gamma, \varphi, \bar{\varphi}$

Inference rules

$$\frac{\Gamma, \bot}{\Gamma}\,\bot \qquad \frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}\,\wedge \qquad \frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi}\,\vee \qquad \frac{\Gamma, \Diamond\Gamma, \varphi}{\Diamond\Gamma, \Box\varphi, \Delta}\,4$$

where $\Diamond\Gamma$ abbreviates $\{\Diamond\varphi_1, \Diamond\varphi_2, \ldots, \Diamond\varphi_n\}$ for $\Gamma = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$.

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- **axiom 4:** $\Box\varphi \to \Box\Box\varphi$
- **Löb's axiom:** $\Box(\Box p \to p) \to \Box p$.

A cyclic proof of Löb's axiom

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box p \wedge \bar{p}, \Diamond(\Box p \wedge \bar{p}), p}{\Box p, \Diamond(\Box p \wedge \bar{p}), p}\,4 \qquad \bar{p}, \Diamond(\Box p \wedge \bar{p}), p}{\Box p \wedge \bar{p}, \Diamond(\Box p \wedge \bar{p}), p}\,\wedge}{\Diamond(\Box p \wedge \bar{p}), \Box p}\,4}{\Diamond(\Box p \wedge \bar{p}) \vee \Box p}\,\vee$$

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Theorem (Shamkanov 2014)

K4 + circularity = Gödel-Löb logic

Gödel-Löb logic axiomatisable by

Boolean tautologies + $\Box$-distribution + Löb axiom + MP

A very general definition

Definition. A cyclic proof of Γ in $\mathscr{S}$ is a derivation tree which can have both axiomatic and **non-axiomatic** leaves as long as the latter satisfy the property:

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What is progress? How is it determined?

Non-example

$$\dfrac{\dfrac{\dfrac{}{\bot}\,\text{weak.}}{\bot, \bot}}{\bot}\,\text{contr.}$$

Fixpoint-example

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- p is reachable $\quad R =_\mu p \vee \Diamond R$
- p is reachable in an even number of steps $\quad E =_\mu p \vee \Diamond\Diamond E$
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Proof by induction vs cyclic proofs

- principle of induction
- local
- non-algorithmic

- principle of infinite decent
- global
- algorithmic
- symmetric treatment of inductive and co-inductive properties

Outline

Section 4

Modal μ-calculus

Syntax: $p \mid \bar{p} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi$

Semantics: For directed labelled graph $T = (S, \to, \lambda)$:

$$\|p\|^T = \{u \in S \mid p \in \lambda(u)\}$$

$$\|\varphi \wedge \psi\|^T = \|\varphi\|_V^T \cap \|\psi\|^T$$

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Expressive limitation: properties along arbitrary (finite or infinite) paths in a structure cannot be presented

Infinite path quantification:

1. $s \models \bigvee_n \Diamond^n q$ – q is reachable from s
2. $s \models \bigwedge_n \Box^n \Diamond p$ – All paths from s always satisfy $\Diamond p$

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A note on expressiveness

$$\mu \mathsf{x}(p \vee \Box \mathsf{x}) : \text{all paths contain a } p$$
$$\mu \mathsf{x}(p \vee \Box \mathsf{x}) \neq p \vee \Box p \vee \Box\Box p \vee \cdots$$

1. $s \models q \vee \Diamond q \vee \Diamond\Diamond q \vee \cdots$ iterating $\mathsf{x} \mapsto q \vee \Diamond \mathsf{x}$ on $\bot$

Given $\varphi(\mathsf{x})$ (positive in x) we introduce

- $\mu \mathsf{x}\, \varphi$ = least fixed point of $\mathsf{x} \mapsto \varphi(\mathsf{x})$ Liveness **inductive**
- $\nu \mathsf{x}\, \varphi$ = greatest fixed point of $\mathsf{x} \mapsto \varphi(\mathsf{x})$ Safety **co-inductive**

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$$\mu \mathsf{x}(p \vee \Box \mathsf{x}) = p \vee \Box p \vee \Box\Box p \vee \cdots \vee \Box^{\omega} p \vee \cdots$$

Modal μ-calculus

Syntax $p \mid \overline{p} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mathsf{x} \mid \mu\mathsf{x}\,\varphi \mid \nu\mathsf{x}\,\varphi$

Semantics Fix $T = (S, \to, \lambda)$. Given $\varphi(\mathsf{x})$ define

$$f_\varphi \colon 2^S \to 2^S$$
$$U \mapsto \|\varphi(U)\|^T$$

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Duality Define $\overline{\varphi}$ as the De Morgan dual of φ:

$$\overline{\mu\mathsf{x}\varphi(\mathsf{x})} = \nu\mathsf{x}\overline{\varphi}(\mathsf{x}) \qquad \|\overline{\varphi}\|^T = S \setminus \|\varphi\|^K$$

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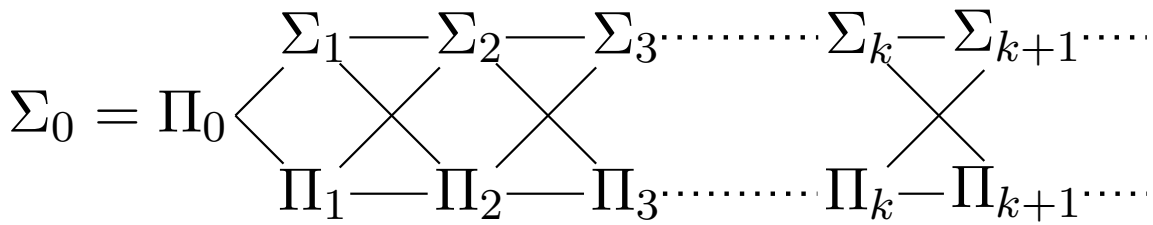
- $\nu \mathsf{x}\big(\mu \mathsf{y}(p \vee \Diamond \mathsf{y}) \wedge \Diamond \mathsf{x}\big)$: a path along which p **is always reachable**

- Count alternations between the two quantifiers **Simple hierarchy**
- Count the *genuine* alternations between least and greatest fixed point quantifiers (**depth**) **Alternation hierarchy**



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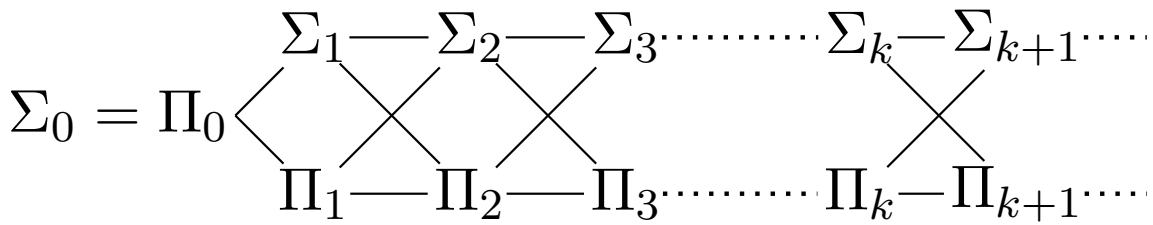

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Connexion:

- LTL can be captured by μ-calculus formulas of depth 2
- CTL, PDL can be captured by alternation-free μ-calculus
- CTL*: depth 3

- Expressive: LTL, PDL, CTL, CTL*
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- Equivalent to: alternating parity automata, parity games
- Tree Model Property: every satisfiable formula has a tree model.
- Finite Model Property: every satisfiable formula has a finite model.

Model Checking Problem *Given an LTS T, a state s, and a closed modal mu-calculus formula φ, does $s \in \|\varphi\|^T_\varnothing$ hold?*

Decidable (when transition systems T is finite), though its exact complexity is a long standing open problem. The best known bound is $\mathbf{NP} \cap \mathbf{co\text{-}NP}$. The problem is conjectured to be in $\mathbf{P}$.

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Small Model Property: *If a modal mu-calculus formula is satisfiable, then it is satisfiable by an LTS of size exponential in size of the formula.*

Outline

Section 5

Proof Systems

$$\frac{}{\varphi, \overline{\varphi}}\ \varphi \qquad \frac{\Gamma}{\Gamma, \varphi}\ \text{weak}$$

$$\frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi}\ \vee \qquad \frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}\ \wedge \qquad \frac{\Gamma, \varphi}{\Diamond\Gamma, \Box\varphi}\ \text{mod}$$

$$\frac{\Gamma, \varphi(\sigma \mathsf{x} \varphi(\mathsf{x}))}{\Gamma, \sigma \mathsf{x} \varphi}\ \sigma \qquad \frac{\psi, \varphi(\overline{\psi})}{\psi, \nu \mathsf{x} \varphi}\ \text{ind} \qquad \frac{\Gamma, \overline{\varphi} \qquad \Gamma, \varphi}{\Gamma}\ \text{cut}$$

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Abbreviation.

- " , " read as disjunction
- $\Gamma = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ finite set of formulas
- $\Diamond\Gamma := \{\Diamond\varphi_1, \Diamond\varphi_2, \ldots, \Diamond\varphi_n\}$
- $\overline{\Gamma} := \overline{\varphi}_1 \wedge \overline{\varphi}_2 \wedge \ldots \wedge \overline{\varphi}_n$
- $\sigma \in \{\mu, \nu\}$

Validity and proofs

A closed formula φ is **valid** iff $\|\varphi\|^T = S$ for every $T = (S, \to, \lambda)$.

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Soundness Proved by Kozen (1983).
Completeness aconjuntive fragment; full μ-calculus

1. Completeness of disjunctive formulas: tableaux.
2. Provable equivalence between disjunctive and μ-formulas: tableaux, games, automata.

$$p \mid \overline{p} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mathsf{x} \mid \mu\mathsf{x}\,\varphi \mid \nu\mathsf{x}\,\varphi$$

μ-quantifier: least fixed point

- $\varphi(\mu\mathsf{x}\varphi) \to \mu\mathsf{x}\varphi$
- $\varphi(\psi) \to \psi \vdash \mu\mathsf{x}\varphi \to \psi$

ν-quantifier: greatest fixed point

- $\nu\mathsf{x}\varphi \to \varphi(\nu\mathsf{x}\varphi)$
- $\psi \to \varphi(\psi) \vdash \psi \to \nu\mathsf{x}\varphi$

ν-regenration

$$\frac{\Gamma, \varphi(\nu\mathsf{x}\varphi(\mathsf{x}))}{\Gamma, \nu\mathsf{x}\varphi}\ \nu$$

ν-induction

$$\frac{\psi, \varphi(\overline{\psi})}{\psi, \nu\mathsf{x}\varphi}\ \text{ind}$$

Axioms and rules of proof system Fix

$$\frac{}{\varphi, \bar{\varphi}} \qquad \frac{\Gamma}{\Gamma, \varphi}\ \text{weak}$$

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A **pre-tableau** is a finitely branching tree in axioms and rules of Fix.

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A **pre-tableau** is a finitely branching tree in axioms and rules of Fix.

Principal and **residual** formulas are defined analogous to modal logic.

Examples of pre-tableaux

The formula $(\nu \mathsf{x} \Diamond \mathsf{x}) \vee (\mu \mathsf{y} \Box \mathsf{y})$ has (essentially) one pre-tableau:

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Fix a pre-tableau $t = \langle S, \to, \lambda \rangle$ for Γ and a path $\mathbb{P}$ through t. A finite sequence of formulas $\varphi_0, \varphi_1, \ldots, \varphi_n$ is a **trace** through $\mathbb{P}$ if

- $\varphi_i \in \lambda(\mathbb{P}(i))$ for each $i \leq n$;
- $\varphi_{i+1} = \varphi_i$ if φ_i is not principal in the rule from $\mathbb{P}(i)$ to $\mathbb{P}(i+1)$
- φ_{i+1} is the residual subformula of φ_i in the label of $\mathbb{P}(i+1)$.

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An infinite seq. of formulas $(\varphi_i)_i$ is a **trace** if every initial seq. is a trace.

Lemma

For every infinite trace there exists a formula $\sigma x \varphi$ that appears infinitely often and is the shortest formula occurring infinitely often.

Example of traces

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\nu x \Diamond x, \mu y \Box y}{\Diamond \nu x \Diamond x, \Box \mu y \Box y}\,\text{mod}}{\Diamond \nu x \Diamond x, \mu y \Box y}\,\mu}{\nu x \Diamond x, \mu y \Box y}\,\nu}{\nu x \Diamond x \vee \mu y \Box y}\,\vee$$

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μ-traces and ν-traces

In each infinite trace the unique $\sigma x \varphi$ identified by the lemma will be referred to as the "**most significant formula**" of the trace.

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- a **μ-trace** if $\sigma = \mu$
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A pre-tableau is a **tableau** if

1. the sequent at each leaf is an axiom, and
2. every infinite path contains a **ν-trace** (progressive thread).

Examples

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\nu x \Diamond x, \mu y \Box y}{\Diamond \nu x \Diamond x, \Box \mu y \Box y}\,\text{mod}}{\Diamond \nu x \Diamond x, \mu y \Box y}\,\mu}{\nu x \Diamond x, \mu y \Box y}\,\nu}{\nu x \Diamond x \vee \mu y \Box y}\,\vee \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\nu x \Diamond x, \mu y \Box y}{\Diamond \nu x \Diamond x, \Box \mu y \Box y}\,\text{mod}}{\Diamond \nu x \Diamond x, \mu y \Box y}\,\mu}{\nu x \Diamond x, \mu y \Box y}\,\nu}{\nu x \Diamond x \vee \mu y \Box y}\,\vee$$

Question What kind of traces do we have here?

It is not hard to check (semantically) that the following formula is valid.

$$\mu \mathsf{x} \nu \mathsf{y} \varphi(\mathsf{x}, \mathsf{y}) \to \nu \mathsf{y} \mu \mathsf{x} \varphi(\mathsf{x}, \mathsf{y})$$

Let $\psi_1 := \nu \mathsf{x} \mu \mathsf{y} \overline{\varphi}(\mathsf{x}, \mathsf{y})$ and $\psi_2 := \nu \mathsf{z} \mu \mathsf{w} \varphi(\mathsf{z}, \mathsf{w})$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mu \mathsf{y} \overline{\varphi}(\psi_1, \mathsf{y}), \mu \mathsf{w} \varphi(\mathsf{w}, \psi_2)}{\psi_1, \mu \mathsf{w} \varphi(\mathsf{w}, \psi_2)} \nu_{\mathsf{x}} \qquad \mu \mathsf{y} \overline{\varphi}(\psi_1, \mathsf{y}), \psi_2}{\vdots}}{\overline{\varphi}(\psi_1, \mu \mathsf{y} \overline{\varphi}(\psi_1, \mathsf{y})), \varphi(\mu \mathsf{w} \varphi(\mathsf{w}, \psi_2), \psi_2)} \mu_{\mathsf{y}}, \mu_{\mathsf{w}}}{\dfrac{\mu \mathsf{y} \overline{\varphi}(\psi_1, \mathsf{y}), \mu \mathsf{w} \varphi(\mathsf{z}, \psi_2)}{\mu \mathsf{y} \overline{\varphi}(\psi_1, y), \psi_2} \nu_{\mathsf{x}}}}{\psi_1, \psi_2} \nu_{\mathsf{z}}$$

It is not hard to check (semantically) that the following formula is valid.

$$\mu x \nu y \varphi(x, y) \to \nu y \mu x \varphi(x, y)$$

Let $\psi_1 := \nu x \mu y \overline{\varphi}(x, y)$ and $\psi_2 := \nu z \mu w \varphi(z, w)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mu y \overline{\varphi}(\psi_1, y), \mu w \varphi(w, \psi_2)}{\psi_1, \mu w \varphi(w, \psi_2)}\,\nu_x \qquad \mu y \overline{\varphi}(\psi_1, y), \psi_2}{\vdots}}{\overline{\varphi}(\psi_1, \mu y \overline{\varphi}(\psi_1, y)), \varphi(\mu w \varphi(w, \psi_2), \psi_2)}\,\mu_y, \mu_w}{\mu y \overline{\varphi}(\psi_1, y), \mu w \varphi(z, \psi_2)}\,\nu_z}{\mu y \overline{\varphi}(\psi_1, y), \psi_2}\,\nu_x}{\psi_1, \psi_2}$$

Question significance order? kind of traces?

Theorem (Niwinski, Walukiewicz 1996)

For every closed formula φ, there exists a tableau-proof for φ iff φ is valid.

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Main challenge Finding regular representations and turn the infinitary proofs into cyclic proofs.

Build an automaton $\mathcal{A}$ accepting precisely the ill-founded proofs. By Rabin Basis Theorem $\mathcal{A}$ accepts a regular tree, i.e. a cyclic proof.

Disadvantage: Need the automaton

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Direct 'Annotate' sequents with more information so that repetitions along paths can be identified. Use it to mark a frontier (finite initial subtree) that when viewed as a cyclic derivation tree witnesses the validity

- modal μ-calculus: Jungteerapanich, 2009; Stirling, 2014
- LTL: Brünnler and Lange 2008; Kokkinis and Studer 2016
- CTL*: A., Leigh & Turata 2023
- μFOL: A., Enqvist & Leigh 2022
- PDL (w.i.p)

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What sort of questions?

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What sort of questions?

- designing sound (and complete) axiomatisation
- investigating metalogical properties such as decidability, consistency, interpolation, cut-elimination, ordinal analysis, applications in other areas such as databases