Lifting final coalgebras and initial algebras, a reconstruction

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The research developed in the preprint [11] was partly motivated by recent research on the algebraic and categorical semantics of linear logic with fixed points [3, 2, 8, 4] extending previous work by one of authors on categorical semantics of fixed-point logics and on circular proof systems [10, 9, 5]. An important model of proofs of linear logic with fixed points that has been considered is the category Nuts of non uniform totality spaces. Other models that motivated us are the categories of the form P-Set introduced in [12]. All these categories arise as total categories over the category Rel of sets and relations, i.e. they are form $\int Q$ for a functor $Q : \mathsf{Rel} \longrightarrow \mathsf{Pos}$ (where Pos is the category of posets and monotone functions).

For a category C and a functor $Q : \mathsf{C} \longrightarrow \mathsf{Pos}$, the total category $\int Q$ (the Grothendieck construction of Q) is so described: an object is a pair (X, α) with $\alpha \in Q(X)$, an arrow $f : (X, \alpha) \longrightarrow (Y, \beta)$ is an arrow $f : X \longrightarrow Y$ of C such that $Q(f)(\alpha) \leq \beta$. The first projection $\pi : \int Q \longrightarrow \mathsf{C}$ is an op-fibration. We give in [11] a theorem for lifting various kind of functors (n-ary, covariant, contravariant,...) from C to $\int Q$ in such way that the canonical op-fibration $\pi : \int Q \longrightarrow \mathsf{C}$ strictly preserves them. This allows us to give conditions for lifting monoidal, closed, $*$-autonomous structures on C. For unary functors the statement sounds as follows:

Lemma. *For $F : \mathsf{C} \longrightarrow \mathsf{C}$, the following data are in bijection:*

(i) a functor $\overline{F} : \int Q \longrightarrow \int Q$ such that the diagram below on the left commutes,

(ii) a collection of arrows $\psi_X : Q(X) \longrightarrow Q(F(X))$ such that, for any arrow $f : X \longrightarrow Y$ in C, the diagram below on the right half commutes.

$$\begin{array}{ccc} \int Q & \xrightarrow{\overline{F}} & \int Q \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathsf{C} & \xrightarrow{F} & \mathsf{C} \end{array} \qquad\qquad \begin{array}{ccc} Q(X) & \xrightarrow{\psi_X} & Q(F(X)) \\ \downarrow{\scriptstyle Q(f)} & \not\geq & \downarrow{\scriptstyle Q(F(f))} \\ Q(Y) & \xrightarrow[\psi_Y]{} & Q(F(Y)) \end{array}$$

Above, $\overline{F}$ is called a lifting of F, and $\psi : Q \longrightarrow Q \circ F$ is a *lax* natural transformation. If $\overline{F}$ and ψ correspond under the bijection, then we write $\overline{F}^{\psi}$ in place of $\overline{F}$. We have that ψ is natural if and only if $\overline{F}^{\psi}$ preserves op-cartesian arrows.

Naturally, we have investigated in [11] the question of lifting initial algebras and final coalgebras of lifted functors, ending up rediscovering some results recently presented in [4] and also by other authors [6]. We aim with this talk at highlighting convergences and divergences of the approach taken in [4] with ours. This relies on the recognition of an op-fibration $\mathrm{CoAlg}_{\int Q}(\overline{F}) \longrightarrow \mathrm{CoAlg}_{\mathsf{C}}(F)$ and on the duality of the category SLatt of complete lattices and sup-preserving functions, by which we can also pinpoint an op-fibration $\mathrm{Alg}_{\int Q}(\overline{F}) \longrightarrow \mathrm{Alg}_{\mathsf{C}}(F)$ when Q is of the form $\mathsf{C} \longrightarrow \mathsf{SLatt}$. Firstly, we rediscovered the following proposition, folklore in coalgebra theory, see e.g. [6, 7, 1, 13].

Proposition. *Let $Q : \mathsf{C} \longrightarrow \mathsf{Pos}$, $F : \mathsf{C} \longrightarrow \mathsf{C}$, and consider a lifting $\overline{F}^{\psi} : \int Q \longrightarrow \int Q$ of F. For a coalgebra $(X, \gamma : X \longrightarrow F(X))$, define $Q^{\nu}_{\psi}(X, \gamma) := \{\, \alpha \in Q(X) \mid Q(\gamma)(\alpha) \leq \psi_X(\alpha) \,\}$. Then Q^{ν}_{ψ} extends (in an obvious way) to a functor $Q^{\nu}_{\psi} : \mathrm{CoAlg}_{\mathsf{C}}(F) \longrightarrow \mathsf{Pos}$ and we have an isomorphism*

$$\mathrm{CoAlg}_{\int Q}(\overline{F}^{\psi}) \simeq \int Q^{\nu}_{\psi}\,.$$

Moreover, if $Q : \mathsf{C} \longrightarrow \mathsf{SLatt}$, then so $Q^{\nu}_{\psi} : \mathrm{CoAlg}_{\mathsf{C}}(F) \longrightarrow \mathsf{SLatt}$.

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Let us stress, however, that for an (op-)fibration $E \longrightarrow \mathsf{C}$ and a predicate lifting $\overline{F}^{\psi} : \mathsf{E} \longrightarrow \mathsf{E}$ of $F : \mathsf{C} \longrightarrow \mathsf{C}$, $\overline{F}^{\psi}$ is most often required to preserve Cartesian arrows, so ψ is most often required to be natural. This condition is not needed for the proposition.

The above statement allows to reduce reasoning on lifting final coalgebras to lifting terminal objects to the total category (and thus use the general machinery of (op-)fibrations). It is easily seen that, for a functor $G : \mathsf{D} \longrightarrow \mathsf{Pos}$, *if* 1 *is a terminal object of* D *and* $\top \in G(1)$ *is the greatest element of this poset, then* $(1, \top)$ *is a terminal object of* $\int G$. Letting $G = Q^{\nu}_{\psi} : \mathrm{CoAlg}_{\mathsf{C}}(F) \longrightarrow \mathsf{Pos}$, we obtain:

Proposition 1 (c.f. [4, Theorem 2.6] and [6, Corollary 4.3])**.** *Given a final coalgebra* $(\nu.F, \xi)$*, recall that* ξ *is invertible. If the greatest fixed point* $\nu.f$ *of* $f := Q(\xi^{-1}) \circ \psi_{\nu.F} : Q(\nu.F) \longrightarrow Q(F(\nu.F)) \longrightarrow Q(\nu.F)$ *exists, then* $(\nu.F, \nu.f, \xi)$ *is a final coalgebra of* $\overline{F}^{\psi}$*.*

Let us further illustrate the strength of the op-fibration $\int Q^{\nu}_{\psi} \longrightarrow \mathrm{CoAlg}_{\mathsf{C}}(F)$. In principle, for a final $\overline{F}$-coalgebra (X, α, ξ), either (X, ξ) might not be a final F-coalgebra, or α might not be the greatest fixed point of some corresponding map. However these cases cannot arise when $Q : \mathsf{C} \longrightarrow \mathsf{SLatt}$. This is an immediate consequence of the following general statement, to be instantiated with $G = Q^{\nu}_{\psi} : \mathrm{CoAlg}_{C}(F) \longrightarrow \mathsf{SLatt}$: *for a functor* $G : \mathsf{D} \longrightarrow \mathsf{Pos}$*, if, for each object* Y *of* D*, the poset* $G(Y)$ *has a greatest element, then a terminal object in the category* $\int G$ *is necessarily of the form* $(1, \top)$ *for a final object* 1 *of* D *and a greatest element* $\top$ *of* $G(1)$*.*

We do not have, in general, a similar representation for $\mathrm{Alg}_{\int Q}(\overline{F})$ but we can still obtain it if we exploit the internal duality of SLatt. Thus we assume next that $Q : \mathsf{C} \longrightarrow \mathsf{SLatt}$ and let $Q^{*} : \mathsf{C}^{op} \longrightarrow \mathsf{SLatt}$ be defined by $Q^{*}(X) := Q(X)^{op}$ and $Q^{*}(f) := Q(f)^{*}$ being right adjoint to $Q(f)$.

Lemma. *We have* $(\int Q)^{op} = \int Q^{*}$ *and, moreover,* $(\overline{F}^{\psi})^{op}$ *is the lifting of* $F^{op} : \mathsf{C}^{op} \longrightarrow \mathsf{C}^{op}$ *to* $\int Q^{*}$ *via* $\psi^{op} : Q(X)^{op} \longrightarrow Q(F(X))^{op}$*.*

The lemma presents the key remarks for the following equalities:

$$\mathrm{Alg}_{\int Q}(\overline{F}^{\psi}) = \left[\mathrm{CoAlg}_{(\int Q)^{op}}(\overline{F}^{\psi\, op})\right]^{op} = \left[\mathrm{CoAlg}_{\int Q^{*}}(\overline{F^{op}}^{(\psi^{op})})\right]^{op} .$$

Using these equalities, we can readily derive the expected statement:

Proposition 2. *Given an initial algebra* $(\mu.F, \xi)$*, if the least fixed point* $\mu.f$ *of* $f := Q(\xi) \circ \psi_{\mu.F} : Q(\mu.F) \longrightarrow Q(F(\mu.F)) \longrightarrow Q(\mu.F)$ *exists, then* $(\mu.F, \mu.f, \xi)$ *is an initial algebra of* $\overline{F}^{\psi}$*.*

The proposition is a dualisation of Proposition 1, considering that a final coalgebra of F^{op} is an initial algebra of F, a greatest fixed point of $f^{op} : P^{op} \longrightarrow P^{op}$ is a least fixed point of f, and that the right adjoint of $Q(\xi^{-1})$ is $Q(\xi)$, i.e. $Q^{*}(\xi^{-1}) = Q(\xi^{-1})^{*} = Q(\xi)$. As it only relies on properties of adjunctions, the proposition generalises to the case where the functor has, as codomain, the category of posets and adjunctions thus yielding a bifibration. We can further consider the isomorphism

$$\mathrm{Alg}_{\int Q}(\overline{F}^{\psi}) = \left[\mathrm{CoAlg}_{\int Q^{*}}(\overline{F^{op}}^{(\psi^{op})})\right]^{op} \simeq \left[\int Q^{*\nu}\right]^{op} = \int Q^{*\nu *} ,$$

thus arriving to a description of the category of algebras of a lifted functor as the total category of a functor. Working with an op-fibration $\pi : \int G \longrightarrow \mathsf{D}$ with $G : \mathsf{D} \longrightarrow \mathsf{Pos}$, the following statement is easily verified: *if* 0 *is an initial object of* D*,* $\perp$ *is a least element of* $G(0)$*, and, for each object* X *of* D*, the unique arrow* $?_X : 0 \longrightarrow X$ *is such that* $G(?_X)(\perp)$ *is the least element of* $G(X)$*, then* $(0, \perp)$ *is an initial object of* $\int G$. The reader will have noticed the difference with the respective statement for lifting the terminal object, in that the maps $G(?_X)$ are asked here to preserve the least element. This preservation property is not mentioned in Proposition 2 since the functor $Q^{*\nu *}$ has SLatt as codomain.

Let us put $Q^{\mu}_{\psi} := Q^{*\nu *} : \mathrm{Alg}_{\mathsf{C}}(F) \longrightarrow \mathsf{SLatt}$. An explicit description of this functor is as follows:

$$Q^{\mu}_{\psi}(X, \gamma : F(X) \longrightarrow X) = \{\, \alpha \in Q(X) \mid Q(\gamma)(\psi_X(\alpha)) \leq \alpha \,\} ,$$
$$Q^{\mu}_{\psi}(\, f : (X, \gamma) \longrightarrow (Y, \delta)\,)(\alpha) = \text{least } \beta \in Q^{\mu}_{\psi}(Y, \delta) \text{ such that } Q(f)(\alpha) \leq \beta .$$

We might ask what happens with the expression defined above if we do not rely on the duality of SLatt. Say that $Q(X)$ is a complete lattice but $Q(f)$ does not preserve all the suprema. If ψ is a natural transformation, then $Q^{\mu}_{\psi}(f)(\alpha) = Q(f)(\alpha)$. In general, Q^{μ}_{ψ} is only an oplax functor, that is, it satisfies $Q^{\mu}_{\psi}(g \circ f) \leq Q^{\mu}_{\psi}(g) \circ Q^{\mu}_{\psi}(f)$. However, this is all what is needed to define the category $\int Q^{\mu}_{\psi}$ which again is isomorphic to $\mathrm{Alg}_{\int Q}(\overline{F}^{\psi})$.Whether or not ψ is natural, in order to derive a statement as in Proposition 2, we only need preservation of least fixed points of the operators $Q(\gamma) \circ \psi_X$. This can be achieved by means of the following lemma on fixed points, which we wish to explicitly state despite its easy proof, since it makes it possible to connect to [4] where it is implicit in Lemma 2.10.

Lemma. *Consider a half-commuting diagram of posets as the one below on the left. If A and B are complete lattices and f preserves suprema of (possibly empty) chains, then $f(\mu.g^A) \leq \mu.g^B$.*

$$
\begin{array}{ccc}
A & \xrightarrow{g^A} & A \\
{\scriptstyle f}\downarrow & \leq & \downarrow{\scriptstyle f} \\
B & \xrightarrow[g^B]{} & B
\end{array}
\qquad\qquad
\begin{array}{ccccc}
Q(X) & \xrightarrow{\psi_X} & Q(F(X)) & \xrightarrow{Q(\gamma)} & Q(X) \\
{\scriptstyle Q(f)}\downarrow & \leq & \downarrow{\scriptstyle Q(F(f))} & & \downarrow{\scriptstyle Q(f)} \\
Q(Y) & \xrightarrow[\psi_Y]{} & Q(F(Y)) & \xrightarrow{Q(\delta)} & Q(Y)
\end{array}
$$

Assuming that $Q(f)$ preserves suprema of chains, the lemma is applied to the situation above on the right and yields that the image by $Q(f)$ of the least element of $Q^{\mu}_{\psi}(X)$ is below the least element of $Q^{\mu}_{\psi}(Y)$, whence $Q^{\mu}_{\psi}(f)$ preserves least elements. As emphasized in [4], the lemma can be strengthened to a constructive setting, by assuming that A and B are CPOs with a least element, by requiring that the poset $\{\, a \in A \mid f(a) \leq \mu.g^B \,\}$ is a CPO containing the least element of A, and relying on Pataraya's theorem.

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