Combining fixpoint and differentiation theory

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We axiomatize the notion of Cartesian differential categories with a fixpoint operator by introducing an additional axiom relating the derivative of a fixpoint with the fixpoint of the derivative. We show how the standard examples of Cartesian differential categories where we can compute least or greatest fixpoints provide canonical models of this notion.

Cartesian differential categories were introduced by Blute, Cockett, and Seely in [4], and provide the categorical foundations of multivariable differential calculus. Briefly, a Cartesian differential category (Definition 1) is a category with finite products that comes equipped with a differential combinator $\mathbf{D}$, which for every map $f : A \to B$ produces its derivative $\mathbf{D}[f] : A \times A \to B$, satisfying seven axioms that are analogues of the fundamental properties of the total derivative, including the chain rule. Cartesian *closed* differential categories provide the categorical semantics of the differential λ-calculus, introduced by Ehrhard and Regnier in [8]. Cartesian (closed) differential categories have been quite successful in formalizing various notions related to differentiation, and have also found numerous applications in both mathematics and computer science. Moreover, Cartesian differential categories belong to a larger story of categorical foundations of differentiation, which include differential categories [5], coherent differential categories [9], and tangent categories [6].

One aspect that has not yet been studied in full detail is the interaction between fixpoint operators and differential operators. We present work in progress towards a general account of fixpoint operators such as parametrized fixpoint operators (Definition 2), Conway operators, and trace operators for various categorial frameworks of differentiations. In this abstract, we focus on Cartesian differential categories equipped with a *parametrized* fixpoint operator. This axiomatization provides a guideline to introducing differentials to λ-calculi with fixpoint operators such as PCF [12] and the λ-Y-calculus [15].

Related Work: In [9, Theorem 5.29], Ehrhard proves a compatibility relation in a cpo-enriched setting between the least fixpoint operator (in the coKleisli category) and the tangent bundle functor of a Scott coherent differential category. In [14], Sprunger and Katsumata construct Cartesian differential categories with delayed trace operators, which are related to trace operators (and hence Conway operators) but no longer assume the fixpoint axiom.

1 Cartesian Differential Categories

Cartesian differential categories are categories which come equipped with a differential combinator, which is an operator which sends maps to their derivative. The axioms of a differential combinator are analogues of the basic properties of the total derivative from multivariable differential calculus. For a more in-depth introduction to Cartesian differential categories, we refer the reader to the original paper [4].

Definition 1. *A* Cartesian differential category *is a Cartesian left additive category [4, Definition 1.2.1]* $\mathbb{C}$ *equipped with a differential combinator* $\mathbf{D}$*, which is a family of functions:*

$$\mathbf{D} : \mathbb{C}(A,B) \to \mathbb{C}(A \times A, B)$$

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*satisfying the seven axioms in [4, Definition 2.1.1]. For a morphism $f : A \to B$, the map $\mathsf{D}[f] : A \times A \to B$ is called the **derivative** of f.*

Briefly, the axioms of a differential combinator are that: (1) the derivative of a sum is the sum of the derivatives, (2) derivatives are additive in their second argument, (3) the derivative of identity maps and projections are projections, (3) the derivative of a pairing is the pairing of the derivatives, (5) the chain rule for the derivative of a composition, (6) the derivative is linear in its second argument, and lastly (7) the symmetry of the partial derivatives. There is a sound and complete term logic for Cartesian differential categories [4, Section 4], which is useful for intuition. So we write:

$$\mathbf{D}[f](a,b) := \frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b$$

In particular, the chain rule axiom is that:

$$\mathsf{D}[g \circ f] = \mathsf{D}[g] \circ \langle f \circ \pi_1, \mathsf{D}[f] \rangle$$

which using the term logic is expressed as:

$$\frac{\mathsf{d}g\left(f(x)\right)}{\mathsf{d}x}(a) \cdot b = \frac{\mathsf{d}g(y)}{\mathsf{d}y}(f(a)) \cdot \left(\frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b \right)$$

2 Fixpoint Operators and Differentiation

Parametrized fixpoint operators [3, 13] axiomatize the notion of fixpoints for morphisms in context. For a morphism of type $A \times X \to X$, the parameter A is viewed as representing the context of the term, then taking the parametrized fixpoint gives a morphism of type $A \to X$. Parametrized fixpoint operators are axiomatized by two axioms: (1) the fixpoint axiom and (2) a naturality in the context axiom.

Definition 2. *For a category $\mathbb{C}$ with finite products, a* parametrized fixpoint operator **fix** *is a family of functions*

$$\mathbf{fix}_A^X : \mathbb{C}(A \times X, X) \to \mathbb{C}(A, X)$$

indexed by pairs of objects in $\mathbb{C}$ such that:

1. ***Fixpoint**: for all maps $f : A \times X \to X$:*

$$f \circ \langle 1_A, \mathbf{fix}_A^X(f) \rangle = \mathbf{fix}_A^X(f)$$

2. ***Naturality**: for all maps $g : A \to B$ and $f : B \times X \to X$:*

$$\mathbf{fix}_B^X(f) \circ g = \mathbf{fix}_A^X(f \circ (g \times 1_X))$$

We shall use term calculus notation to write the parametrized fixpoint operator as follows:

$$\mathbf{fix}_A^X(f)(a) = \mu x. f(x,a)$$

where the variable x is bounded. So in particular, the fixpoint axiom is expressed as:

$$\mu x. f(a,x) = f(a, \mu x. f(a,x))$$

So how should a parametrized fixpoint operator and differential operator interact? In particular, what should the derivative of a parametrized fixpoint be? Well consider the following computation using the parametrized fixpoint and the chain rule:

$$\frac{\mathsf{d}\mu x.f(u,x)}{\mathsf{d}u}(a)\cdot b = \frac{\mathsf{d}f(u,\mu x.f(u,x))}{\mathsf{d}u}(a)\cdot b = \frac{\mathsf{d}f(u,v)}{\mathsf{d}(u,v)}(a,\mu x.f(a,x))\cdot\left(b,\frac{\mathsf{d}\mu x.f(u,x)}{\mathsf{d}u}(a)\cdot b\right)$$

Thus, our compatibility relation between a parametrized fixpoint operator and differential operator is asking that $\frac{\mathsf{d}\mu x.f(u,x)}{\mathsf{d}u}(a)\cdot b$ to be equal to the second component of the nested fixpoint:

$$\mu(x,y).\left(f(a,x),\frac{\mathsf{d}f(u,v)}{\mathsf{d}(u,v)}(a,x)\cdot(b,y)\right)$$

In a Cartesian differential category equipped with a fixpoint operator, this corresponds to requiring the following two operations to be equal:

$$\dfrac{\dfrac{A\times X\xrightarrow{f}X}{A\xrightarrow{\mathbf{fix}_A^X(f)}X}}{A\times A\xrightarrow{\mathbf{D}[\mathbf{fix}_A^X(f)]}X} \quad = \quad \dfrac{\dfrac{\dfrac{A\times X\xrightarrow{f}X}{A\times X\times A\times X\xrightarrow{\langle f\circ\pi_1,\mathbf{D}[f]\rangle}X\times X}}{A\times A\times X\times X\xrightarrow{c}A\times X\times A\times X\xrightarrow{\langle f\circ\pi_1,\mathbf{D}[f]\rangle}X\times X}}{A\times A\xrightarrow{\mathbf{fix}_{A\times A}^{X\times X}(\langle f\circ\pi_1,\mathbf{D}[f]\rangle\circ c)}X\times X\xrightarrow{\pi_2}X}$$

where $c = A\times\langle\pi_2,\pi_1\rangle\times X : A\times A\times X\times X\to A\times X\times A\times X$ is the canonical isomorphism swapping the middle two terms.

Definition 3. *A* Cartesian differential fixpoint category *is a Cartesian differential category equipped with a parametrized fixpoint operator such that the following equality holds:*

$$\mathbf{D}[\mathbf{fix}(f)] = \pi_2\circ\mathbf{fix}(\langle f\circ\pi_1,\mathbf{D}[f]\rangle\circ c) \tag{1}$$

which in the term calculus is expressed as follows:

$$\frac{\mathsf{d}\mu x.f(u,x)}{\mathsf{d}u}(a)\cdot b = \pi_2\left(\mu(x,y).\left(f(a,x),\frac{\mathsf{d}f(u,v)}{\mathsf{d}(u,v)}(a,x)\cdot(b,y)\right)\right)$$

If the parametrized fixpoint operator is also a **Conway operator** [13, Definition 2.4], by Bekić's property [13, Proposition 2.5] we have that:

$$\mu(x,y).\left(f(a,x),\frac{\mathsf{d}f(u,v)}{\mathsf{d}(u,v)}(a,x)\cdot(b,y)\right) = \left(\mu x.f(a,x),\frac{\mathsf{d}\mu x.f(u,x)}{\mathsf{d}u}(a)\cdot b\right)$$

Therefore for a Conway operator, (1) is equivalent to:

$$\frac{\mathsf{d}\mu x.f(u,x)}{\mathsf{d}u}(a)\cdot b = \mu y.\frac{\mathsf{d}f(u,v)}{\mathsf{d}(u,v)}(a,\mu x.f(a,x))\cdot(b,y) \tag{2}$$

Since a category with a Conway operator is a *traced* Cartesian monoidal category [10, Theorem 3.1], we may call a Cartesian differential fixpoint category with a Conway operator a **traced Cartesian differential category**.

3 Example: Relations

Let **Rel** be the category of sets and binary relations. The operation mapping a set A to the set of finite multisets over A, $!A := \{m : A \to \mathbb{N} \mid m \text{ has finite support}\}$, can be equipped with a comonad structure on **Rel**. The induced co-Kleisli category $\mathbf{Rel}_!$ is a Cartesian differential category whose differential operator takes a relation $R \subseteq\, !A \times B$ to the relation

$$\mathbf{D}R = \{((m,[a]),b) \mid (m+[a],b) \in R\} \subseteq\, !A \times !A \times B$$

The category $\mathbf{Rel}_!$ also has a parametrized fixpoint operator mapping a relation $R \subseteq\, !A \times !X \times X$ to the relation $\mathbf{fix}\, R \subseteq\, !A \times X$ inductively defined as:

$$\mathbf{fix}\, R := \bigvee_{n \in \mathbb{N}} \mathbf{fix}_n\, R$$

where $\mathbf{fix}_0\, R = \emptyset$ and $\mathbf{fix}_{n+1}\, R$ is given by:

$$\{(m_0 + \cdots + m_k, x) \mid \exists [x_1,\ldots,x_k] \in\, !X, (m_0,[x_1,\ldots,x_k],x) \in R, \forall 1 \le i \le n, (m_i,x_i) \in \mathbf{fix}_n\, R\}.$$

This fixpoint operator and the differential operator verify (1), and therefore $\mathbf{Rel}_!$ is a Cartesian differential fixpoint category. Moreover, the fixpoint operator is a Conway operator, and therefore $\mathbf{Rel}_!$ is a traced Cartesian differential category.

The extensions of the relational models to weighted relations or matrices over a continuous semiring [11] are also instances of Cartesian differential fixpoint categories. In general, (1) holds for a Cartesian differential category where the fixpoint operator is computed as a least or greatest fixpoint operator in an ordered-enriched setting. We also aim to extend our results to *guarded* fixpoint operators to accommodate for Cartesian differential categories based on metric spaces [1], where fixpoints can be computed via the Banach fixpoint theorem.

Future Work

1. One objective is to find and study other examples of Cartesian differential fixpoint categories or traced Cartesian differential categories. One possible way of constructing more examples is via the Faa di Bruno construction [7], which is a way of constructing cofree Cartesian differential categories.
2. An important source of examples of Cartesian differential categories are the coKleisli categories of differential categories [5] (the latter of which provide the categorical semantics of Differential Linear Logic). The relational model above is such an example. In certain cases, there are interesting differential categories that are compact closed, and therefore have a canonical trace. Thus it is natural to study differential categories that are also traced, and try to provide a unified setting for the interactions between fixpoints, trace and differentiation.
3. Cartesian differential categories axiomatize differential calculus over Euclidean spaces, while tangent categories [6] axiomatize differential calculus over smooth manifolds. In a Cartesian differential category, the differential combinator induces a functor $\mathbf{T} : \mathbb{C} \to \mathbb{C}$, called the tangent bundle functor, defined on object as $\mathbf{T}(A) = A \times A$ and on maps as $\mathbf{T}(f) = \langle f \circ \pi_1, \mathsf{D}[f] \rangle$. If the parametrized fixpoint operator satisfies the additional Bekić's axiom [2], we can reformulate the differential axiom (1) as
$$\mathbf{T}(\mathbf{fix}(f)) = \mathbf{fix}(\mathbf{T}(f) \circ c)$$
giving us a notion of tangent fixpoint category.

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