Coinductive control of inductive data types

Paige Randall North[1] and Maximilien Péroux[2]

[1] Utrecht University
p.r.north@uu.nl
[2] Michigan State University
peroux@msu.edu

Abstract

We combine the theory of inductive data types with the theory of universal measurings. By doing so, we find that many categories of algebras of endofunctors are actually enriched in the corresponding category of coalgebras of the same endofunctor. The enrichment captures all possible partial algebra homomorphisms, defined by measuring coalgebras. Thus this enriched category carries more information than the usual category of algebras which captures only total algebra homomorphisms. We specify new algebras besides the initial one using a generalization of the notion of initial algebra.

This is based on our paper [4] and subsequent unpublished work.

1 Introduction

In both the tradition of functional programming and categorical logic, one takes the perspective that most data types should be obtained as initial algebras of certain endofunctors (to use categorical language). For instance, the natural numbers are obtained as the initial algebra of the endofunctor $X \mapsto X + 1$, assuming that the category in question (often the category of sets) has a terminal object 1 and a coproduct $+$. Much theory has been developed around this approach, which culminated in the notion of W-types [2].

In another tradition, for k a field, it has been long understood (going back at least to Wraith, according to [1], and Sweedler [7]) that the category of k-algebras is naturally enriched over the category of k-coalgebras, a fact which has admitted generalization to several other settings (e.g. [1, 8, 6, 3]). In this work, we extend this theory to the setting of an endofunctor on a category – in particular those endofunctors that are considered in the theory of W-types.

This work is thus the beginning of a development of an analogue of the theory of W-types – not based on the notion of initial objects in a *category* of algebras, but rather on generalized notions of initial objects in a *coalgebra enriched category* of algebras. Our main result states that the categories of algebras of endofunctors considered in the theory of W-types are often enriched in their corresponding categories of coalgebras. The hom-coalgebras of our enriched category carry more information than the hom-sets in the unenriched category that is usually considered in the theory of W-types. Because of our passage to the enriched setting, we have more precise control than in the unenriched setting, and we are able to specify more data types than just those which are captured by the theory of W-types. We do this by generalizing the notion of initial algebra, taking inspiration from the notion of weighted limits.

We illustrate our theory with several enlightening examples which serves as illustrations of the relevance of our enriched theory and as a motivation for the more general setting. In particular, we provide explicit calculations for the case of algebras over the endofunctors whose initial algebras are natural numbers, lists, and binary trees, respectively. In those examples, we illustrate that it is appropriate to interpret the elements of the hom-coalgebras as *partial homomorphisms*.

Indeed, in the classical Sweedler theory, the enrichment in coalgebras can also be understood as encoding a notion of partial homomorphism. Though we do not study k-algebras in this work, we conclude this introduction with details of that classical theory. A *measuring* from a k-algebra A to a k-algebra B, in the sense of Sweedler [7], is a k-coalgebra C together with a linear homomorphism $\phi\colon C\otimes_k A\to B$ that is compatible with the multiplication and identities of A and B. A measuring from A to B is equivalently a k-coalgebra C together with a k-linear map $\phi : C\to A\to B$ such that

$$\phi_c(aa') = \sum_{i=1}^{n} \phi_{c_i^{(1)}}(a)\phi_{c_i^{(2)}}(a'), \quad \text{and} \quad \phi_c(1_A) = \varepsilon(c)1_B,$$

for all $c\in C$ and $a, a'\in A$ where $\Delta(c) = \sum_{i=1}^{n} c_i^{(1)}\otimes c_i^{(2)}$ is the comultiplication $\Delta\colon C\to C\otimes_k C$ and $\varepsilon\colon C\to k$ is the counit of C. Therefore the k-linear maps $\phi_c\colon A\to B$ can be regarded as partial algebra homomorphisms, and the elements $c\in C$ can be interpreted as measuring how far each partial homomorphism ϕ_c is from being a total homomorphism. For instance when $\Delta(c) = c\otimes c$, we have that $\phi_c\colon A\to B$ is a total algebra homomorphism. Now we proceed to tell an analogous story about endofunctors.

2 General results

Our main theorem is the following.

Theorem. *Let $(\mathsf{C}, \otimes, \mathbb{I}, \underline{\mathsf{C}}(-,-))$ be a locally presentable symmetric monoidal closed category. Let $F : \mathsf{C}\to\mathsf{C}$ be an accessible lax symmetric monoidal endofunctor. Then the category Alg_F of F-algebras is enriched, tensored, and powered over the symmetric monoidal category CoAlg_F of F-coalgebras.*

We show that many endofunctors of interest in the theory of W-types satisfy these hypotheses (see section 3) Thus, there is much interesting structure in the category of algebras of these endofunctors that has yet to be exploited.

We then observe that there is an implicit parameter in the notion of initial algebra which we may now vary. One might think of an initial object as a certain *colimit*, but in reality, an initial object in a category C is usually (equivalently) defined as an object I with the property that $\mathsf{hom}(I, X) = \{*\}$ for every $X\in\mathsf{C}$. That is, I is the vertex of a cone over the identity functor on C with the special property that each leg of the cone (at an object $X\in\mathsf{C}$) is the only morphism of $\mathsf{hom}(I, X)$. The reader might know that as such, an initial object can always be defined as the *limit* of the identity functor on C. In this enriched setting, we can ask that there is a unique morphism of coalgebras $F\to\mathsf{hom}(I, X)$ for every X, where F is the terminal coalgebra (so playing the role of $*$). In general, however, we can vary this F.

Definition (C-initial algebra)**.** *Suppose that C is locally presentable and closed and that F is accessible.*

*Given a coalgebra C, we say an algebra A is a C-*initial algebra *if there exists a unique map $C\to\underline{\mathsf{Alg}}(C, X)$, for all algebras X (where $\underline{\mathsf{Alg}}(C, X)$ denotes the hom-coalgebra between C and X given by the enrichment of the above theorem).*

The terminal *C-initial algebra is the terminal object, if it exists, in the subcategory of Alg spanned by the C-initial algebras.*

In examples, this produces many interesting generalized initial algebras. We interpret them as encoding a notion of *partial induction*.

3 Examples

There are many examples that validate the hypotheses of our main theorem. For example, the endofunctors on Set whose initial algebras are natural numbers, lists, and binary trees produce examples.

Examples. *Suppose that* C *is locally presentable and closed. The following endofunctors on* C *are accessible and lax symmetric monoidal.*

(id) *The identity endofunctor* id_{C}*.*

(A) *The constant endofunctor that sends each object to a fixed commutative monoid* A *in* C.

(GF) *The composition* GF *of accessible, lax symmetric monoidal endofunctors* F *and* G*.*

$(F \otimes G)$ *The pointwise tensor product* $F \otimes G$ *of accessible, lax symmetric monoidal endofunctors* F *and* G*, assuming* C *is closed.*

$(F + G)$ *The pointwise coproduct* $F + G$ *of an accessible, lax symmetric monoidal endofunctor* F *and an accessible endofunctor* G *equipped with natural transformations* $GX \otimes GY \to G(X \otimes Y)$, $\lambda : FX \otimes GY \to G(X \otimes Y)$, $\rho : GX \otimes FY \to G(X \otimes Y)$ *satisfying the axioms described in [5, Appendix A.7], assuming* C *is closed.*

(id^A) *The exponential* id^A *for any object* A *of* C*, assuming the monoidal product on* C *is cartesian closed.*

(W-types) *A polynomial endofunctor associated to a morphism* $f : X \to Y$ *in* Set*, given a commutative monoid structure on* Y *and an oplax symmetric monoidal structure on the preimage functor* $f^{-1} : C \to \mathsf{Set}$.

(d.e.s.) *A discrete equational system, assuming that the monoidal structure on* C *is cocartesian and that* C *has binary products that preserve filtered colimits.*

For example, consider the endofunctor on Set given by $X \mapsto X + 1$, the endofunctor whose initial algebra is the natural numbers.

The category $\underline{\mathsf{Alg}}$ of algebras for this endofunctor is naturally enriched in its coalgebras: sets equipped with a partial endofunction. One recovers the usual (unenriched) category Alg of algebras by taking the points of every hom-coalgebra in $\underline{\mathsf{Alg}}$: that is, to recover the hom-set $\mathsf{Alg}(A, B)$ from the hom-coalgebra $\underline{\mathsf{Alg}}(A, B)$ (for two algebras A, B), one takes the set $\mathsf{CoAlg}(\infty, \underline{\mathsf{Alg}}(A, B))$.

Above, ∞ denotes the coalgebra whose underlying set is the singleton and whose partial endofunction is the identity; it is the monoidal unit so $\mathsf{CoAlg}(\infty, C)$ gives us the points of any coalgebra C. Here, one can see that $\mathbb{N}^\infty$ is a rather special coalgebra, and points of a coalgebra C might be hard to come by, even if C has interesting structure. For example, the terminal coalgebra $\mathbb{N}^\infty$ has only one point, and the subterminal coalgebras $\mathbb{n}^\circ$ (with underlying set $\{0, ..., n\}$ and partial endofunction $i \mapsto i - 1$) have no points.

For some interesting examples of hom-coalgebras between algebras that capture more information than the usual homsets, consider the following preinitial algebras. Let $\mathbb{n}$ denote the algebra with underlying set $\{0, ..., n\}$, 'zero' element 0, and 'successor' endofunction $i \mapsto i + 1$ for $i < n$ and $n \mapsto n$. Then $\underline{\mathsf{Alg}}(\mathbb{n}, \mathbb{N}) \cong \mathbb{n}^\circ$ but $\mathsf{Alg}(\mathbb{n}, \mathbb{N}) = \emptyset$. We interpret the value $\mathbb{n}^\circ$ for $\underline{\mathsf{Alg}}(\mathbb{n}, \mathbb{N})$ as expressing the fact that the construction of an algebra homomorphism $\mathbb{n} \to \mathbb{N}$ can be inductively attempted by specifying where 0 is sent, where 1 is sent, etc, but fails at

n. We can think of this as an n-*partial* algebra homomorphism that is generated by n-*partial* induction.

We then find that the $\mathbb{n}^{\circ}$-initial algebra is $\mathbb{n}$. We interpret this as confirming our intuition that $\mathbb{n}$ is initial with respect to n-partial induction.

References

[1] Martin Hyland, Ignacio López Franco, and Christina Vasilakopoulou. Hopf measuring comonoids and enrichment. *Proc. Lond. Math. Soc. (3)*, 115(5):1118–1148, 2017. doi:10.1112/plms.12064.

[2] Per Martin-Löf. Intuitionistic type theory. *Naples: Bibliopolis*, 1984.

[3] Dylan McDermott, Exequiel Rivas, and Tarmo Uustalu. Sweedler theory of monads. In Patricia Bouyer and Lutz Schröder, editors, *Foundations of Software Science and Computation Structures*, pages 428–448, Cham, 2022. Springer International Publishing. doi:10.1007/978-3-030-99253-8_22.

[4] Paige Randall North and Maximilien Péroux. Coinductive Control of Inductive Data Types. In Paolo Baldan and Valeria de Paiva, editors, *10th Conference on Algebra and Coalgebra in Computer Science (CALCO 2023)*, volume 270 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 15:1–15:17, Dagstuhl, Germany, 2023. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. URL: https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.CALCO.2023.15, doi:10.4230/LIPIcs.CALCO.2023.15.

[5] Paige Randall North and Maximilien Péroux. Coinductive control of inductive data types, arxiv:2303.16793, 2023. arXiv:2303.16793.

[6] Maximilien Péroux. The coalgebraic enrichment of algebras in higher categories. *Journal of Pure and Applied Algebra*, 226(3):106849, 2022. doi:https://doi.org/10.1016/j.jpaa.2021.106849.

[7] Moss E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.

[8] Christina Vasilakopoulou. Enriched duality in double categories: $\mathcal{V}$-categories and $\mathcal{V}$-cocategories. *J. Pure Appl. Algebra*, 223(7):2889–2947, 2019. doi:10.1016/j.jpaa.2018.10.003.