The μ-calculus' Alternation Hierarchy is Strict over Non-Trivial Fusion Logics

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The modal μ-calculus is obtained by adding least and greatest fixed-point operators to modal logic. It's alternation hierarchy classifies the μ-formulas by their alternation depth: a measure of the codependence of their least and greatest fixed-point operators. The μ-calculus' alternation hierarchy is strict over the class of all Kripke frames: for all n, there is a μ-formula with alternation depth $n+1$ which is not equivalent to any formula with alternation depth n. This does not always happen if we restrict the semantics. For example, every μ-formula is equivalent to a formula without fixed-point operators over $\mathsf{S5}$ frames. We show that the multimodal μ-calculus' alternation hierarchy is strict over non-trivial fusions of modal logics. We also comment on two examples of multimodal logics where the μ-calculus collapses to modal logic.

1 Introduction

The modal μ-calculus is obtained by adding least and greatest fixed-point operators to modal logic. One measure of complexity for μ-formulas is their alternation depth, which measures the codependence of least and greatest fixed-point operators. Bradfield [2] showed that the μ-calculus' alternation hierarchy is strict: for all $n \in \mathbb{N}$, there is a formula with alternation depth $n+1$ which is not equivalent over unimodal frames to any formula with alternation depth n. On the other hand, Alberucci and Facchini [1] proved that, over $\mathsf{S5}$ frames, every μ-formula is equivalent to a formula without fixed-point operators. See Chapter 2 of [10] for a survey on the μ-calculus' alternation hierarchy over various classes of frames.

Let L_0 and L_1 be modal logics with disjoint signatures. The fusion $\mathsf{L}_0 \otimes \mathsf{L}_1$ is the smallest modal logic containing both L_0 and L_1. If L_0 and L_1 are respectively characterized by the Kripke frames in F_0 and F_1, then the fusion $\mathsf{L}_0 \otimes \mathsf{L}_1$ is characterized by frames which are in F_i when restricted to the signature of L_i, for $i = 0, 1$. Fusion logics are commonly used for multi-agent epistemic logics and on the specification of computer systems. We show that, over fusions of non-trivial classes of frames, the μ-calculus' alternation hierarchy is strict.

Let F be a class of unimodal Kripke frames. We say $\circ \leftarrow \circ \rightarrow \circ$ is a subframe of F iff there is some frame $F = \langle W, R\rangle \in \mathsf{F}$ with pairwise different $w_0, w_1, w_2 \in W$ such that $w_0 R w_1$ and $w_0 R w_2$. We analogously define $\circ \rightarrow \circ \rightarrow \circ$ is a subframe of F and $\circ \rightarrow \circ$ is a subframe of F. We will define multimodal versions W_n of the winning region formulas W_n' to prove:

Main Theorem. *Let F_0, F_1, and F_2 be classes of unimodal Kripke frames closed under isomorphic copies and disjoint unions. If*

1. $\circ \leftarrow \circ \rightarrow \circ$ *is a subframe of* F_0 *and* $\circ \rightarrow \circ$ *a subframe of* F_1*; or*

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2. $\circ \to \circ \to \circ$ is a subframe of F_0 and $\circ \to \circ$ a subframe of F_1;

then the μ-calculus' alternation hierarchy is strict over $\mathsf{F}_0 \otimes \mathsf{F}_1$. If

3. $\circ \to \circ$ is a subframe of F_0, F_1, and F_2;

then the μ-calculus' alternation hierarchy is strict over $\mathsf{F}_0 \otimes \mathsf{F}_1 \otimes \mathsf{F}_2$.

Corollary. *Let $\{\mathsf{L}_0, \mathsf{L}_1\} \subseteq \{\mathsf{K}, \mathsf{K4}, \mathsf{S4}, \mathsf{KD45}, \mathsf{S5}, \mathsf{GL}\}$, then the μ-calculus' alternation hierarchy is strict over $\mathsf{L}_0 \otimes \mathsf{L}_1$.*

While the hypotheses of the Main Theorem looks *ad hoc*, we conjecture that they are optimal.

Conjecture. *Let F_0 and F_1 be classes of unimodal Kripke frames closed under isomorphic copies and disjoint unions. Suppose $\circ \to \circ$ is a subframe of F_0 and F_1. Then every μ-formula is equivalent to one with alternation depth 1 over $\mathsf{F}_0 \otimes \mathsf{F}_1$.*

As a counterpoint, we comment on two multimodal logics where the μ-calculus collapses to modal logic. GLP is a provability logic which contains countably many modal operators; its fixed-point property was proved by Ignatiev [6]. IS5 is an intuitionistic version of S5 which can be thought as fragment of a bimodal logic; the μ-calculus' collapse to modal logic over IS5 was proved by Pacheco [11].

Outline In Section 2, we review some basic definitions. In Sections 3, 4, and 5, we give a detailed proof of Item 1 of the Main Theorem: we first show that evaluation games for the μ-calculus are also parity games; then define the formulas W_n and show how parity games can be encoded as multimodal Kripke models; and, at last, show that W_n is not equivalent to any formula with lower alternation depth. In Section 6, we sketch how to modify the proof to show Items 2 and 3 of the Main Theorem. In Section 7, we describe two examples of multimodal logics where the μ-calculus collapses to modal logic.

2 Preliminaries

The μ-calculus Fix a set Prop of propositional symbols, a set Var of variable symbols, and a non-empty signature Λ. The *μ-formulas* are generated by the following grammar:

$$\varphi := P \mid \neg P \mid X \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box_i \varphi_i \mid \Diamond_i \varphi \mid \mu X.\varphi \mid \nu X.\varphi,$$

where $P \in$ Prop, $X \in$ Var is a variable symbol, and $i \in \Lambda$. We write $\eta X.\varphi$ for $\mu X.\varphi$ or $\nu X.\varphi$. The *set of subformulas* of a formula φ is denoted by $\mathrm{Sub}(\varphi)$.

Given a signature Λ, a *Kripke frame* is a pair $M = \langle W, \{R_i\}_{i \in \Lambda} \rangle$ where: W is the set of *possible worlds*; and each R_i is a binary relation on W, the *accessibility relations*. A *Kripke model* is a triple $M = \langle W, \{R_i\}_{i \in \Lambda}, V \rangle$ obtained by extending a Kripke frame with a function V from propositional symbols to subsets of W; V is called a *valuation function*. Given a set $A \subseteq W$, the *augmented model* $M[X := A]$ is obtained by setting $V(X) := A$. A *pointed Kripke model* is a pair (M, w) consisting of a Kripke model M and a world w of M.

Fix a Kripke model $M = \langle W, \{R_i\}_{i \in \Lambda}, V \rangle$. Given a μ-formula $\varphi(X)$ with a distinguished variable X, let $\Gamma_{\varphi(X)} : \mathscr{P}(W) \to \mathscr{P}(W)$ be the operator which maps $A \subseteq W$ to $\|\varphi(X)\|^{M[X:=A]}$. We define the valuation $\|\varphi\|^M$ on M inductively on the structure of μ-formulas:

- $\|P\|^M := V(P)$;
- $\|X\|^{M[X:=A]} := A$;
- $\|\neg\varphi\|^M := W \setminus \|\varphi\|^M$;
- $\|\varphi \wedge \psi\|^M := \|\varphi\|^M \cap \|\psi\|^M$;
- $\|\varphi \vee \psi\|^M := \|\varphi\|^M \cup \|\psi\|^M$;
- $\|\Box_i \varphi\|^M := \{w \in W \mid \forall v. wR_i v \to v \in \|\varphi\|^M\}$;
- $\|\Diamond_i \varphi\|^M := \{w \in W \mid \exists v. wR_i v \wedge v \in \|\varphi\|^M\}$;
- $\|\mu X.\varphi\|^M$ is the least fixed-point of $\Gamma_{\varphi(X)}$;
- $\|\nu X.\varphi\|^M$ is the greatest fixed-point of $\Gamma_{\varphi(X)}$.

Note that, the operator $\Gamma_{\varphi(X)}$ is monotone for all formula $\varphi(X)$: if $A \subseteq B \subseteq W$, then $\Gamma_{\varphi(X)}(A) \subseteq \Gamma_{\varphi(X)}(B)$. By the Knaster–Tarski Theorem, the least and greatest fixed-points of $\Gamma_{\varphi(X)}$ are well-defined. We say a formula φ is *valid on a Kripke model M* iff φ holds on all worlds of M. We say a formula φ is *valid on a Kripke frame F* iff φ is valid on all Kripke models obtained by adding valuations to F. When convenient, we write $M, w \models \varphi$ for $w \in \|\varphi\|^M$. See [3] for more information on the μ-calculus.

Fusions Fix $n \in \mathbb{N}$. A (normal) modal logic is a set of formulas (without fixed-point operators) closed containing all the propositional tautologies and closed under *modus ponens*, necessitation, and substitution. Let $\{\mathsf{L}_j\}_{j\leq n}$ be a collection of modal logics with pairwise disjoint signatures. The fusion $\bigotimes_{j\leq n} \mathsf{L}_j$ is the smallest modal logic containing the logics $\{\mathsf{L}_j\}_{j\leq n}$. Let $\{\mathsf{F}_j\}_{j\leq n}$ be classes of frames with pairwise disjoint signatures $\{\Lambda_j\}_{j\leq n}$. Put $\Lambda = \bigcup_{j\leq n} \Lambda_j$. Define $\bigotimes_{j\leq n} \mathsf{F}_j$ as the class of frames $F = \langle W, \{R_i\}_{i\in\Lambda}\rangle$ such that $\langle W, \{R_i\}_{i\in\Lambda_j}\rangle$ is a frame of F_j for all $j \leq n$.

Suppose the modal logic L_j is characterized by the class of frames F_j, for all $j \leq n$. Then $\bigotimes_{j\leq n} \mathsf{L}_j$ is characterized by $\bigotimes_{j\leq n} \mathsf{F}_j$. Furthermore, if all the L_j have the finite model property, then $\bigotimes_{j\leq n} \mathsf{L}_j$ also has the finite model property. Similarly, if all the L_j are decidable, so is $\bigotimes_{j\leq n} \mathsf{L}_j$. On the other hand, fusions do not preserve the complexity of the logics: almost all interesting fusions are PSPACE-hard. See [8, 4] for more on fusions of modal logics and other combinations of modal logics.

Alternation Hierarchy The μ-calculus' alternation hierarchy classifies the μ-formulas according to the co-dependence of its least and greatest fixed-point operators. We define it as follows:

- $\Sigma^\mu_0 (= \Pi^\mu_0)$ is the set of all μ-formulas with no fixed-point operators.
- Σ^μ_{n+1} is the closure of $\Sigma^\mu_n \cup \Pi^\mu_n$ under propositional operators, modal operators, μX, and the substitution: if $\varphi(X) \in \Sigma^\mu_{n+1}$ and $\psi \in \Sigma^\mu_{n+1}$ are such that no free variable of ψ becomes bound in $\varphi(\psi)$, then $\varphi(\psi) \in \Sigma^\mu_{n+1}$.
- Π^μ_{n+1} is the closure of $\Sigma^\mu_n \cup \Pi^\mu_n$ under propositional symbols, modal operators, νX, and the analogous substitution: if $\varphi(X) \in \Pi^\mu_{n+1}$ and $\psi \in \Pi^\mu_{n+1}$ are such that no free variable of ψ becomes bound in $\varphi(\psi)$, then $\varphi(\psi) \in \Pi^\mu_{n+1}$.

Let F be a class of Kripke frames. The μ-calculus' alternation hierarchy is strict over F iff, for all n, there is a formula in $\Sigma^\mu_{n+1} \cup \Pi^\mu_{n+1}$ which is not equivalent to any formula in $\Sigma^\mu_n \cup \Pi^\mu_n$ over F. The μ-calculus collapses to modal logic over F iff every μ-formula is equivalent to a formula without fixed-point operators over F.

Game Semantics The μ-calculus also has an equivalent game semantics. Fix a μ-formula φ, a Kripke model $M = \langle W, \{R_i\}_{i\in\Lambda}, V\rangle$, and a world w. For notational simplicity, we suppose each variable occurring in φ has only one occurrence and is bound by some fixed-point operator.[1] The evaluation game $\mathscr{G}(M, w \models \varphi)$ is a game for two players: Verifier and Refuter, denoted by V and R respectively. The positions of the game are of the form $\langle \psi, v\rangle$ with $\psi \in \mathrm{Sub}(\varphi)$ and $v \in W$. The initial position is $\langle \varphi, w\rangle$. Each position $\langle \psi, v\rangle$ is owned by a player, who makes the next move. Table 1 summarizes the ownership of $\langle \psi, v\rangle$ and admissible moves on it; both are determined by the construction of ψ. On Table 1, ψ_X denotes the unique subformula of φ such that X occurs freely in ψ_X and $\eta X.\psi_X \in \mathrm{Sub}(\varphi)$.

Let ρ be a run of an evaluation game $\mathscr{G}(M, w \models \varphi)$. If ρ is finite, V wins ρ iff R cannot make a move and R wins ρ iff V cannot make a move. If ρ is infinite, let $\eta X.\psi \in \mathrm{Sub}(\varphi)$ be a formula such

[1] This statement is not problematic as we are interested in metamathematical properties of the μ-calculus. More care is needed when one is interested in the complexity of algorithms related to the μ-calculus. See [7].

that: positions of the form $\langle \eta X.\psi, v\rangle$ appear infinitely many often in ρ; and, for all formula θ such that positions $\langle \theta, v\rangle$ appear infinitely often in ρ, $\theta \in \mathrm{Sub}(\eta X.\psi)$. Then V wins ρ iff η is ν and R wins ρ iff η is μ. A strategy is a function indicating how a player should move. A winning strategy for V is a strategy σ for V such that V win all runs where they follow σ. We define winning strategies for R similarly.

Relational semantics and game semantics are equivalent:

Proposition 1. *Let $M = \langle W, \{R_i\}_{i\in\Lambda}, V\rangle$ be a Kripke model, $w \in W$ be a world, and φ be a μ-formula. Then $M, w \models \varphi$ iff V has a winning strategy in the evaluation game $\mathscr{G}(M, w \models \varphi)$; and $M, w \not\models \varphi$ iff R has a winning strategy in the evaluation game $\mathscr{G}(M, w \models \varphi)$.*

Proof. See [3] or [11] □

Table 1: The rules of evaluation game for modal μ-calculus.

Verifier: Position	Verifier: Admissible moves	Refuter: Position	Refuter: Admissible moves
$\langle \psi_1 \vee \psi_2, w\rangle$	$\{\langle \psi_1, w\rangle, \langle \psi_2, w\rangle\}$	$\langle \psi_1 \wedge \psi_2, w\rangle$	$\{\langle \psi_1, w\rangle, \langle \psi_2, w\rangle\}$
$\langle \Diamond_i \psi, w\rangle$	$\{\langle \psi, v\rangle \mid \langle w, v\rangle \in R_i\}$	$\langle \Box_i \psi, w\rangle$	$\{\langle \psi, v\rangle \mid \langle w, v\rangle \in R_i\}$
$\langle P, w\rangle$ and $w \notin V(P)$	$\emptyset$	$\langle P, w\rangle$ and $w \in V(P)$	$\emptyset$
$\langle \neg P, w\rangle$ and $w \in V(P)$	$\emptyset$	$\langle \neg P, w\rangle$ and $w \notin V(P)$	$\emptyset$
$\langle \mu X.\psi_X, w\rangle$	$\{\langle \psi_X, w\rangle\}$	$\langle \nu X.\psi_X, w\rangle$	$\{\langle \psi_X, w\rangle\}$
$\langle X, w\rangle$	$\{\langle \mu X.\psi_X, w\rangle\}$	$\langle X, w\rangle$	$\{\langle \nu X.\psi_X, w\rangle\}$

Parity games A parity game is a tuple $\mathscr{P} = \langle V_\exists, V_\forall, v_0, E, \Omega\rangle$ where two players $\exists$ and $\forall$ move a token in the graph $\langle V_\exists \cup V_\forall, E\rangle$. We suppose $V_\exists$ and $V_\forall$ are disjoint sets of *vertices*; $E \subseteq (V_\exists \cup V_\forall)^2$ is a set of *edges*; and $\Omega : V_\exists \cup V_\forall \to n$ is a *parity function*. If a player has no available move, then the other player wins. In an infinite play ρ, the winner is determined by the following parity condition: $\exists$ wins ρ iff the greatest parity which appears infinitely often in ρ is even; otherwise, $\forall$ wins ρ. $\exists$ wins the parity game $\mathscr{P}$ iff $\exists$ has a winning strategy; a winning strategy for $\exists$ is a function σ from $V_\exists$ to $V_\exists \cup V_\forall$, where, if $\exists$ follows σ, all resulting plays are winning for them. Similarly, $\forall$ wins $\mathscr{P}$ iff $\forall$ has a winning strategy.

Fix a parity game $\mathscr{P} = \langle V_\exists, V_\forall, v_0, E, \Omega\rangle$. The set of winning positions for $\exists$ in $\mathscr{P}$ is the set of positions v where $\exists$ wins the parity game if the players start at v. That is, $v \in V_\exists \cup V_\forall$ is a winning position for $\exists$ iff $\exists$ wins $\mathscr{P}_v = \langle V_\exists, V_\forall, v, E, \Omega\rangle$.

Sometimes it is convenient to suppose that all parity games are *tree-like*. That is, for all $v \in V_\exists \cup V_\forall$, there is no path $v = v_0 E \cdots E v_n = v$, for all $n \in \mathbb{N}$. Any parity game $\mathscr{P} = \langle V_\exists, V_\forall, v_0, E, \Omega\rangle$ can be unfolded into a tree-like parity game. In the unfolded game, instead of moving to a node v, the players move to a fresh copy of v. The unfolded parity game is bisimilar to the original game.

3 Evaluation games as parity games

Fix a model $M = \langle W, \{R_i\}_{i\in\Lambda}, V\rangle$, a world $w \in W$ and a μ-formula φ. We define a parity game $\mathscr{G}^{\mathrm{P}} = \mathscr{G}^{\mathrm{P}}(M, w \models \varphi) = \langle V_\exists, V_\forall, v_0, E, \Omega\rangle$ which is equivalent to $\mathscr{G} = \mathscr{G}(M, w \models \varphi)$.

The set of positions $V_\exists$ consists of the positions owned by V in $\mathscr{G}$. Similarly $V_\forall$ consists of the positions owned by $\forall$ in $\mathscr{G}$. The set of edges E consists of the transitions in $\mathscr{G}$. The initial position v_0 is $\langle\varphi, w\rangle$. Define the parity function:

- $\Omega(\langle\mu X.\psi, v\rangle) = 2(i+\varepsilon)-1$ if $\mu X.\psi \in \Sigma^\mu_{2i+\varepsilon}\setminus\Pi^\mu_{2i+\varepsilon}$;
- $\Omega(\langle\nu X.\psi, v\rangle) = 2i$ if $\nu X.\psi \in \Pi^\mu_{2i+\varepsilon}\setminus\Sigma^\mu_{2i+\varepsilon}$;
- $\Omega(\langle\psi, v\rangle) = 0$ for ψ not of the form $\eta X.\psi$;

where $\varepsilon \in \{0,1\}$.

Proposition 2. *Let $M = (W, \{R_i\}_{i\in\Lambda}, V)$ be a Kripke model, $w \in W$, and φ a μ-formula. Then:*

$$\mathsf{V} \text{ wins } \mathscr{G}(M,w\models\varphi) \iff \exists \text{ wins } \mathscr{G}^{\mathrm{P}}(M,w\models\varphi).$$

Proof. Denote $\mathscr{G}(M,w\models\varphi)$ by $\mathscr{G}$ and $\mathscr{G}^{\mathrm{P}}(M,w\models\varphi)$ by $\mathscr{G}^{\mathrm{P}}$. As both games are on the same board, strategies for V and R in $\mathscr{G}$ are strategies for $\exists$ and $\forall$ in $\mathscr{G}^{\mathrm{P}}$. As any position is owned by V in $\mathscr{G}$ iff it is owned by $\exists$ in $\mathscr{G}$, any finite run is winning for V iff it is winning for $\exists$.

Consider an infinite run ρ. The parity $\Omega(\langle\psi, v\rangle)$ is odd iff $\psi \in \Sigma^k\setminus\Pi^k$ for some $k \in \mathbb{N}$. If the greatest infinitely often occurring parity in ρ is odd, then some $\mu X.\psi$ is the outermost infinitely often occurring fixed-point formula. Otherwise, if $\mu X.\psi \in \mathrm{Sub}(\nu \mathrm{Y}.\theta)$ and $\nu Y.\theta$ is the outermost infinitely occurring fixed-formula formula, then $\Omega(\langle\nu Y.\theta, v\rangle) \geq \Omega(\langle\mu X.\psi, v\rangle)$ and $\Omega(\langle\nu Y.\theta, v\rangle)$ is even. Similarly, if the greatest infinitely often occurring parity in ρ is even, then some $\nu X.\psi$ is the outermost infinitely often occurring fixed-point formula. Either way, ρ is winning for V in $\mathscr{G}$ iff ρ is winning for $\exists$ in $\mathscr{G}^{\mathrm{P}}$. □

4 Winning region formulas

Let F_0 and F_1 be classes of frames with signatures $\{0\}$ and $\{1\}$, respectively. Suppose that $\circ \leftarrow \circ \rightarrow \circ$ is a subframe of F_0 and that $\circ \rightarrow \circ$ is a subframe of F_1. Fix $F_0 \in \mathsf{F}_0$ and $F_1 \in \mathsf{F}_1$ witnessing these facts. Given a parity game $\mathscr{P}$ we will define an associated Kripke model $\mathscr{P}^{\mathrm{K}}$ with frame in $\mathsf{F}_0 \otimes \mathsf{F}_1$. We will also define winning region μ-formulas W_n, for all $n \in \mathbb{N}$. If $\mathscr{P}$ is a parity game which uses parities up to n, then $\exists$ wins $\mathscr{P}$ starting at v iff $\mathscr{P}^{\mathrm{K}}, v \models W_n$.

Let $\mathscr{P} = \langle V_\exists, V_\forall, v, E, \Omega\rangle$ be a parity game. We represent $\mathscr{P}$ as a birelational Kripke model $\mathscr{P}^K = \langle W, R_0, R_1, V\rangle$. The set W of possible worlds will consist of a world $\underline{v}$ for each state $v \in V_\exists, V_\forall$ and a countable supply of other worlds. If $v \in V_\exists, V_\forall$ and $vE = \{v_0, \dots v_n\}$, then we will represent the connection between v and the v_i using fresh isomorphic copies of F_0 and F_1. We first use a copy of F_0 to choose between v_0 and the other vertices, then we use copies of F_1 to confirm the choices. Similarly, we use a copy of F_0 to choose between v_1 and the other vertices, and copies of F_1 to confirm the choices. We repeat this procedure until we use up all the v_i. By using fresh copies of F_0 and F_1, we guarantee that the resulting frame is in $\mathsf{F}_0 \otimes \mathsf{F}_1$. We denote by $\underline{v}, \underline{v_0}, \cdots$ the worlds of $\mathscr{P}^{\mathrm{K}}$ corresponding to the positions $v, v_0, \cdots$; we do not name the other worlds connecting them. An example of this construction is depicted in Figure 1.

To control the flow of the evaluation game, we will use fresh propositional symbols bd, pos, pre_0, pre_1, nxt_0, and nxt_1. The proposition symbol bd indicate that a world is used to represent the parity game. That is, only the isomorphic copies of $\circ \leftarrow \circ \rightarrow \circ$ and $\circ \rightarrow \circ$ used in the paragraph above satisfy bd. The proposition symbol pos indicates that a world corresponds to a position in the parity game. That is, it holds only on worlds which are $\underline{v}$ for some $v \in V_\exists \cup V_\forall$. The proposition symbols pre_0, pre_1, nxt_0, and nxt_1 represent the intended flow of the parity game in our model. pre_0 holds when we are making

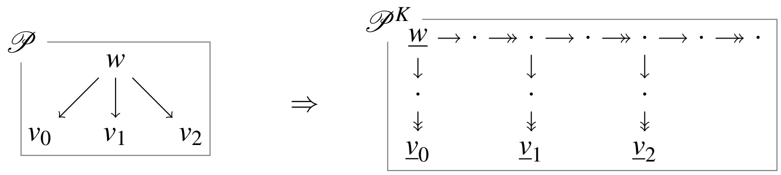

Figure 1: Example of a parity game $\mathscr{P}$ and the corresponding bimodal model $\mathscr{P}^K$, built using copies of $\circ \leftarrow \circ \rightarrow \circ$ and $\circ \rightarrow \circ$. We show only worlds satisfying bd and omit the other propositional symbols.

a choice and nxt_0 holds after we make a choice. Similarly, pre_1 holds when we are confirming a choice and nxt_1 holds after we confirmed a choice. Overall, these propositional symbols allow us to stay in the part of the model which represents the parity game, and that we respect the flow of the game.

The proposition symbols $P_\exists$ and $P_\forall$ indicate the ownership of the positions: $P_\exists$ holds at $\underline{v}$ iff $v \in V_\exists$ and $P_\forall$ holds at $\underline{v}$ iff $v \in V_\forall$. The proposition symbols $P_0, \dots, P_n$ indicate the parities of the positions: P_i holds at $\underline{v}$ iff $\Omega(v) = i$. At each $\underline{v}$, exactly one of the P_i will hold. The proposition symbols $P_\exists$, $P_\forall$, $P_0, \dots, P_n$ are false at worlds which are not of the form $\underline{v}$ for some $v \in V_\exists \cup V_\forall$. This finishes the definition of $\mathscr{P}^{\mathrm{K}}$.

To define the winning region formulas W_n, we use the following shorthand formulas:

- $\blacklozenge\varphi := \mu Y.\mathrm{pre}_0 \wedge \mathrm{bd} \wedge \Diamond_0(\mathrm{nxt}_0 \wedge \mathrm{pre}_1 \wedge \mathrm{bd} \wedge \Diamond_1(\mathrm{nxt}_1 \wedge \mathrm{bd} \wedge ((Y \wedge \neg\mathrm{pos}) \vee (\varphi \wedge \mathrm{pos}))))$; and
- $\blacksquare\varphi := \nu Y.\mathrm{pre}_0 \wedge \mathrm{bd} \to \Box_0(\mathrm{nxt}_0 \wedge \mathrm{pre}_1 \wedge \mathrm{bd} \to \Box_1(\mathrm{nxt}_1 \wedge \mathrm{bd} \to ((Y \wedge \neg\mathrm{pos}) \wedge (\varphi \wedge \mathrm{pos}))))$,

where Y is a fresh variable symbol. We use these modalities to represent a move in $\mathscr{P}$ as multiple moves in evaluation game $\mathscr{P}^{\mathrm{K}}, \underline{v} \models W_n$. Given $n \in \mathbb{N}$, define:

$$W_n := \eta X_n \dots \nu X_2 \mu X_1 \nu X_0. \bigvee_{0 \le j \le n} [(P_j \wedge P_\exists \wedge \blacklozenge X_j) \vee (P_j \wedge P_\forall \wedge \blacksquare X_j)].$$

The formula W_n defines the winning positions of $\exists$ in parity games using parities up to n:

Proposition 3. *Let $\mathscr{P} = \langle V_\exists, V_\forall, v_0, E, \Omega\rangle$ be a parity game. If $\max\{\Omega(v) \mid v \in W\} \le n$, then*

$$\mathscr{P}^K, \underline{v}_0 \models W_n \textit{ iff } \exists \textit{ wins } \mathscr{P}.$$

Proof. Suppose $\mathscr{P}^{\mathrm{K}}, \underline{v}_0 \models W_n$. Let σ be a winning strategy for V in the evaluation game $\mathscr{G} := \mathscr{G}(\mathscr{P}^{\mathrm{K}}, \underline{v}_0 \models W_n)$. We define a winning strategy σ' for $\exists$ in $\mathscr{P}$ while playing simultaneous runs of $\mathscr{G}$ and $\mathscr{P}$.

The games $\mathscr{G}$ and $\mathscr{P}$ start at positions $\langle W_n, \underline{v}_0\rangle$ and v_0, respectively. First, have the players move to the position

$$\left\langle \bigvee_{0 \le j \le n} [(P_j \wedge P_\exists \wedge \blacklozenge X_j) \vee (P_j \wedge P_\forall \wedge \blacksquare X_j), \underline{v}_0 \right\rangle$$

in $\mathscr{G}$.

Now, suppose the players are at positions

$$\left\langle \bigvee_{0 \le j \le n} [(P_j \wedge P_\exists \wedge \blacklozenge X_j) \vee (P_j \wedge P_\forall \wedge \blacksquare X_j), \underline{v} \right\rangle$$

in $\mathscr{G}$ and v in $\mathscr{P}$, respectively. As σ is winning for V in $\mathscr{G}$, σ does not make any immediately losing move. That is, V picks the disjuncts according to v's parity and owner. We also have $\forall$ make non-immediately losing moves. The players eventually reach one of two possible cases:

Case 1. The players are in the position $\langle\blacklozenge X_j,\underline{v}\rangle$ in $\mathscr{G}$, with $v\in V_\exists$. By our choice of σ, V plays respects the flow of the parity game in in $\mathscr{G}$. As $\blacklozenge X_j$ is a Σ_1^μ-formula, the players are eventually in a position of the form $\langle X_j,\underline{v'}\rangle$. Then $\exists$ moves to v' in $\mathscr{P}$.

Case 2. The players are in the position $\langle\blacksquare X_j,\underline{v}\rangle$ in $\mathscr{G}$ and $v\in V_\forall$ in $\mathscr{P}$. If $\forall$ moves to v', have R move to $\langle X_j,\underline{v'}\rangle$ in $\mathscr{G}$ while following the flow of the game.

Now, have the players regenerate X_j in $\mathscr{G}$ and move until they get to positions of the form

$$\left\langle \bigvee_{0\leq j\leq n}[(P_j\wedge P_\exists\wedge\blacklozenge X_j)\vee(P_j\wedge P_\forall\wedge\blacksquare X_j),\underline{v'}\right\rangle \text{ and } v'$$

in $\mathscr{G}$ and $\mathscr{P}$, respectively. We are back to the initial situation, and we repeat this process to define σ'.

We consider parallel runs ρ in $\mathscr{G}$ and ρ' in $\mathscr{P}$ played according to σ and σ', respectively. Then either both runs are finite or both runs are infinite. If ρ' is finite, this means that one of the players didn't have a move available to play at a position v in $\mathscr{P}$. Therefore, one of the players couldn't find a valid position to play after $\langle\blacklozenge X_{\Omega(v)},\underline{v}\rangle$ or $\langle\blacksquare X_{\Omega(v)},\underline{v}\rangle$. The former is not possible by our choice of σ, so it must be $\forall$ who could not make a move. Therefore $\exists$ wins ρ'. If ρ is infinite, then the outermost infinitely often regenerated fixed-point operator is some νX_{2k}. By the construction of σ' the greatest infinitely often occurring parity must be $2k$. Therefore $\exists$ wins ρ'. We can now conclude that σ' is a winning strategy for $\exists$ in $\mathscr{P}$.

On the other hand, suppose $\exists$ wins $\mathscr{P}$ via σ'. We define σ for V in $\mathscr{G}$. At vertices of the form $\langle\blacklozenge X_j,\underline{v}\rangle$ in $\mathscr{G}$, have V move to

$$\sigma(\langle\blacklozenge X_j,\underline{v}\rangle):=\langle X_j,\underline{v'}\rangle,$$

with $v'=\sigma'(v)$. On other positions, have σ be the non-immediately losing moves for V.

Consider parallel runs ρ in $\mathscr{G}$ and ρ' in $\mathscr{P}$ played according to σ and σ', respectively. If ρ is finite, then one of the players made a move not respecting the flow of the parity game, or did not have an adequate moves after a position of the form $\langle\blacklozenge X_j,\underline{v}\rangle$ or $\langle\blacksquare X_j,\underline{v}\rangle$. By the choice of σ' and definition of σ, V makes no such move. So it must be R's mistake, and so V wins. If ρ is infinite, the greatest parity appearing infinitely often in ρ' is even. Therefore the outermost infinitely often occurring fixed-point operator in ρ is a ν-operator. ρ is winning for V. Therefore σ is a winning strategy for V in $\mathscr{G}$. □

Given an evaluation game $\mathscr{G}(M,w\models\varphi)$, we define the Kripke model $\mathscr{G}^{\mathrm{K}}(M,w\models\varphi)$ as $(\mathscr{G}^{\mathrm{P}}(M,w\models\varphi))^{\mathrm{K}}$. As evaluation games are also parity games, the W_n also define winning regions for V in evaluation games:

Proposition 4. *Let $M=(W,R_0,R_1,V)$ be a bimodal Kripke model, $w\in W$, and φ a bimodal μ-formula. If $n\geq 1$ and the greatest parity used in $\mathscr{G}^{\mathrm{P}}(M,w\models\varphi)$ is less or equal than n, then:*

$$M,w\models\varphi \textit{ iff } \mathscr{G}^{K}(M,w\models\varphi),\langle\varphi,w\rangle\models W_n.$$

Proof. We have:

$$\begin{aligned} M,w\models\varphi &\text{ iff V wins } \mathscr{G}(M,w\models\varphi)\\ &\text{ iff } \exists \text{ wins } \mathscr{G}^{\mathrm{P}}(M,w\models\varphi)\\ &\text{ iff } \mathscr{G}^{\mathrm{K}}(M,w\models\varphi),\langle\varphi,w\rangle\models W_n. \end{aligned}$$

The first equivalence follows from Proposition 1, the second one follows from Proposition 2, the third one follows from Proposition 3. □

5 Strictness

Fix classes of frames F_0 and F_1 with signatures $\{0\}$ and $\{1\}$, respectively. We show that, if $\circ \leftarrow \circ \rightarrow \circ$ is a subframe of F_0 and that $\circ \rightarrow \circ$ is a subframe of F_1, then the μ-calculus' alternation hierarchy is strict over $\mathsf{F}_0 \otimes \mathsf{F}_1$.

Let $(M,w) = \langle W, R_0, R_1, V, w\rangle$ and $(M',w') = \langle W', R_0', R_1', V', w'\rangle$ be pointed Kripke models without loops in their graphs. (M,w) is *isomorphic* to (M',w') iff there is a bijection $I : W \rightarrow W'$ such that:

- $I(w) = w'$;
- for all $v, v' \in W$, vR_0v' iff $I(v)R_0'I(v')$;
- for all $v, v' \in W$, vR_1v' iff $I(v)R_1'I(v')$; and
- for all $v \in W$, $v \in V(P)$ iff $I(v) \in V'(P)$.

For all $n \in \mathbb{N}$, let $(M \upharpoonright n, w)$ be the submodel of (M,w) obtained by restricting W to worlds with distance less than n from w. We say (M,w) is *n-isomorphic* to (M',w') if and only if $(M \upharpoonright n, w)$ is isomorphic to $(M' \upharpoonright n, w')$. For any (M,w), $(M \upharpoonright 0, w)$ is an empty Kripke model. We assume the empty Kripke model is isomorphic to itself.

Given a μ-formula φ, let f_φ be the function mapping a pointed model to the pointed Kripke model representing its evaluation game with respect to φ. That is $f_\varphi(M,w) = (\mathscr{G}^{\mathsf{K}}(M, w \models \varphi), \langle \varphi, w\rangle)$, for all pointed models (M,w).

Lemma 5. *Fix a μ-formula φ. If (M,w) and (M',w') are n-isomorphic via a function I, then $f_{\varphi\wedge\varphi}(M,w)$ and $f_{\varphi\wedge\varphi}(M',w')$ are $(n+1)$-isomorphic via the function J defined by:*

$$J(\langle \psi, w\rangle) = (\langle \psi, I(w)\rangle),$$

for all world w of M and subformula ψ of φ.

Proof. As (M,w) and (N,v) are n-isomorphic, the evaluation games $\mathscr{G}(M, w \models \varphi \wedge \varphi)$ and $\mathscr{G}(N, v \models \varphi \wedge \varphi)$ are going to be same up to n-many plays of the form $\langle \triangle\psi, w\rangle$, with $\triangle \in \{\Box_0, \Diamond_0, \Box_1, \Diamond_1\}$. As the first move in an evaluation game for the formula $\varphi \wedge \varphi$ is to choose between a conjunction, we can guarantee that the two games above are the same up to $n+1$ moves. $\Box$

Lemma 6. *For all μ-formula φ, the function $f_{\varphi\wedge\varphi}$ has a fixed-point (up to isomorphism). That is, there is a model (M,w) such that $f_\varphi(M,w)$ is isomorphic to (M,w).*

Proof. Let (M_0, w_0) be a fixed arbitrary pointed Kripke model. We define $(M_{n+1}, w_{n+1}) = f_{\varphi\wedge\varphi}(M_n, w_n)$ inductively on $n \in \mathbb{N}$. If $n = 0$, then (M_0, w_0) and (M_1, w_1) are trivially 0-isomorphic. By induction on n, (M_n, w_n) and (M_{n+1}, w_{n+1}) are n-isomorphic via Lemma 5. Therefore, if $m > n$ then (M_n, w_n) is n-isomorphic to (M_m, w_m).

We can now define a pointed Kripke model (M,w) which is n-isomorphic to (M_n, w_n) for all n. We identify $(M_n \upharpoonright n, w_n)$ and $(M_{n+1} \upharpoonright n, w_{n+1})$, since they are n-isomorphic. Take the graph of M to be the union of the graph of the $M_n \upharpoonright n$, the valuation of M to be the union of the valuation of the $M_n \upharpoonright n$ and w as w_0. Finally, we have that (M,w) is isomorphic to $f_{\varphi\wedge\varphi}(M,w)$. Otherwise, there world be n such that (M_n, w_n) is not n-isomorphic to (M_{n+1}, w_{n+1}). $\Box$

Proof of Item 1 of the Main Theorem. Let F_0 and F_1 be classes of unimodal Kripke frames closed under isomorphic copies and disjoint unions. Suppose $\circ \leftarrow \circ \rightarrow \circ$ is a subframe of F_0 and $\circ \rightarrow \circ$ a subframe of F_1.

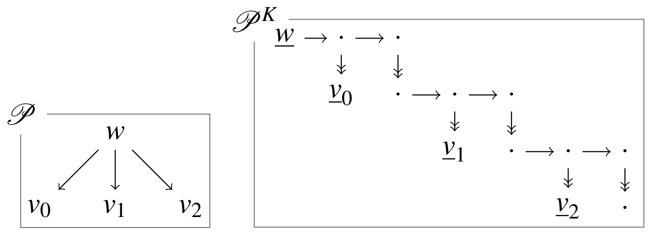

Figure 2: Example of a parity game $\mathscr{P}$ and the corresponding bimodal model $\mathscr{P}^K$, built using copies of $\circ \to \circ \to \circ$ and $\circ \to \circ$. We show only worlds satisfying bd and omit the other propositional symbols.

If n is even, then $W_n \in \Pi^\mu_{n+1}$. For a contradiction, suppose that W_n is equivalent to some formula in Π^μ_n over $\mathsf{F}_0 \otimes \mathsf{F}_1$. Let $\varphi \in \Sigma^\mu_n$ be equivalent to $\neg W_n$. Let (M,w) be a fixed-point of $f_{\varphi\wedge\varphi}$. Then

$$\begin{aligned} M,w \models \neg W_n &\iff M,w \models \varphi \wedge \varphi \\ &\iff f_{\varphi\wedge\varphi}(M,w) \models W_n \iff M,w \models W_n. \end{aligned}$$

This is a contradiction. The case for n odd is symmetric: $W_n \in \Sigma^\mu_{n+1}$ and is not equivalent to any formula in Σ^μ_n. □

6 Finishing the proof of the Main Theorem

To prove Items 2 and 3 of the Main Theorem, we modify two points in the proof above: first, we define new functions transforming parity games into Kripke models; second, we supply new versions of the modalities ⧫ and ■.

We first consider the case of Item 2. Let F_0 and F_1 be classes of unimodal Kripke frames closed under isomorphic copies and disjoint unions. Suppose $\circ \to \circ \to \circ$ is a subframe of F_0 and $\circ \to \circ$ is a subframe of F_1. When we define a Kripke model $\mathscr{P}^{\mathrm{K}}$ from a parity game $\mathscr{P}$, we change the way we use the copies of $\circ \to \circ \to \circ$ and $\circ \to \circ$. Suppose $v \in V_\exists, V_\forall$ and $vE = \{v_0, \dots v_n\}$. The players choose the next position as follows: they first move once in a copy of $\circ \to \circ \to \circ$; they then confirm some v_i using a copy of $\circ \to \circ$ or move along the current copy $\circ \to \circ \to \circ$; if they moved along $\circ \to \circ \to \circ$, they must confirm this move via a copy of $\circ \to \circ$. See Figure 2 for an example.

To control the flow of the game over copies of $\circ \to \circ \to \circ$, we use three propositional symbols pre_0, mid_0, and nxt_0. Here, pre_0 holds at the first world of the copies of $\circ \to \circ \to \circ$, mid_0 holds at the second world, and nxt_0 holds at the third world. We define ⧫ and ■ as follows:

- $\blacklozenge\varphi := \mu Y.\mathrm{pre}_0 \wedge \mathrm{bd} \wedge \Diamond_0[\mathrm{mid}_0 \wedge \mathrm{pre}_1 \wedge \wedge \mathrm{bd} \wedge \Diamond_1(\mathrm{nxt}_1 \wedge \mathrm{bd} \wedge ((Y \wedge \neg\mathrm{pos}) \vee (\varphi \wedge \mathrm{pos}))) \vee \Diamond_0(\mathrm{nxt}_0 \wedge \mathrm{pre}_1 \wedge \mathrm{bd} \wedge \Diamond_1(\mathrm{nxt}_1 \wedge \mathrm{bd} \wedge ((Y \wedge \neg\mathrm{pos}) \vee (\varphi \wedge \mathrm{pos}))))]$; and
- $\blacksquare\varphi := \mu Y.\mathrm{pre}_0 \wedge \mathrm{bd} \to \Box_0[\mathrm{mid}_0 \wedge \mathrm{pre}_1 \wedge \wedge \mathrm{bd} \to \Box_1(\mathrm{nxt}_1 \wedge \mathrm{bd} \wedge ((Y \wedge \neg\mathrm{pos}) \vee (\varphi \wedge \mathrm{pos}))) \wedge \Box_0(\mathrm{nxt}_0 \wedge \mathrm{pre}_1 \wedge \mathrm{bd} \to \Box_1(\mathrm{nxt}_1 \wedge \mathrm{bd} \wedge ((Y \wedge \neg\mathrm{pos}) \vee (\varphi \wedge \mathrm{pos}))))]$,

where Y is a fresh variable symbol. The definition of the winning region formulas W_n are the same as above, where ⧫ and ■ use their new definitions.

Now for the proof of Item 3. Let F_0, F_1, and F_2 be classes of unimodal Kripke frames closed under isomorphic copies and disjoint unions. Suppose $\circ \to \circ$ is a subframe of F_0, F_1, and F_2. Given $v \in V_\exists, V_\forall$ and $vE = \{v_0, \dots v_n\}$, we build a Kripke model as in the proof of Item 2, but instead of using a copy of $\circ \to \circ \to \circ$, we use two copies of $\circ \to \circ$, one from F_0 and one from F_1; we use copies of $\circ \to \circ$ from

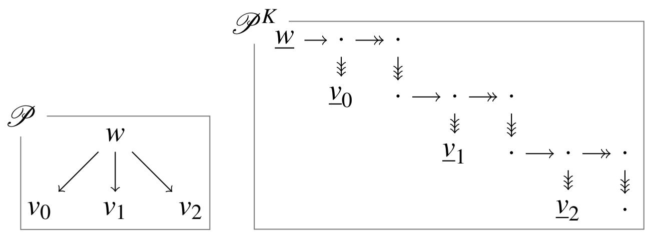

Figure 3: Example of a parity game $\mathscr{P}$ and the corresponding tri-modal model $\mathscr{P}^K$, built using copies of $\circ \to \circ$. We show only worlds satisfying bd and omit the other propositional symbols.

F_2 to confirm the choices. See Figure 3 for an example. This time we will also use fresh proposition variables pre_2 and nxt_2 to control the flow of the evaluation games along copies of $\circ \to \circ$ in F_2. Here, we define $\blacklozenge$ and $\blacksquare$ as follows:

- $\blacklozenge\varphi := \mu Y.\mathrm{pre}_0 \wedge \mathrm{bd} \wedge \Diamond_0[\mathrm{nxt}_0 \wedge \mathrm{pre}_1 \wedge \mathrm{pre}_2 \wedge \mathrm{bd} \wedge \Diamond_2(\mathrm{nxt}_2 \wedge \mathrm{bd} \wedge ((Y \wedge \neg\mathrm{pos}) \vee (\varphi \wedge \mathrm{pos}))) \vee \Diamond_1(\mathrm{nxt}_1 \wedge \mathrm{pre}_2 \wedge \mathrm{bd} \wedge \Diamond_2(\mathrm{nxt}_2 \wedge \mathrm{bd} \wedge ((Y \wedge \neg\mathrm{pos}) \vee (\varphi \wedge \mathrm{pos}))))]$; and
- $\blacksquare\varphi := \mu Y.\mathrm{pre}_0 \wedge \mathrm{bd} \to \Box_0[\mathrm{nxt}_0 \wedge \mathrm{pre}_1 \wedge \mathrm{pre}_2\mathrm{bd} \to \Box_2(\mathrm{nxt}_2 \wedge \mathrm{bd} \wedge ((Y \wedge \neg\mathrm{pos}) \vee (\varphi \wedge \mathrm{pos}))) \wedge \Box_1(\mathrm{nxt}_1 \wedge \mathrm{pre}_2 \wedge \mathrm{bd} \to \Box_2(\mathrm{nxt}_2 \wedge \mathrm{bd} \wedge ((Y \wedge \neg\mathrm{pos}) \vee (\varphi \wedge \mathrm{pos}))))]$,

where Y is a fresh variable symbol. The definition of the winning region formulas W_n are the same as above, where $\blacklozenge$ and $\blacksquare$ use their new definitions.

7 Case studies on the collapse over multimodal logics

We now comment on two where the μ-calculus collapses to modal logic. These are not originally framed in the context of multimodal μ-calculus.

Provability Logic GLP is a multimodal provability logic with signature $\mathbb{N}$, first defined by Japaridze. One of the possible arithmetical interpretations for each $\Box_n$ is as a provability predicate for $\mathsf{I}\Sigma_n$. Each modality $\Box_n$ satisfies the necessitation rule and the axioms for the provability GL: $\Box(P \to Q) \to (\Box P \to \Box Q)$ and $\Box(\Box P \to P) \to \Box P$. While GLP contains the fusion of infinitely many copies of GL, it is not a fusion logic: it also includes the axioms $\Box_m P \to \Box_n \Box_m P$, $\Diamond_m P \to \Box_n \Diamond_m P$, and $\Box_m P \to \Box_n P$, for all $m \leq n$.

Ignatiev [6] proved that GLP has the fixed-point property: if X is in the scope of some $\Box_i$ in $\varphi(X)$, then there is ψ such that $\mathsf{GLP} \vdash \psi \leftrightarrow \varphi(\psi)$. This implies that we do not get a more expressive logic if we add to it the operators μ and ν. While the additional conditions on the relation between the modalities makes it possible to have the fixed-point property, GLP is not complete over any class of Kripke models.

Intuitionistic Modal Logic IS5 is an intuitionistic variation of S5; it is also known as MIPQ. It consists of closure under necessitation and *modus ponens* of the set of formulas containing the intuitionistic tautologies along with the axioms $T := \Box\varphi \to \varphi \wedge \varphi \to \Diamond\varphi$, $4 := \Box\varphi \to \Box\Box\varphi \wedge \Diamond\Diamond\varphi \to \Diamond\varphi$, and $5 := \Diamond\varphi \to \Box\Diamond\varphi \wedge \Diamond\Box\varphi \to \Box\varphi$. An IS5 model is a tuple $\langle W, \preceq, R, V\rangle$ satisfying: $\preceq$ is a pre-order; R is an equivalence relation; $wR;\preceq v$ implies $w \preceq;Rv$; and $w \preceq v$ and $w \in V(P)$ implies $v \in V(P)$. IS5 can be thought as a bimodal logic, where $\Box$ and $\Diamond$ are abbreviations for $\Box_{\preceq}\Box_R$ and $\Box_{\preceq}\Diamond_R$, respectively. Ono [9] and Fischer Servi [5] proved that IS5 is complete over IS5 frames.

Pacheco [11] prove that the μ-calculus collapses to constructive modal logic over IS5 frames using game semantics for the constructive μ-calculus. This example shows that, if we add restrictions on how we use multiple modalities, then we may still have the collapse to modal logic. Note that the relation $\langle W, \preceq \rangle$ is an S4 frame, and the μ-calculus does not collapse to modal logic over S4 frames [1]. So the restriction on the usage of the modalities here is quite strong.

Problem. *When does the μ-calculus collapse to modal logic over multimodal frames?*

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