Cyclic Proofs for iGL via Corecursion

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1 Introduction

Cyclic proof theory studies a notion of proof where we allow the trees representing proofs to have loops (see [1]). This is really useful for logics with fixpoints operators: these cycles can be used to represent the unfolding of a fixpoint providing natural axiomatizations for these systems. However, this cyclic character is not unique to such explicit fixpoints. For example, modal logics whose frames have a Noetherian (conversely wellfounded) condition, such as GL ([6],[3]),$\mathsf{S4Grz}$ ([5]) and $\mathsf{K4Grz}$ ([8]); also have cyclic proof systems.

Particularly, in [6], Shamkanov introduces a non-wellfounded and a cyclic sequent system for provability logic (GL). He proves the equivalence of these two systems with an acyclic finite system via proof translations. In order to go from the finite system to the non-wellfounded system and from the non-wellfounded to the cyclic he uses corecursion.

In [3], Iemhoff generalized the work of Shamkanov studying when, for a given modal logic proof system, there exists another modal logic proof system such that proofs in the first are equivalent to cyclic proofs in the second. There, she shows that iGL, an[1] intuitionistic version of GL, also haves a natural cyclic proof system. Iemhoff does not use corecursion, which should not be surprising since when defining a function from finite trees to non-wellfounded trees we have recursive and corecursive tools at hand.

Our main interest is to explore corecursion as a tool for non-wellfounded and cyclic proof theory. For this reason, in [7], we have provided an alternative proof of this equivalence using a corecursive translation from the finite

[1]The use of "an" instead of "the" is deliberated. Check the footnote of page 5 for an explanation.

acyclic system to the non-wellfounded system. Let us briefly describe what we do in each section of this abstract:

- In the second section we introduce non-wellfounded trees, together with the methods of recursion and corecursion, using category theory.
- In the third section we define the sequent rules we are going to need. We also define non-wellfounded and cyclic proofs.
- In the fourth section we define some conditions that suffice to define a corecursive proof translation from finite proofs to non-wellfounded proofs.
- In the fifth section we finally show how to convert finite proofs with rules of iGL into cyclic proofs with rules of iK4. We also briefly explain how to show the reverse direction.

2 Algebras and Coalgebras: (co)recursion[2]

We need to work with (possibly) not wellfounded trees. We are going to use the representation of trees with a non-empty set of finite sequences of natural numbers. Given a set A we will write A^* to denote the set of finite sequences over A, i.e. $\bigcup_{n\in\mathbb{N}}\{f \mid f : \{0,\dots,n-1\} \longrightarrow A\}$. Elements of ω^* will be denoted by w, v, u.

Definition 1. An *A-labelled tree* is a pair $T = (N, \ell)$, where:

1. $N \subseteq \omega^*$, non-empty and closed under initial segments.
2. For any $w \in N$ there exists a natural number m such that for any i, $w, i \in N$ iff $i < m$. Given w, this number can be shown to be unique and we call it the *arity of w in T*.
3. $\ell : N \longrightarrow A$, called the labelling function of T.

T is said to be *finite* iff N is finite. A branch of T is just an infinite path of T. We denote the collection of L-labelled trees as Tree_A and the collection of finite L-labelled trees as $\mathrm{FinTree}_A$.

[2]We are just going to introduce the necessary principles for our work, particularly infinite trees and corecursion to them. These ideas can already found in [1], so the methods we use can be considered to be standard for the treatment of infinite trees. For a more modern view on coalgebras the reader can consult for example [9].

Given a finite or infinite sequence w and a natural number i, we will write $w{\restriction}i$ to mean the restriction of w to $\{0, \ldots, i-1\}$. If X is a set of finite sequences, we say that w is maximal in X iff there is no sequence in X strictly extending w. Let us define some usual notions of trees in this formalism.

Definition 2. A *branch* of T is just an infinite sequence $b \in \omega^\omega$ such that for any $i \in \omega$, $b{\restriction}i$ is a node of T.

A *leaf* of T is a maximal sequence in the nodes of T. An *internal node* is any node that is not a leaf.

Let $T = (N, \ell)$ be a tree and w one of its nodes. We define the *T-subtree generated at w*, as $T\text{-}\mathsf{subtree}(w) = (N', \ell')$ where:

$$\begin{aligned} N' &= \{v \in \omega^* \mid wv \in N\}, \\ \ell'(v) &= \ell(wv). \end{aligned}$$

We need to explain how to do corecursion over trees. In order to do so, we are going to use category theory and define recursion at the same time. First we need to define algebras, coalgebras and the morphisms between them.

Definition 3 (Algebra/Coalgebra)**.** Let F be an endofunctor of the category **Set**. An F-algebra is a pair (A, α) of a set A and a function $\alpha : F(A) \longrightarrow A$. An algebra morphism from (A, α) to (B, β) is just a function $f : A \longrightarrow B$ such that the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{Ff} & F(B) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow[f]{} & B \end{array}$$

Similarly, an F-coalgebra is a pair (C, γ) of a set C and a function $\gamma : C \longrightarrow F(C)$. A coalgebra morphism from (C, γ) to (D, δ) is a function $g : C \longrightarrow D$ such that the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\delta} \\ F(C) & \xrightarrow[Fg]{} & F(D) \end{array}$$

An intuition to think about algebras and coalgebras is to image the functor F is representing some structure. Then, the objects of an algebra A are *created* using the structures of shape F over A. Conversely, the objects of a coalgebra C are *destructed* into structures of shape F over C. Using (co)algebras as objects and (co)algebra morphisms as arrows, we can define a category we will call it F-**(Co)Alg**. We say that a (co)algebra is initial (final), in case it is the[3] initial (final) object of the corresponding category. Finally, we can define what it means to define a function by recursion or corecursion.

Definition 4 (Recursion/Corecursion)**.** Let (A, α) be the initial algebra of an endofunctor F and B be a set. We say that $f : A \longrightarrow B$ has been defined by *recursion* iff there exists a function $\beta : F(B) \longrightarrow B$ such that f is the only algebra morphism from (A, α) to (B, β).

Let (D, δ) be the final coalgebra of an endofunctor F and C be a set. We say that $g : C \longrightarrow D$ has been defined by *corecursion* iff there exists a function $\gamma : C \longrightarrow F(C)$ such that g is the only coalgebra morphism from (C, γ) to (D, δ).

We can define an endofunctor of **Set**, $\mathcal{T}_L$, such that (possibly) non-wellfounded (finitely branching) L-labelled trees are its final coalgebra and finite L-labelled trees are its initial algebra.

Definition 5 (Tree endofunctor)**.** Let L be a set of labels. We define the **Set** endofunctor $\mathcal{T}_L$ as:

$$\mathcal{T}_L(A) = L \times A^*,$$
$$\mathcal{T}_L(f : A \longrightarrow B) = \mathrm{id}_L \times \mathrm{map}_f,$$

where $\mathrm{map}_f : A^* \longrightarrow B^*$ is the pointwise application of f.

We need to find the functions that make the finite trees an inital algebra and the non-wellfounded trees a final coalgebra. These functions are well-known, we call them $\mathsf{construct} : L \times \mathrm{Tree}^* \longrightarrow \mathrm{Tree}$ and $\mathsf{destruct} : \mathrm{Tree} \longrightarrow L \times \mathrm{Tree}^*$.

Definition 6. We define the function $\mathsf{construct} : L \times \mathrm{Tree}^* \longrightarrow \mathrm{Tree}$ as $\mathsf{construct}(a, ((N_0, \ell_0), \ldots, (N_{n-1}, \ell_{n-1}))) = (N, \ell)$ where:

$$N = \{\epsilon\} \cup \bigcup_{i<n} \{iw \mid w \in N_i\},$$
$$\ell(w) = \begin{cases} a & \text{if } w = \epsilon, \\ \ell_i(v) & \text{if } w = iv. \end{cases}$$

[3] We are justified to talk about the inial (final) (co)algebra, since if it exists it is unique up to (co)algebra isomorphism.

We define the function $\mathsf{destruct} : \text{Tree} \longrightarrow L \times \text{Tree}^*$ as:

$$\mathsf{destruct}(N, \ell) = (\ell(\epsilon), (\mathsf{succ}_i(N, \ell))_{i<T\text{-arity}(\epsilon)})$$

where $\mathsf{succ}_i(N, \ell) = (\{w \mid iw \in N\}, w \mapsto \ell(iw))$.

In simple words, given a label a and a finite sequence of trees $T_0, \ldots, T_{n-1}$ we have that $\mathsf{construct}$ creates the tree whose root has a as label and T_i as the i-th successor. Similarly given a tree T, $\mathsf{destruct}$ will return a pair with the label of the root and the sequence of trees which are successors of the root, in order. Note that if we restrict the domain to finite trees, we will obtain finite trees in the codomain for both functions. It is easy to check that:

Lemma 7. $(\text{Tree}, \mathsf{destruct})$ is the final coalgebra of $\mathcal{T}_L$ and $(\text{FinTree}, \mathsf{construct})$ is the initial algebra of $\mathcal{T}_L$.

In other words, we can define functions to trees by corecursion and from finite trees by recursion.

3 Non-wellfounded and cyclic proofs

We work with formulas in the language described by the following BNF:

$$\phi ::= p \mid \bot \mid \phi \to \phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Box\phi,$$

where p is a propositional variable. Note that, since our base logic is intuitonistic, $\Diamond$ is not definable with $\Box$. In other words, we are working in the $\Box$-fragment of modal logic.[4]

A sequent is just a pair (Γ, ϕ) where Γ is a finite multiset of formulas and ϕ is a formula. We will use Seqto denote the set of all sequents. In other words, we work with 2-sided single conclusion sequents. A *rule instance* is a pair consisting in a finite sequence of sequents and a sequence, called premises and conclusion. A *rule* is just a set of rule instances, let Rul be the set consisting in all rules (i.e. the set with all sets of rule instances). We are interested in the following rules:

[4] In particular for us iGL will be the smallest modal logic with intuitionistic propositional logic and the non-logical axioms of GL formulated with $\Box$. There is an alternative approach and define iGL to include also the diamond intuitionistic versions of the diamond axioms. Note that even the $\Box$-fragment of these logics is not the same, so they must not be identified.

$$\frac{}{\Gamma, p \Rightarrow p}\text{Prop} \qquad \frac{}{\Gamma, \bot \Rightarrow \phi}\text{Abs}$$

$$\frac{\Gamma, \phi, \psi \Rightarrow \chi}{\Gamma, \phi \wedge \psi \Rightarrow \chi}\wedge\text{L} \qquad \frac{\Gamma \Rightarrow \phi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \wedge \psi}\wedge\text{R}$$

$$\frac{\Gamma, \phi \Rightarrow \chi \quad \Gamma, \psi \Rightarrow \chi}{\Gamma, \phi \vee \psi \Rightarrow \chi}\vee\text{L} \qquad \frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \phi \vee \psi}\vee\text{R}_1 \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \vee \psi}\vee\text{R}_2$$

$$\frac{\Gamma, \phi \to \psi \Rightarrow \phi \quad \Gamma, \psi \Rightarrow \chi}{\Gamma, \phi \to \psi \Rightarrow \chi}\to\text{L} \qquad \frac{\Gamma, \phi \Rightarrow \psi}{\Gamma \Rightarrow \phi \to \psi}\to\text{R}$$

$$\frac{\Gamma, \Box\Gamma \Rightarrow \phi}{\Pi, \Box\Gamma \Rightarrow \Box\phi}\Box_{\mathsf{K4}} \qquad \frac{\Gamma, \Box\Gamma, \Box\phi \Rightarrow \phi}{\Pi, \Box\Gamma \Rightarrow \Box\phi}\Box_{\mathsf{GL}}$$

where Γ, Π are multisets of formulas, p is a propositional variable and ϕ, ψ, χ are formulas. The first 9 rules are called propositional rules, while the last 2 are called modal rules.

From now on, we assume that with tree we mean $\text{Seq} \times \text{Rul}$-labelled tree. This permits us to talk about the premises, conclusion and rule of any node w as follows.

Definition 8. Let π be a tree and w one of its nodes. We define:

$$\begin{aligned} \pi\text{-prem}(w) &= (\mathsf{fst}(\ell(wi)))_{i<\pi\text{-arity}(w)}, \\ \pi\text{-concl}(w) &= (\mathsf{fst}(\ell(w))), \\ \pi\text{-rule}(w) &= (\mathsf{snd}(\ell(w))), \end{aligned}$$

where fst is the first projection from an ordered pair and snd the second projection.

A proof in iGL is just a standard finite proof-tree with the propositional logic rules and the modal rule $\Box_{\mathsf{GL}}$. Let us define non-wellfounded and cyclic proofs in $\mathsf{iK4}$.

Definition 9. A *non-wellfounded proof in iK4* is a tree π such that:

1. For any node w, we have that $(\pi\text{-prem}(w), \pi\text{-concl}(w))$ is an instance of the rule $\pi\text{-rule}(w)$ and $\pi\text{-rule}(w)$ is either a propositional rule or $\Box_{\mathsf{K4}}$.

2. For any branch w, there are infinitely many i's with $\pi\text{-rule}(w\restriction i) = \Box_{\mathsf{K4}}$.

We will write $\vdash_{\mathsf{iK4}_\infty} S$ to mean that π is a non-wellfounded proof in $\mathsf{iK4}$ and π-concl$(\epsilon) = S$. Also, we will denote the collection of non-wellfounded proofs in $\mathsf{iK4}$ as Proof($\mathsf{iK4}_\infty$).

Definition 10. A *cyclic proof in iK4* is a pair (τ, b) such that:

1. τ is a finite tree and b is a partial function from leaves of τ to the internal nodes of τ called the *backlink function*.
2. For any node $w \notin \mathrm{dom}(b)$, we have that (π-prem(w), π-concl(w)) is an instance of the rule π-rule(w) and π-rule(w) is a propositional rule or $\Box_{\mathsf{K4}}$.
3. For any node $w \in \mathrm{dom}(b)$, we have that:
   (a) $b(w)$ is a (strict) inital segment of w.
   (b) π-concl$(w) = \pi$-concl$(b(w))$ and π-rule$(w) = \varnothing$.
   (c) There is a v between $b(w)$ and w ($b(w)$ initial segment of v and v initial segment of w), such that π-rule$(v) = \Box_{\mathsf{K4}}$.

We will write $\vdash_{\mathsf{iK4}_\circ} S$ to mean that π is a cyclic proof in $\mathsf{iK4}$ and π-concl$(\epsilon) =$ S. Also, we will denote the collection of cyclic proofs in $\mathsf{iK4}$ as Proof($\mathsf{iK4}_\circ$).

4 Infinitary Proof Translation

Thanks to the definition of corecursion we will be able to define a proof translation from a function $\alpha : \mathrm{FinTree} \longrightarrow (\mathrm{Seq} \times \mathrm{Rul}) \times \mathrm{FinTree}^*$. However, not any function of that shape will give a function from proofs to proofs, in the following definition we enumerate the necessary conditions for this to happen.

Definition 11 (Infinitary Proof Translation)**.** Let $\alpha : \mathrm{FinTree} \longrightarrow (\mathrm{Seq} \times \mathrm{Rul}) \times \mathrm{FinTree}^*$. We say that it is an infinitary proof translation iff for any $\pi \in$ Proof(iGL), if we denote:

$$\alpha(\pi) = ((S, R), (\tau_0, \ldots, \tau_{n-1})),$$
$$\alpha(\tau_i) = ((S_i, R_i), \ldots),$$

then the following conditions are satisfied:

1. $\tau_0, \ldots, \tau_{n-1} \in$ Proof(iGL).

2. The following is a rule instance of $\mathsf{iK4}$:

$$\frac{S_0 \quad \cdots \quad S_{n-1}}{S} R$$

3. If $R \neq \Box_{\mathsf{K4}}$, then $\text{height}(\tau_0), \ldots, \text{height}(\tau_{n-1}) < \text{height}(\pi)$.

Given such α we define trans_α as the only coalgebra morphism from $(\text{FinTree}, \alpha)$ to $(\text{Tree}, \mathsf{destruct})$. This implies that

$$\text{trans}_\alpha = \mathsf{construct} \circ (\text{id} \times \mathsf{map}_{\text{trans}_\alpha}) \circ \alpha.$$

We want to show that if α is an infinitary proof translation and π is a (finite) proof in iGL, then $\text{trans}_\beta(\pi)$ is a non-wellfounded proof in $\mathsf{iK4}$. First we need the following technical lemma:

Lemma 12. Let α be an infinitary proof translation and $\pi \in \text{Proof}(\mathsf{iGL})$. If w is a node of $\text{trans}_\alpha(\pi)$, then there is an unique sequence of finite iGL-proofs, $(\iota_i)_{i \leq \text{length}(w)}$, such that:

1. $\iota_0 = \pi$.

2. For any $i \leq \text{length}(w)$, $(\text{trans}_\alpha(\pi))\text{-}\mathsf{subtree}(w\restriction i) = \text{trans}_\alpha(\iota_i)$.

3. For any $i < \text{length}(w)$, $\iota_{i+1} = \mathsf{succ}_{w_i}(\alpha(\iota_i))$.

Similarly, if w is a branch of $\text{trans}_\alpha(\pi)$, then there is an unique sequence of finite iGL-proofs, $(\iota_i)_{i \in \omega}$, such that:

1. $\iota_0 = \pi$.

2. For any i, $(\text{trans}_\alpha(\pi))\text{-}\mathsf{subtree}(w\restriction i) = \text{trans}_\alpha(\iota_i)$.

3. For any i, $\iota_{i+1} = \mathsf{succ}_{w_i}(\alpha(\iota_i))$.

Proof. See Lemma 17 in appendix. □

With this lemma we can show that infinitary proof translations transform finite proofs in iGL into infinitary proofs in $\mathsf{iK4}$:

Theorem 13. If α is an infinitary proof translation, then

$$\text{trans}_\alpha : \text{Proof}(\mathsf{iGL}) \longrightarrow \text{Proof}(\mathsf{iK4}_\infty).$$

Proof. In appendix. □

5 From finitary to cyclic, and vice versa

In order to define the corecursion from the finite acyclic system into the cyclic system first we need to obtain non-wellfounded proofs of certain shape. For this, we need the admissibility of the following two rules in iGL:

$$\frac{\Gamma, \psi, \psi \Rightarrow \phi}{\Gamma, \psi \Rightarrow \phi}\,\text{Contract} \qquad \frac{\Gamma, \Box\Gamma, \Box\phi \Rightarrow \phi}{\Gamma, \Box\Gamma \Rightarrow \phi}\,\text{Löb}$$

The admissibility of these rules is proven in [2]. If Γ is a finite multiset it can be split into a set (a multiset where all its elements appear exactly once) and a multiset in an unique way. Γ^s will be the resulting set of this splitting and Γ^m will be the multiset.

Thanks to admissibility of contraction we can define a function that, given an iGL-proof π of $\Gamma, \Box\Gamma \Rightarrow \phi$, returns a proof $\mathsf{contract}(\pi)$ of $\Gamma^s, \Box\Gamma^s \Rightarrow \phi$. Similarly, thanks to admissibility of Löb we can define a function that given an iGL-proof π of $\Gamma, \Box\Gamma, \Box\phi \Rightarrow \phi$, returns a proof $\mathsf{löb}(\pi)$ of $\Gamma, \Box\Gamma \Rightarrow \phi$.

Using these two functions and the theorem of proof translations, we have the following result:

Theorem 14. There is a unique $h : \text{Proof}(\mathsf{iGL}) \longrightarrow \text{Proof}(\mathsf{iK4}_\infty)$, such that h is the identity in initial sequents, commutes with the logical rules and:

$$\frac{\begin{array}{c}\pi\\ \Gamma, \Box\Gamma, \Box\phi \Rightarrow \phi\end{array}}{\Pi, \Box\Gamma \Rightarrow \Box\phi}\,\Box_{\mathsf{GL}} \longmapsto \frac{\begin{array}{c}h(\mathsf{contract}(\mathsf{löb}(\pi)))\\ \Gamma^s, \Box\Gamma^s \Rightarrow \phi\end{array}}{\Pi, \Box\Gamma^m, \Box\Gamma^s \Rightarrow \Box\phi}\,\Box_{\mathsf{K4}}$$

In addition, if π is a proof of $\Gamma \Rightarrow \phi$, then $h(\pi)$ is also a proof of $\Gamma \Rightarrow \phi$.

Proof. See Appendix. □

Finally, thanks to this proof translation we can show that any finite acyclic proof can be transformed into a cyclic proof.

Corollary 15. Let S be a sequent. If $\vdash_{\mathsf{iGL}} S$, then $\vdash_{\mathsf{iK4}_\circ} S$.

Proof. This result is based in two observations. First, that thanks to the shape of the rule we have the subformula property. Second, that any branch must have infinitely many applications of the $\Box_{\mathsf{K4}}$ rule, but choosing the translation we defined before the premise of such an application is determined by a set. Using these two facts together we can get the required repetition of sequents to define a cyclic proof (check the Appendix). □

The argument to obtain a proof from the cyclic system to the finite acyclic system is totally analogous to the classical case ([6]). The only difference being that now the interpretation of a sequent $\Gamma \Rightarrow \phi$ as a formula should be $\bigwedge \Gamma \Rightarrow \phi$. We can conclude the desired result:

Theorem 16. Let S be a sequent. Then $\vdash_{\mathsf{iGL}} S$ iff $\vdash_{\mathsf{iK4}} S$.

6 Conclusion and Future Work

We have provided a proof of the equivalence between a finite acyclic system with rules of iGL and a (finite) cyclic system with rules of iK4 using a corecursive translation of proofs. In order to do this, we exploited that while performing a corecursion we can use the admissible rules in the finite acyclic system to obtain non-wellfounded proofs of the desired shape. We propose two possible extensions of this work:

1. We do not provide a proof of the admissibility of contraction in the non-wellfounded system. Shamkanov's proof in the classical case ([6]) does not work due to the shape of intuitionistic propositional rules, neither does induction in the height of the tree works. So it remains open to show admissibility of contraction in this system, and also to show that it is equivalent to the other finite acyclic and finite cyclic systems.

2. Applying the same idea of using the admissibility of a rule in the finite system to produce well-shaped non-wellfounded proofs with other rules and systems. In particular, we propose to study cut and check if there is some non-wellfounded system with cut-elimination where this idea can be applied. This would provide an alternative method of cut-elimination for the non-wellfounded system, if we show in addition how to transform non-wellfounded proofs with cut into finite acyclic proofs with cut. A candidate logic where to study this is S4Grz, ([5]).

References

[1] James Brotherston. “Sequent calculus proof systems for inductive definitions”. In: (2006).

[2] Iris van der Giessen and Rosalie Iemhoff. “Sequent Calculi for Intuitionistic Gödel–Löb Logic”. In: *Notre Dame Journal of Formal Logic* 62.2 (2021), pp. 221–246. DOI: 10.1215/00294527-2021-0011. URL: https://doi.org/10.1215/00294527-2021-0011.

[3] Rosalie Iemhoff. “Reasoning in circles”. In: *Liber Amicorum Alberti: A Tribute to Albert Visser* (2016).

[4] D. Kőnig. “Über eine Schlussweise aus dem Endlichen ins Unendliche”. In: *Acta Sci. Math. (Szeged), 3 (2–3): 121–130* (1983).

[5] Yury Savateev and Daniyar Shamkanov. “Non-Well-Founded Proofs for the Grzegorczyk Modal Logic”. In: *The Review of Symbolic Logic* 14 (Apr. 2018). DOI: 10.1017/S1755020319000510.

[6] D. S. Shamkanov. “Circular proofs for the Gödel-Löb provability logic”. In: *Mathematical Notes* 96.3–4 (Sept. 2014), pp. 575–585. ISSN: 1573-8876. DOI: 10.1134/s0001434614090326. URL: http://dx.doi.org/10.1134/S0001434614090326.

[7] Borja Sierra-Miranda. *Cyclic Proofs for iGL via Corecursion.* 2023. arXiv: 2310.10785 [math.LO].

[8] Guillermo Menéndez Turata. *Cyclic Proof Systems for Modal Fixpoint Logics,* Ph.D. Thesis, University of Amsterdam, to appear.

[9] Yde Venema. “6 Algebras and coalgebras”. In: *Handbook of Modal Logic.* Ed. by Patrick Blackburn, Johan Van Benthem, and Frank Wolter. Vol. 3. Studies in Logic and Practical Reasoning. Elsevier, 2007, pp. 331–426. DOI: https://doi.org/10.1016/S1570-2464(07)80009-7. URL: https://www.sciencedirect.com/science/article/pii/S1570246407800097.

A Proofs

First, we prove a technical lemma that given a node of a tree defined by corecursion give us the "story" about how it was created.

Lemma 17. Let $(C, \alpha : C \longrightarrow A \times C^*)$ be a $\mathcal{T}_A$-coalgebra and $\beta : C \longrightarrow \text{Tree}$ be the only coalgebra morphism from this coalgebra to the (non-wellfounded) tree coalgebra. Given $c \in C$, $\pi := \beta(c)$ and $w \in \text{Node}(\pi)$ there exists a sequence $(c_i)_{i \leq \text{length}(w)}$ of elements of C such that:

1. For any $i \leq \text{length}(w)$, $\beta(c_i) = \pi\text{-}\mathsf{subtree}(w \restriction i)$.
2. For any $i < \text{length}(w)$, $c_{i+1} = \mathsf{succ}_{w_i}(\beta(c_i))$.

Let b be a branch of π. Then, there exists a sequence $(c_i)_{i \in \omega}$ of elements of C such that:

1. For any $i \in \omega$, $\beta(c_i) = \pi\text{-}\mathsf{subtree}(w \restriction i)$.
2. For any $i \in \omega$, $c_{i+1} = \mathsf{succ}_{w_i}(\beta(c_i))$.

Proof. We show the result for nodes by induction in the length of w. If the length is 0, $w = \epsilon$, the empty sequence. Then, the sequence is simply $c_0 = c$. If the length is $n + 1$, then $w = w', i$. By induction hypothesis there is a sequence $(c_i)_{i \leq n}$ fulfilling the conditions for w'. Let $\alpha(c_n) = (a, (c'_0, \dots, c'_{k-1}))$, we want to define $c_{n+1} = c'_i$, which will fulfill the conditions. In order to do this, we need to be sure that $i < k$. Notice that k is the arity of ϵ in $\beta(c_n) = \pi\text{-}\mathsf{subtree}(w')$, which is the same as the arity of w' in π. Thanks to w', i being a node of π we have that the arity of $i < k$.

The result for branches follows from the result from nodes, using that if b is a branch then $b \restriction i$ is a node. So for each restriction of the branch we get a sequence of elements of C. Thanks to the second condition in the node case, we know that if $i \leq j$ then the sequence of $b \restriction j$ extends the sequence of $b \restriction i$. Taking the union of all the sequences of $b \restriction i, i \in \omega$, we get the desired infinite sequence. □

Theorem 13. If α is an infinitary proof translation, then

$$\text{trans}_\alpha : \text{Proof}(\mathsf{iGL}) \longrightarrow \text{Proof}(\mathsf{iK4}_\infty).$$

Proof. We need to show that every node is the instance of a rule and that the branch condition is fulfilled. For both things we will use Lemma 12. So, let α be an infinitary proof translation, π by a (finite) proof in iGL, $\tau = \text{trans}_\alpha(\pi)$ and w be a node of τ of length n. Thanks to the lemma, we get a sequence $(\iota_i)_{i \leq n}$ such that:

1. For $i \leq n$, $\mathrm{trans}_\alpha(\iota_i) = \tau\text{-}\mathsf{subtree}(w\restriction i)$.

2. For $i < n$, $\iota_{i+1} = \mathsf{succ}_{w_i}(\alpha(\iota_i))$.

Thanks to 2 above and the first condition on infinitary proof translations, we have that each ι_i is a (finite) proof in iGL. Then, using Lemma 12 for each w, i with i smaller than the arity of w in τ and the second condition of initary proof translation with ι_n, we get the desired rule instance condition on w.

Let b be a branch of τ and let us show that the branch condition is fulfilled. By reductio ad absurdum, assume that it is not i.e. if $(R_i)_{i\in\omega}$ is the sequence of rules in the nodes $(b\restriction i)_{i\in\omega}$ in τ, then there is a k such that for $i \geq k$ $R_i \neq \Box_{\mathsf{K4}}$ (i). Use Lemma 12 to get an infinite sequence $(\iota_i)_{i\in\omega}$ such that

1. For $i \in \omega$, $\mathrm{trans}_\alpha(\iota_i) = \tau\text{-}\mathsf{subtree}(w\restriction i)$.

2. For $i \in \omega$, $\iota_{i+1} = \mathsf{succ}_{w_i}(\alpha(\iota_i))$.

Again by 2 above and the first condition of infinitary proof translation, that all the ι_i are finite proofs of iGL. By 1 above we get that, if $\alpha(\iota_i) = ((S_i, R'_i), (\iota'_0, \ldots, \iota'_{n_i}))$ then $R'_i = R_i$. So, if $i \geq k$ we know that $R_i \neq \Box_{\mathsf{K4}}$ and by the third condition of infinitary proof translation we have that $\iota'_0, \ldots, \iota'_{n_i}$ are of strictly smaller height than ι_i. Since ι_{i+1} must be equal to one of these (by 2 above), in fact we have proven that for $i \geq k$, $\mathrm{height}(\iota_i) > \mathrm{height}(\iota_{i+1})$. But then $(\mathrm{height}(\iota_i))_{i\geq k}$ is an infinite (strictly) descending sequence of natural numbers, impossible. $\Box$

Theorem 14 . There is a unique $h : \mathrm{Proof}(\mathsf{iGL}) \longrightarrow \mathrm{Proof}(\mathsf{iK4}_\infty)$, such that h is the identity in initial sequents, commutes with the logical rules and:

$$\dfrac{\begin{matrix}\pi\\ \vdots\\ \Gamma, \Box\Gamma, \Box\phi \Rightarrow \phi\end{matrix}}{\Pi, \Box\Gamma \Rightarrow \Box\phi}\,\Box_{\mathsf{GL}} \quad\longmapsto\quad \dfrac{\begin{matrix}h(\mathsf{contract}(\mathsf{l\ddot{o}b}(\pi)))\\ \vdots\\ \Gamma^s, \Box\Gamma^s \Rightarrow \phi\end{matrix}}{\Pi, \Box\Gamma^m, \Box\Gamma^s \Rightarrow \Box\phi}\,\Box_{\mathsf{K4}}$$

In addition, if π is a proof of $\Gamma \Rightarrow \phi$, then $h(\pi)$ is also a proof of $\Gamma \Rightarrow \phi$.

Proof. The idea is to define $\alpha : \text{FinTree} \longrightarrow (\text{Seq} \times \text{Rul}) \times \text{FinTree}^*$ as:

$$\alpha\left(\dfrac{\begin{array}{ccc}\pi_0 & & \pi_{n-1}\\ S_0 & \cdots & S_{n-1}\end{array}}{S}\,R\right) = ((S, R), (\pi_0, \dots, \pi_{n-1})) \text{ if } R \neq \Box_{\mathsf{GL}},$$

$$\alpha\left(\dfrac{\begin{array}{c}\pi_0\\ \Gamma, \Box\Gamma, \Box\phi \Rightarrow \phi\end{array}}{\Pi, \Box\Gamma \Rightarrow \Box\phi}\,\Box_{\mathsf{GL}}\right) = ((\Pi, \Box\Gamma \Rightarrow \Box\phi, \Box_{\mathsf{K4}}), \mathsf{contract}(\mathsf{l\ddot{o}b}(\pi))).$$

And it is straightforward to check that indeed, it is an infinitary proof translation. Then, the desired h is just the one defined by corecursion using this infinitary proof translation. □

Corollary 15. Let S be a sequent. If $\vdash_{\mathsf{iGL}} S$, then $\vdash_{\mathsf{iK4}_\circ} S$

Proof. Let π be a iGL proof of S, and $\tau = (N, \ell) := h(\pi)$ (h is defined in PUT) is a $\mathsf{iK4}_\infty$ proof of S. Let Φ be the set of subformulas of formulas in S, note that this is a finite set. First, we notice that we have the subformula property, i.e. each sequents has only formulas in Φ. This and the definition of h guarantees that for any $w \in \text{Node}(\tau)$ such that $\tau\text{-rule}(w) = \Box_{\mathsf{K4}}$ there exists an unique set $\Gamma \subseteq \Phi$ and a unique $\phi \in \Phi$ such that $\tau\text{-premise}(w) = (\Gamma, \Box\Gamma \Rightarrow \phi)$. We call this Γ as Γ_w and this ϕ as ϕ_w. Let us define the cyclic proof.

First, we define the following collections of nodes of τ:

$$N_0 = \{w \in N \mid \forall i < j < \text{length}(w) \,.\, \tau\text{-rule}(w{\restriction}i) = \tau\text{-rule}(w{\restriction}j) = \Box_{\mathsf{K4}}, \\ \text{then } \tau\text{-prem}(w{\restriction}i) \neq \tau\text{-prem}(w{\restriction}j)\},$$

$$N_1 = \{(w, 0) \in N \mid w \in N_0 \text{ and } \exists j < \text{length}(w) \,.\, \tau\text{-rule}(w{\restriction}j) = \Box_{\mathsf{K4}}, \\ \text{and } \tau\text{-prem}(w{\restriction}j) \neq \tau\text{-prem}(w)\},$$

In words, N_0 is the collection of nodes of π such that if you look to their path to the root there is no premise of $\Box_{\mathsf{K4}}$ being repeated. N_1 is the nodes which are a successor of an N_0 which ended in an application of $\Box_{\mathsf{K4}}$ whose premise is repeated.

Then, we define the tree $\iota = (N', \ell')$ where $N' = N_0 \cup N_1$ and

$$\ell' = \begin{cases} \ell(w) & \text{if } w \in N_0, \\ (S, \varnothing) & \text{if } w \in N_1 \text{ and } \ell(w) = (S, R). \end{cases}$$

Note that this is well-defined since $N_0 \cap N_1 = \varnothing$. In words, the tree is just the result of only keeping the nodes of $N_0 \cup N_1$ of τ and change the rule of the leaves which are a premise of $\Box_{\mathsf{K4}}$ reapeated for $\varnothing$. We notice, that thanks to the definition of N_1 if $w \in N_1$ there is a $j < \text{length}(w) - 1$ such that $\tau\text{-rule}(w{\restriction}j) = \Box_{\mathsf{K4}}$ and $\tau\text{-premise}(w{\restriction}j) = \tau\text{-premise}(w{\restriction}(\text{length}(w)-1))$ and it is in fact unique. We will denote this unique j as j_w. Then we define the backlink function of the cyclic proof as $b : N_1 \longrightarrow N'$, $b(w) = w{\restriction}(j_w+1)$. Clearly $b(w)$ is a strict initial segment of w and they share the same sequent and $\iota\text{-rule}(w) = \varnothing$, by definition of ℓ'. Also, between $b(w)$ and w there is an application of $\Box_{\mathsf{K4}}$, particularly in the node $w{\restriction}(\text{length}(w) - 1)$. This means that b fulfills all the conditions of backlink function (condition 3 of cyclic proof).

Also, we note that since ι is just a part of the tree τ whose only changes in the labels occurs in the domain of the backlink function, and thanks to τ being a non-wellfounded proof of $\mathsf{K4}$, we get that for any node w which is not in the domain of the backlink function, $(\iota\text{-prem}(w), \iota\text{-concl}(w)) \in \iota\text{-rule}(w)$ and the rule is propositional or $\Box_{\mathsf{K4}}$ (condition 2 of cyclic proof). Also, it is clear that the conclusion of ι is still S.

The only thing left to show that (ι, b) is the desired cyclic proof of S is to show that ι is indeed a finite tree. Assume otherwise, since our trees are finitely branching we know by König's lemma (see [4]) that it must have an infinite branch, let it be $(b_i)_{i\in\omega}$. Note that in such a branch all the nodes $b{\restriction}i$ must belong to N_0, since the definition of N_1 must be leaves. Then, this infinite branch would also be an infinite branch of τ with the same sequents and rules, so thanks to the branch condition, we get that there are infinitely many i's such that $\iota\text{-rule}(b{\restriction}i) = \Box_{\mathsf{K4}}$. For any such i we know that there is a $\Gamma_i, \phi_i) \in \wp(\Phi) \times \Phi$ determining the premise of w. But, Φ is a finite set, so $\wp(\Phi) \times \Phi$ is also finite. Since there were infinitely many i's with this propery, we get that for some $i \neq j$, $(\Gamma_i, \phi_i) = (\Gamma_j, \phi_j)$. In other words, along the branch there are two distinct nodes which have the same sequence and are premises of the $\Box_{\mathsf{K4}}$ rule. But then, $(b{\restriction}i)_{i\in\omega}$ cannot be a sequence of nodes in N_0, contradiction. We conclude that the tree is necessarily finite. $\Box$