The computational strength of fixed points

Anupam Das

University of Birmingham

12th International Workshop on Fixed Points in Computer Science,

Naples, Italy,
19th February 2024.

For $(L, \leqslant)$ a complete lattice & $F : L \to L$ monotone:

Theorem (Tarski '55)

The fixed points of F form a complete lattice under $\leqslant$.

$\rightsquigarrow$ *F has a least fixed point μF and a greatest fixed point νF.*

For $(L, \leqslant)$ a complete lattice & $F : L \to L$ monotone:

Theorem (Tarski '55)
The fixed points of F form a complete lattice under $\leqslant$.
$\rightsquigarrow$ *F has a least fixed point μF and a greatest fixed point νF.*

Applications across Computer Science:

- Game Theory.
- (Finite) Model Theory.
- Automata Theory.
- *Typed Programming Languages*.

Outline

1. Types with fixed points
2. Circular proofs
3. Main results
4. Conclusions

Outline

Simple types

Types: $\sigma, \tau, \ldots \;::=\; \bot \mid X \mid \sigma + \tau \mid \sigma \times \tau \mid \sigma \to \tau$

Types: $\sigma, \tau, \ldots \;::=\; \bot \mid X \mid \sigma + \tau \mid \sigma \times \tau \mid \sigma \to \tau$

Type	**Set theoretic model**
$\bot$	$\varnothing$
$\sigma_0 + \sigma_1$	$\sigma_0 \uplus \sigma_1$
$\sigma_0 \times \sigma_1$	$\sigma_0 \times \sigma_1$
$\sigma \to \tau$	τ^{σ}

Types: $\sigma, \tau, \ldots \ ::= \ \bot \mid X \mid \sigma + \tau \mid \sigma \times \tau \mid \sigma \to \tau$

Type	**Set theoretic model**
$\bot$	$\varnothing$
$\sigma_0 + \sigma_1$	$\sigma_0 \uplus \sigma_1$
$\sigma_0 \times \sigma_1$	$\sigma_0 \times \sigma_1$
$\sigma \to \tau$	τ^σ

- 'Formulas-as-types' $\leadsto$ *Curry-Howard correspondence*.

Types: $\sigma, \tau, \ldots \ ::= \ \bot \mid X \mid \sigma + \tau \mid \sigma \times \tau \mid \sigma \rightarrow \tau \mid \mu X\,\sigma \mid \nu X\,\sigma$

In $\mu X\,\sigma$ and $\nu X\,\sigma$, the variable X must occur positively in σ.

Type	Set theoretic model
$\bot$	$\varnothing$
$\sigma_0 + \sigma_1$	$\sigma_0 \uplus \sigma_1$
$\sigma_0 \times \sigma_1$	$\sigma_0 \times \sigma_1$
$\sigma \rightarrow \tau$	τ^{σ}
$\mu X\,\sigma(X)$	**?**
$\nu X\,\sigma(X)$	**?**

- 'Formulas-as-types' $\rightsquigarrow$ *Curry-Howard correspondence*.
- No set theoretic interpretation of $\nu X\,X$ and $\mu X((X \rightarrow \sigma) \rightarrow \tau)$.
  $\rightsquigarrow$ impredicativity.

Types: $\sigma, \tau, \ldots \; ::= \; \bot \mid X \mid \sigma + \tau \mid \sigma \times \tau \mid \sigma \to \tau \mid \mu X\, \sigma \mid \nu X\, \sigma$

In $\mu X\, \sigma$ and $\nu X\, \sigma$, the variable X must occur positively in σ.

Type	Set theoretic model	Computability theoretic model
$\bot$	$\varnothing$	$\varnothing$
$\sigma_0 + \sigma_1$	$\sigma_0 \uplus \sigma_1$	$\{\langle i, M\rangle : M \in \sigma_i\}$
$\sigma_0 \times \sigma_1$	$\sigma_0 \times \sigma_1$	$\{\langle M_0, M_1\rangle : M_i \in \sigma_i\}$
$\sigma \to \tau$	τ^σ	$\{M : \forall N \in \sigma.\, MN\!\downarrow \in \tau\}$
$\mu X\, \sigma(X)$	**?**	
$\nu X\, \sigma(X)$	**?**	

- 'Formulas-as-types' $\rightsquigarrow$ *Curry-Howard correspondence*.
- No set theoretic interpretation of $\nu X\, X$ and $\mu X((X \to \sigma) \to \tau)$.
  $\rightsquigarrow$ impredicativity.

Types: $\sigma, \tau, \ldots ::= \bot \mid X \mid \sigma + \tau \mid \sigma \times \tau \mid \sigma \to \tau \mid \mu X\, \sigma \mid \nu X\, \sigma$

In $\mu X\, \sigma$ and $\nu X\, \sigma$, the variable X must occur positively in σ.

Type	Set theoretic model	Computability theoretic model
$\bot$	$\varnothing$	$\varnothing$
$\sigma_0 + \sigma_1$	$\sigma_0 \uplus \sigma_1$	$\{\langle i, M\rangle : M \in \sigma_i\}$
$\sigma_0 \times \sigma_1$	$\sigma_0 \times \sigma_1$	$\{\langle M_0, M_1\rangle : M_i \in \sigma_i\}$
$\sigma \to \tau$	τ^σ	$\{M : \forall N \in \sigma.\, MN{\downarrow} \in \tau\}$
$\mu X\, \sigma(X)$	**?**	$\mu\, [A \mapsto \sigma(A)]$
$\nu X\, \sigma(X)$	**?**	$\nu\, [A \mapsto \sigma(A)]$

- 'Formulas-as-types' $\rightsquigarrow$ *Curry-Howard correspondence*.
- No set theoretic interpretation of $\nu X\, X$ and $\mu X((X \to \sigma) \to \tau)$.
  $\rightsquigarrow$ impredicativity.

Sequents: $\sigma_1, \ldots, \sigma_n \Rightarrow \tau$ (interpret as $\sigma_1 \times \cdots \times \sigma_n \to \tau$)

Each type can be constructed and destructed. E.g.

$$\to_r \frac{\Gamma, \sigma \Rightarrow \tau}{\Gamma \Rightarrow \sigma \to \tau} \qquad \to_l \frac{\Gamma \Rightarrow \rho \quad \Gamma, \sigma \Rightarrow \tau}{\Gamma, \rho \to \sigma \Rightarrow \tau}$$

$\rightsquigarrow$ *Curry-Howard correspondence*: proofs-as-programs.

Sequents: $\sigma_1, \ldots, \sigma_n \Rightarrow \tau$ (interpret as $\sigma_1 \times \cdots \times \sigma_n \to \tau$)

Each type can be constructed and destructed. E.g.

$$\to_r \frac{\Gamma, \sigma \Rightarrow \tau}{\Gamma \Rightarrow \sigma \to \tau} \qquad \to_l \frac{\Gamma \Rightarrow \rho \quad \Gamma, \sigma \Rightarrow \tau}{\Gamma, \rho \to \sigma \Rightarrow \tau}$$

$\rightsquigarrow$ *Curry-Howard correspondence*: proofs-as-programs.

Fixed point rules:

$$\mu_r \frac{\Gamma \Rightarrow \sigma(\mu X \sigma(X))}{\Gamma \Rightarrow \mu X \sigma(X)} \quad \mu_l \frac{\Gamma, \sigma(\tau) \Rightarrow \tau}{\Gamma, \mu X \sigma(X) \Rightarrow \tau} \quad \nu_r \frac{\Gamma, \tau \Rightarrow \sigma(\tau)}{\Gamma, \tau \Rightarrow \nu X \sigma(X)} \quad \nu_l \frac{\Gamma, \sigma(\nu X \sigma(X)) \Rightarrow \tau}{\Gamma, \nu X \sigma(X) \Rightarrow \tau}$$

Sequents: $\sigma_1, \ldots, \sigma_n \Rightarrow \tau$ (interpret as $\sigma_1 \times \cdots \times \sigma_n \to \tau$)

Each type can be constructed and destructed. E.g.

$$\to_r \frac{\Gamma, \sigma \Rightarrow \tau}{\Gamma \Rightarrow \sigma \to \tau} \qquad \to_l \frac{\Gamma \Rightarrow \rho \quad \Gamma, \sigma \Rightarrow \tau}{\Gamma, \rho \to \sigma \Rightarrow \tau}$$

$\rightsquigarrow$ *Curry-Howard correspondence*: proofs-as-programs.

Fixed point rules:

$$\mu_r \frac{\Gamma \Rightarrow \sigma(\mu X\, \sigma(X))}{\Gamma \Rightarrow \mu X\, \sigma(X)} \quad \mu_l \frac{\Gamma, \sigma(\tau) \Rightarrow \tau}{\Gamma, \mu X\, \sigma(X) \Rightarrow \tau} \quad \nu_r \frac{\Gamma, \tau \Rightarrow \sigma(\tau)}{\Gamma, \tau \Rightarrow \nu X\, \sigma(X)} \quad \nu_l \frac{\Gamma, \sigma(\nu X\, \sigma(X)) \Rightarrow \tau}{\Gamma, \nu X\, \sigma(X) \Rightarrow \tau}$$

Definition ([Cla09])

μLJ is the extension of usual LJ by the fixed point rules above.

Computational model given by cut-reduction.

Examples: natural numbers and streams

$$\underline{N := \mu X(1 + X)}$$

$$\underline{0} := \quad \mu_r\dfrac{\dfrac{\overline{\Rightarrow 1}}{\Rightarrow 1 + N}}{\Rightarrow N} \qquad\qquad \underline{n+1} := \quad \mu_r\dfrac{\dfrac{\dfrac{\underline{n}}{\Rightarrow N}}{\Rightarrow 1 + N}}{\Rightarrow N}$$

$$\underline{N := \mu X(1 + X)}$$

$$\underline{0} := \quad \mu_r\,\dfrac{\dfrac{\overline{\Rightarrow 1}}{\Rightarrow 1 + N}}{\Rightarrow N} \qquad \underline{n+1} := \quad \mu_r\,\dfrac{\dfrac{\begin{array}{c}\underline{n}\\ \vdots \\ \Rightarrow N\end{array}}{\Rightarrow 1 + N}}{\Rightarrow N}$$

$$\underline{\mathsf{add} : N \times N \to N}$$

$$\mu_l\,\dfrac{\dfrac{\dfrac{\mathsf{id}\,\overline{N \Rightarrow N}}{1, N \Rightarrow N} \qquad \mu_r\,\dfrac{\dfrac{\mathsf{id}\,\overline{N \Rightarrow N}}{N \Rightarrow 1 + N}}{N \Rightarrow N}}{1 + N, N \Rightarrow N}}{N, N \Rightarrow N}$$

$$\left(\begin{array}{r} \mathsf{add}(0, n) = n \\ \mathsf{add}(m + 1, n) = \mathsf{add}(m, n) + 1 \end{array}\right)$$

$$\underline{N := \mu X(1+X)}$$

$$\underline{0} := \mu_r\,\dfrac{\dfrac{\overline{\Rightarrow 1}}{\Rightarrow 1+N}}{\Rightarrow N} \qquad \underline{n+1} := \mu_r\,\dfrac{\dfrac{\triangle_{\underline{n}}}{\Rightarrow N}}{\dfrac{\Rightarrow 1+N}{\Rightarrow N}}$$

$$\underline{S := \nu Y(N \times Y)}$$

$$\mathtt{hd} := \nu_l\,\dfrac{\dfrac{\overline{N \Rightarrow N}\ \text{id}}{N \times S \Rightarrow N}}{S \Rightarrow N} \qquad \mathtt{tl} := \nu_l\,\dfrac{\dfrac{\overline{S \Rightarrow S}\ \text{id}}{N \times S \Rightarrow S}}{S \Rightarrow S}$$

$$\underline{\mathtt{add} : N \times N \to N}$$

$$\mu_l\,\dfrac{\dfrac{\dfrac{\overline{N \Rightarrow N}\ \text{id}}{1, N \Rightarrow N} \qquad \mu_r\,\dfrac{\dfrac{\overline{N \Rightarrow N}\ \text{id}}{N \Rightarrow 1+N}}{N \Rightarrow N}}{1+N, N \Rightarrow N}}{N, N \Rightarrow N}$$

$$\left(\begin{array}{c} \mathtt{add}(0, n) = n \\ \mathtt{add}(m+1, n) = \mathtt{add}(m, n) + 1 \end{array}\right)$$

$$\underline{N := \mu X(1 + X)}$$

$$\underline{0} := \dfrac{\dfrac{\dfrac{}{\Rightarrow 1}}{\Rightarrow 1 + N}}{\Rightarrow N}\,\mu_r \qquad \underline{n + 1} := \dfrac{\dfrac{\dfrac{\underline{n}}{\Rightarrow N}}{\Rightarrow 1 + N}}{\Rightarrow N}\,\mu_r$$

$$\underline{S := \nu Y(N \times Y)}$$

$$\mathtt{hd} := \dfrac{\dfrac{\dfrac{}{N \Rightarrow N}\,\mathsf{id}}{N \times S \Rightarrow N}}{S \Rightarrow N}\,\nu_l \qquad \mathtt{tl} := \dfrac{\dfrac{\dfrac{}{S \Rightarrow S}\,\mathsf{id}}{N \times S \Rightarrow S}}{S \Rightarrow S}\,\nu_l$$

$$\underline{\mathtt{add} : N \times N \to N}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{N \Rightarrow N}\,\mathsf{id}}{1, N \Rightarrow N} \qquad \dfrac{\dfrac{\dfrac{}{N \Rightarrow N}\,\mathsf{id}}{N \Rightarrow 1 + N}}{N \Rightarrow N}\,\mu_r}{1 + N, N \Rightarrow N}}{N, N \Rightarrow N}\,\mu_l$$

$$\left(\begin{array}{c} \mathtt{add}(0, n) = n \\ \mathtt{add}(m + 1, n) = \mathtt{add}(m, n) + 1 \end{array} \right)$$

$$\underline{f : n \mapsto [n, n + 1, \ldots]}$$

$$\dfrac{\dfrac{\dfrac{}{N \Rightarrow N}\,\mathsf{id} \qquad \dfrac{\dfrac{\dfrac{}{N \Rightarrow N}\,\mathsf{id}}{N \Rightarrow 1 + N}}{N \Rightarrow N}\,\mu_r}{N \Rightarrow N \times N}}{N \Rightarrow S}\,\nu_r$$

$$\left(\; f(n) \; = \; n :: f(n + 1) \; \right)$$

Computational interpretations relate logic and computation

Computational interpretations relate logic and computation:

System	Computation	Logic
simple types	Extended Polynomials	Pure FO Logic
$+ N$	HO Primitive Recursion (T)	FO Arithmetic (PA)
$+ \forall, \exists$	Polymorphic λ-Calculus (F)	SO Arithmetic ($\mathsf{PA2}$)

Computational interpretations relate logic and computation:

System	Computation	Logic
simple types	Extended Polynomials	Pure FO Logic
$+ N$	HO Primitive Recursion (T)	FO Arithmetic (PA)
$+ \mu, \nu$	**?**	**?**
$+ \forall, \exists$	Polymorphic λ-Calculus (F)	SO Arithmetic ($\mathsf{PA2}$)

Computational interpretations relate logic and computation:

System	Computation	Logic
simple types	Extended Polynomials	Pure FO Logic
$+ N$	HO Primitive Recursion (T)	FO Arithmetic (PA)
$+ \mu, \nu$	**?**	**?**
$+ \forall, \exists$	Polymorphic λ-Calculus (F)	SO Arithmetic (PA2)

What do fixed point types compute?

Outline

1. Types with fixed points
2. Circular proofs
3. Main results
4. Conclusions

- Derivations may be non-wellfounded but regular.
- Correctness by an ω-regular property of infinite branches.

- Derivations may be non-wellfounded but regular.
- Correctness by an ω-regular property of infinite branches.

Cyclic proofs are a bridge between automata, games and proofs.

- Derivations may be non-wellfounded but regular.
- Correctness by an ω-regular property of infinite branches.

Cyclic proofs are a bridge between automata, games and proofs.

Landscape:

- *Algebra*. E.g., [San02, FS13, DP17, DD24].
- *Type systems*. E.g., [Cla09, BDS16, DP18, KPP21, BDKS22].
- *Modal logics*. E.g., [NW96, Lan03, Stu08, Sha14, AL17].
- *Predicate logic*: E.g. [Sim17, BT17a, BT17b, Das20, DG22].

μ'LJ obtained from μLJ by replacing μ_l and ν_r by unfoldings:

μ'LJ obtained from μLJ by replacing μ_l and ν_r by unfoldings:

$$\mu_l' \frac{\Gamma, \sigma(\mu X\, \sigma(X)) \Rightarrow \tau}{\Gamma, \mu X\, \sigma(X) \Rightarrow \tau} \qquad \nu_r' \frac{\Gamma \Rightarrow \sigma(\nu X\, \sigma(X))}{\Gamma \Rightarrow \nu X\, \sigma(X)}$$

$\mu'\mathsf{LJ}$ obtained from $\mu\mathsf{LJ}$ by replacing μ_l and ν_r by unfoldings:

$$\mu_l' \; \frac{\Gamma, \sigma(\mu X\, \sigma(X) \Rightarrow \tau}{\Gamma, \mu X\, \sigma(X) \Rightarrow \tau} \qquad\qquad \nu_r' \; \frac{\Gamma \Rightarrow \sigma(\nu X\, \sigma(X))}{\Gamma \Rightarrow \nu X\, \sigma(X)}$$

- A **coderivation** is generated *coinductively* from rules of $\mu'\mathsf{LJ}$.
- It is **progressing** if every infinite branch has an infinite progressing thread.
  (Precise definition is beyond the scope of this talk.)

μ'LJ obtained from μLJ by replacing μ_l and ν_r by unfoldings:

$$\mu_l' \frac{\Gamma, \sigma(\mu X \sigma(X) \Rightarrow \tau}{\Gamma, \mu X \sigma(X) \Rightarrow \tau} \qquad \nu_r' \frac{\Gamma \Rightarrow \sigma(\nu X \sigma(X))}{\Gamma \Rightarrow \nu X \sigma(X)}$$

- A **coderivation** is generated *coinductively* from rules of μ'LJ.
- It is **progressing** if every infinite branch has an infinite progressing thread.
  (Precise definition is beyond the scope of this talk.)

Definition
CμLJ is the class of regular progressing coderivations.

Computational model again given by cut-reduction.

Examples of progressing coderivations

$\underline{\mathtt{add} : N \times N \to N}$

$$
\mu'_l\ \dfrac{\dfrac{\mathsf{id}\ \dfrac{}{N \Rightarrow N}}{1, N \Rightarrow N} \qquad \mu_r\ \dfrac{\dfrac{\mu'_l\ \dfrac{\vdots}{N, N \Rightarrow N}\ \bullet}{N, N \Rightarrow 1 + N}}{N, N \Rightarrow N}}{\dfrac{1 + N, N \Rightarrow N}{N, N \Rightarrow N}}\ \bullet
$$

$\underline{[n_0, n_1, \ldots]}$

$$
\nu_r\ \dfrac{\dfrac{\dfrac{n_0}{\Rightarrow N} \qquad \nu_r\ \dfrac{\dfrac{\dfrac{n_1}{\Rightarrow N} \quad \vdots}{\Rightarrow N \times S}}{\Rightarrow S}}{\Rightarrow N \times S}}{\Rightarrow S}
$$

$\underline{n \mapsto [n, n+1, \ldots]}$

$$
\nu'_r\ \dfrac{\dfrac{\mathsf{id}\ \dfrac{}{N \Rightarrow N} \qquad \mathsf{cut}\ \dfrac{\mu_r\ \dfrac{\dfrac{\mathsf{id}\ \dfrac{}{N \Rightarrow N}}{N \Rightarrow 1 + N}}{N \Rightarrow N} \qquad \dfrac{\vdots}{N \Rightarrow S}\ \bullet}{N \Rightarrow S}}{N \Rightarrow N \times S}}{N \Rightarrow S}\ \bullet
$$

$\underline{\mathsf{add} : N \times N \to N}$

$$\mu_l'\ \dfrac{\dfrac{\mathsf{id}\ \dfrac{}{N \Rightarrow N}}{1, N \Rightarrow N} \qquad \mu_r\ \dfrac{\dfrac{\mu_l'\ \dfrac{\vdots}{N, N \Rightarrow N}\ \bullet}{N, N \Rightarrow 1 + N}}{N, N \Rightarrow N}}{\dfrac{1 + N, N \Rightarrow N}{N, N \Rightarrow N}}\ \bullet$$

$\underline{[n_0, n_1, \ldots]}$

$$\nu_r\ \dfrac{\dfrac{\dfrac{n_0}{\Rightarrow N} \qquad \nu_r\ \dfrac{\dfrac{\dfrac{n_1}{\Rightarrow N} \quad \vdots}{\Rightarrow N \times S}}{\Rightarrow S}}{\Rightarrow N \times S}}{\Rightarrow S}$$

$\underline{n \mapsto [n, n + 1, \ldots]}$

$$\nu_r'\ \dfrac{\dfrac{\mathsf{id}\ \dfrac{}{N \Rightarrow N} \qquad \mathsf{cut}\ \dfrac{\mu_r\ \dfrac{\dfrac{\mathsf{id}\ \dfrac{}{N \Rightarrow N}}{N \Rightarrow 1 + N}}{N \Rightarrow N} \qquad \dfrac{\vdots}{N \Rightarrow S}\ \bullet}{N \Rightarrow S}}{N \Rightarrow N \times S}}{N \Rightarrow S}\ \bullet$$

Recursion to cycles:

$$\mu_l\ \dfrac{\Gamma, \sigma(\tau) \Rightarrow \tau}{\Gamma, \mu X \sigma(X) \Rightarrow \tau} \quad \rightsquigarrow \quad \mu_l'\ \dfrac{\mathsf{cut}\ \dfrac{\sigma\ \dfrac{\mu_l'\ \dfrac{\vdots}{\Gamma, \mu X \sigma(X) \Rightarrow \tau}\ \bullet}{\Gamma, \sigma(\mu X \sigma(X)) \Rightarrow \sigma(\tau)} \qquad \Gamma, \sigma(\tau) \Rightarrow \tau}{\Gamma, \sigma(\mu X \sigma(X)) \Rightarrow \tau}}{\Gamma, \mu X \sigma(X) \Rightarrow \tau}\ \bullet$$

$$
c\;\dfrac{\mu'_l,\mathsf{s}\;\dfrac{\mu'_l\;\dfrac{\mathsf{wk}\;\dfrac{\mathsf{cut}\;\dfrac{1\;\dfrac{}{\Rightarrow N}\qquad \dfrac{\vdots}{N, N \Rightarrow N}\bullet}{N \Rightarrow N}}{N, \underline{N} \Rightarrow N}\qquad \mathsf{cut}\;\dfrac{\dfrac{\vdots}{N, N \Rightarrow N}\bullet\qquad \dfrac{\vdots}{N, N \Rightarrow N}\bullet}{N, N, N \Rightarrow N}}{N, N, \underline{N} \Rightarrow N}}{\underline{N}, N, N \Rightarrow N}}{\underline{N}, N \Rightarrow N}\bullet
$$

$$
\left(\begin{array}{r}
A(0, n) = n + 1 \\
A(m + 1, 0) = A(m, 1) \\
A(m + 1, n + 1) = A(m, A(n + 1, m))
\end{array}\right)
$$

Are circular and finitary proofs equally expressive?

Are circular and finitary proofs equally expressive?

Several recent advances around first-order arithmetic via metamathematics.

Are circular and finitary proofs equally expressive?

Several recent advances around first-order arithmetic via metamathematics. E.g.

The case of N

Write (C)T for restriction of (C)μLJ to just the fixed point N:

Theorem ([KPP21, Das21b])
CT *and* T *define the same (type 1) functions.*

Theorem ([Das21a])
CT *and* T *define the same functionals (at all types).*

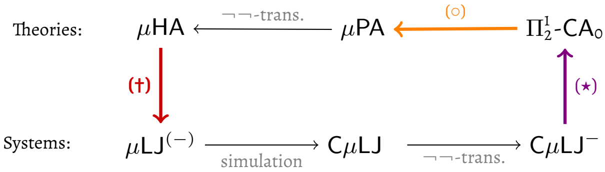
NB: further results on affinity [KPP21] and type levels [Das21b].

Outline

Theorem ([CD23])
μLJ and CμLJ define just the functions provably recursive in Π^1_2-CA_0.

Theorem ([CD23])

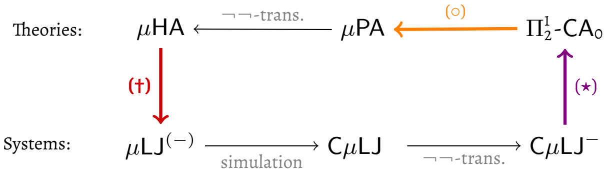
μLJ and CμLJ define just the functions provably recursive in Π^1_2-CA_0.



Main result

Theorem ([CD23])

μLJ and CμLJ define just the functions provably recursive in Π^1_2-CA$_0$.



- (⋆) Formalisation of semantics by *fixed points as fixed points*:
  - Novel reverse mathematics of fixed point theorems, building on [Das21a, DM23].
- (○) A complex black box result due to [Mö02].
- (†) Realisability interpretation by *fixed points as SO types*.
  - (Considerable) specialisation of HA2 → F.

μPA is (informally) the extension of PA by μ and ν formulas.

μPA is (informally) the extension of PA by μ and ν formulas.

Axioms:

(pre) $\varphi(\mu\varphi) \subseteq \mu\varphi$.

(ind) $\varphi(\psi) \subseteq \psi) \to (\mu\varphi \subseteq \psi)$

Perspective: $\mu\varphi = \bigcap\{A \supseteq \varphi(A)\}$.

μPA is (informally) the extension of PA by μ and ν formulas.

Axioms:

(pre) $\varphi(\mu\varphi) \subseteq \mu\varphi$.

(ind) $\varphi(\psi) \subseteq \psi) \to (\mu\varphi \subseteq \psi)$

Perspective: $\mu\varphi = \bigcap\{A \supseteq \varphi(A)\}$.

Theorem (Essentially [Mö02])

Π^1_2-CA_0 *is arithmetically conservative over* μPA.

μPA is (informally) the extension of PA by μ and ν formulas.

Axioms:

(pre) $\varphi(\mu\varphi) \subseteq \mu\varphi$.

(ind) $\varphi(\psi) \subseteq \psi) \to (\mu\varphi \subseteq \psi)$

Perspective: $\mu\varphi = \bigcap\{A \supseteq \varphi(A)\}$.

Theorem (Essentially [Mö02])

Π^1_2-CA_0 *is arithmetically conservative over* μPA.

Proposition (Implicit in [Tup04])

μPA *is* Π^0_2*-conservative over an intuitionistic variant,* μHA.

Idea: specialises Π^0_2-conservativity of PA2 over HA2.

Define a judgement $\cdot\,\mathbf{r}\,\cdot \subseteq \langle \mu\mathsf{LJ}^{-} \rangle \times \mathrm{Form}$ and a $\mathtt{t} : \mathrm{Form} \to \mathrm{Type}$ s.t.:

Theorem

If $\mu\mathsf{HA} \vdash \varphi$ then there is $P \,\mathbf{r}\, \varphi$ with $\mu\mathsf{LJ}^{-} \vdash P : \mathtt{t}(\varphi)$.

Define a judgement $\cdot\mathbf{r}\cdot \subseteq \langle \mu\mathsf{LJ}^{-} \rangle \times \mathrm{Form}$ and a $\mathtt{t} : \mathrm{Form} \to \mathrm{Type}$ s.t.:

Theorem

If $\mu\mathsf{HA} \vdash \varphi$ then there is $P\,\mathbf{r}\,\varphi$ with $\mu\mathsf{LJ}^{-} \vdash P : \mathtt{t}(\varphi)$.

Corollary

The provably total recursive functions of $\mu\mathsf{HA}$ are definable in $\mu\mathsf{LJ}^{-}$.

Realisability candidates:
Binary relation $\cdot A \cdot \subseteq \langle \mu\mathsf{LJ}^- \rangle \times \mathbb{N}$ with nice properties.

- $t \,\mathbf{r}\, \underline{m} = \underline{n}$ if $\underline{m} =_{\mathrm{r}} t =_{\mathrm{r}} \underline{n}$.
- $t \,\mathbf{r}\, \varphi_0 \wedge \varphi_1$ if $\mathsf{p}_0 t \,\mathbf{r}\, \varphi_0$ and $\mathsf{p}_1 t \,\mathbf{r}\, \varphi_1$.
- $t \,\mathbf{r}\, A\underline{n}$ if tAn.
- $t \,\mathbf{r}\, \exists x \varphi(x)$ if there is $n \in \mathbb{N}$ with $\mathsf{p}_0 t =_{\mathrm{r}} \underline{n}$ and $\mathsf{p}_1 t \,\mathbf{r}\, \varphi(\underline{n})$.
- $t \,\mathbf{r}\, \varphi \to \psi$ if, whenever $s \,\mathbf{r}\, \varphi$, we have $ts \,\mathbf{r}\, \psi$.
- $t \,\mathbf{r}\, \forall x(\varphi(x) \to \psi(x))$ if whenever $u \,\mathbf{r}\, (\underline{n} = \underline{n} \wedge \varphi(\underline{n}))$ we have $tu \,\mathbf{r}\, \psi(\underline{n})$.
- $t \,\mathbf{r}\, \forall x \varphi(x)$, where φ is not a $\to$-formula, if for all $n \in \mathbb{N}$ we have $t\underline{n} \,\mathbf{r}\, \varphi(\underline{n})$.
- $t \,\mathbf{r}\, \underline{n} \in \mu X \lambda x \varphi(X, x)$ if $t \,\mathbf{r}\, \forall x(\varphi(A, x) \to Ax) \to A\underline{n}$ for all A.

A varies over realisability candidates, subsets of $\langle \mu\mathsf{LJ}^- \rangle \times \mathbb{N}$ with 'nice' properties.

Realisability candidates:
Binary relation $\cdot A \cdot \subseteq \langle \mu \mathsf{LJ}^- \rangle \times \mathbb{N}$ with nice properties.

- $t \mathbf{r} \underline{m} = \underline{n}$ if $\underline{m} =_r t =_r \underline{n}$.
- $t \mathbf{r} \varphi_0 \wedge \varphi_1$ if $\mathsf{p}_0 t \mathbf{r} \varphi_0$ and $\mathsf{p}_1 t \mathbf{r} \varphi_1$.
- $t \mathbf{r} A\underline{n}$ if tAn.
- $t \mathbf{r} \exists x \varphi(x)$ if there is $n \in \mathbb{N}$ with $\mathsf{p}_0 t =_r \underline{n}$ and $\mathsf{p}_1 t \mathbf{r} \varphi(\underline{n})$.
- $t \mathbf{r} \varphi \to \psi$ if, whenever $s \mathbf{r} \varphi$, we have $ts \mathbf{r} \psi$.
- $t \mathbf{r} \forall x(\varphi(x) \to \psi(x))$ if whenever $u \mathbf{r} (\underline{n} = \underline{n} \wedge \varphi(\underline{n}))$ we have $tu \mathbf{r} \psi(\underline{n})$.
- $t \mathbf{r} \forall x \varphi(x)$, where φ is not a $\to$-formula, if for all $n \in \mathbb{N}$ we have $t\underline{n} \mathbf{r} \varphi(\underline{n})$.
- $t \mathbf{r} \underline{n} \in \mu X \lambda x \varphi(X, x)$ if $t \mathbf{r} \forall x(\varphi(A, x) \to Ax) \to A\underline{n}$ for all A.

A varies over realisability candidates, subsets of $\langle \mu \mathsf{LJ}^- \rangle \times \mathbb{N}$ with 'nice' properties.

Perspectives:

- μ, ν viewed under their SO encodings (unlike [BT21]).
- ...a (considerable) specialisation of $\mathsf{HA2} \to \mathsf{F}$.
- Must inline relativisation to N (unlike [BT21]).

Semantics: a computability model

Definition (Type structure)

A varies over sets of coterms with 'nice' properties.

$$\|\sigma \to \tau\| := \{t \mid \forall s \in \|\sigma\|.\, ts \in \|\tau\|\}$$
$$\|\mu X \sigma(X)\| := \mu[A \mapsto \|\sigma(A)\|]$$

$$\|A\| := A$$
$$\|N\| := \{t \mid \exists n \in \mathbb{N}.\, t =^{\eta}_{r'} \underline{n}\}$$
$$\|\sigma \times \tau\| := \{t \mid \mathsf{p}_0 t \in \|\sigma\| \;\&\; \mathsf{p}_1 t \in \|\tau\|\}$$

Definition (Type structure)

A varies over sets of coterms with 'nice' properties.

$$\|\sigma \to \tau\| := \{t \mid \forall s \in \|\sigma\|.\, ts \in \|\tau\|\}$$
$$\|\mu X \sigma(X)\| := \mu[A \mapsto \|\sigma(A)\|]$$

$$\|A\| := A$$
$$\|N\| := \{t \mid \exists n \in \mathbb{N}.\, t =^{\eta}_{r'} \underline{n}\}$$
$$\|\sigma \times \tau\| := \{t \mid \mathsf{p}_0 t \in \|\sigma\| \,\&\, \mathsf{p}_1 t \in \|\tau\|\}$$

Theorem

If $\mathsf{C}\mu\mathsf{LJ}^- \vdash P : \tau$ *then* $P \in \|\tau\|$.

Definition (Type structure)

A varies over sets of coterms with 'nice' properties.

$$\|\sigma \to \tau\| := \{t \mid \forall s \in \|\sigma\|.\, ts \in \|\tau\|\}$$
$$\|\mu X \sigma(X)\| := \mu[A \mapsto \|\sigma(A)\|]$$

$$\|A\| := A$$
$$\|N\| := \{t \mid \exists n \in \mathbb{N}.\, t =^{\eta}_{r'} \underline{n}\}$$
$$\|\sigma \times \tau\| := \{t \mid \mathsf{p}_0 t \in \|\sigma\| \;\&\; \mathsf{p}_1 t \in \|\tau\|\}$$

Theorem

If $\mathsf{C}\mu\mathsf{LJ}^- \vdash P : \tau$ *then* $P \in \|\tau\|$.

Proof idea.

- *Otherwise*, iteratively construct an *infinite branch* and $\|\cdot\|$-inputs witnessing $\|\cdot\|$-non-membership. Need 'general inputs' for two-sided sequents.
- The progressing condition induces from this an infinite descending sequence of ordinals approximating $\|\cdot\|$-non-membership. □

Definition (Type structure)

A varies over sets of coterms with 'nice' properties.

$$\|\sigma \to \tau\| := \{t \mid \forall s \in \|\sigma\|.\, ts \in \|\tau\|\}$$
$$\|\mu X \sigma(X)\| := \mu[A \mapsto \|\sigma(A)\|]$$

$$\|A\| := A$$
$$\|N\| := \{t \mid \exists n \in \mathbb{N}.\, t =^{\eta}_{r'} \underline{n}\}$$
$$\|\sigma \times \tau\| := \{t \mid \mathsf{p}_0 t \in \|\sigma\| \;\&\; \mathsf{p}_1 t \in \|\tau\|\}$$

Theorem

If $\mathsf{C}\mu\mathsf{LJ}^- \vdash P : \tau$ *then* $P \in \|\tau\|$.

Proof idea.

- *Otherwise*, iteratively construct an *infinite branch* and $\|\cdot\|$-inputs witnessing $\|\cdot\|$-non-membership. Need 'general inputs' for two-sided sequents.
- The progressing condition induces from this an infinite descending sequence of ordinals approximating $\|\cdot\|$-non-membership. □

Corollary

if $\mathsf{C}\mu\mathsf{LJ}^- \vdash P : N \to N$ *then* P *defines a total function* $\mathbb{N} \to \mathbb{N}$.

- Model $\|\cdot\|$ and totality argument can be formalised (non-uniformly) in Π^1_2-CA_0.
- Requires novel reverse mathematics of fixed point theorems.

- Model $\| \cdot \|$ and totality argument can be formalised (non-uniformly) in Π^1_2-CA_0.
- Requires novel reverse mathematics of fixed point theorems.

Theorem (Π^1_2-CA_0)
If $\varphi(X, x) \in \Delta^{1,+}_2(X, \vec{X})$ then $\mu X \lambda x \varphi(X, x) \in \Delta^{1,+}_2(\vec{X})$.
Can be seen as a partial arithmetisation of [Lub93].

Proof crucially relies on $\mu\varphi = \bigcap\{A \supseteq \varphi(A)\} = \bigcup_\alpha \varphi^\alpha$.

- Model $\|\cdot\|$ and totality argument can be formalised (non-uniformly) in Π^1_2-CA_0.
- Requires novel reverse mathematics of fixed point theorems.

Theorem (Π^1_2-CA_0)
If $\varphi(X,x) \in \Delta^{1,+}_2(X,\vec{X})$ *then* $\mu X \lambda x \varphi(X,x) \in \Delta^{1,+}_2(\vec{X})$.

Can be seen as a partial arithmetisation of [Lub93].

Proof crucially relies on $\mu\varphi = \bigcap\{A \supseteq \varphi(A)\} = \bigcup_\alpha \varphi^\alpha$.

Corollary (Π^1_2-CA_0)
$\|\sigma\| \in \Delta^1_2$.

Outline

We settled both questions we started with!

Theorem ([CD23])

μLJ and CμLJ define just the functions provably recursive in Π^1_2-CA_0.

We settled both questions we started with!

Theorem ([CD23])

μLJ and CμLJ define just the functions provably recursive in Π^1_2-CA_0.

Corollary

…and so does μMALL and CμMALL.

We settled both questions we started with!

Theorem ([CD23])

μLJ and CμLJ define just the functions provably recursive in Π^1_2-CA_0.

Corollary

…and so does μMALL and CμMALL.

Emerging consensus: circularity and (co)recursion are *equally powerful*.
…but circular proofs may be more succinct, cf. [Das20, Das21b].

We settled both questions we started with!

Theorem ([CD23])

μLJ and CμLJ define just the functions provably recursive in Π^1_2-CA_0.

Corollary

...and so does μMALL and CμMALL.

Emerging consensus: circularity and (co)recursion are *equally powerful*.
...but circular proofs may be more succinct, cf. [Das20, Das21b].

- What about (very) weak systems? (cf., e.g., [CD22])
- Relating FOL and type systems via proof interpretations? (w.i.p.w. Tom Powell)

We settled both questions we started with!

Theorem ([CD23])

μLJ and CμLJ define just the functions provably recursive in Π^1_2-CA_0.

Corollary

…and so does μMALL and CμMALL.

Emerging consensus: circularity and (co)recursion are *equally powerful*.
…but circular proofs may be more succinct, cf. [Das20, Das21b].

- What about (very) weak systems? (cf., e.g., [CD22])
- Relating FOL and type systems via proof interpretations? (w.i.p.w. Tom Powell)

Thank you.

References I

Bahareh Afshari and Graham E. Leigh.

Cut-free completeness for modal mu-calculus.

In *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017*, pages 1–12. IEEE Computer Society, 2017.

David Baelde, Amina Doumane, Denis Kuperberg, and Alexis Saurin.

Bouncing threads for circular and non-wellfounded proofs: Towards compositionality with circular proofs.

In Christel Baier and Dana Fisman, editors, *LICS '22, Haifa, Israel, August 2 - 5, 2022*, pages 63:1–63:13. ACM, 2022.

David Baelde, Amina Doumane, and Alexis Saurin.

Infinitary proof theory: the multiplicative additive case.

In Jean-Marc Talbot and Laurent Regnier, editors, *CSL '16, August 29 - September 1, 2016, Marseille, France*, volume 62 of *LIPIcs*, pages 42:1–42:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.

Stefano Berardi and Makoto Tatsuta.

Classical system of martin-löf's inductive definitions is not equivalent to cyclic proof system.

In Javier Esparza and Andrzej S. Murawski, editors, *Foundations of Software Science and Computation Structures - 20th International Conference, FOSSACS 2017, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2017, Uppsala, Sweden, April 22-29, 2017, Proceedings*, volume 10203 of *Lecture Notes in Computer Science*, pages 301–317, 2017.

Stefano Berardi and Makoto Tatsuta.

Equivalence of inductive definitions and cyclic proofs under arithmetic.

In *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017*, pages 1–12. IEEE Computer Society, 2017.

Ulrich Berger and Hideki Tsuiki.

Intuitionistic fixed point logic.

Annals of Pure and Applied Logic, 172(3):102903, 2021.

Gianluca Curzi and Anupam Das.

Cyclic implicit complexity.

In Christel Baier and Dana Fisman, editors, *LICS '22: 37th Annual ACM/IEEE Symposium on Logic in Computer Science, Haifa, Israel, August 2 - 5, 2022*, pages 19:1–19:13. ACM, 2022.

Gianluca Curzi and Anupam Das.
Computational expressivity of (circular) proofs with fixed points.
In *LICS*, pages 1–13, 2023.

Pierre Clairambault.
Least and greatest fixpoints in game semantics.
In Ralph Matthes and Tarmo Uustalu, editors, *FICS '09, Coimbra, Portugal, September 12-13, 2009*, pages 39–45. Institute of Cybernetics, 2009.

Anupam Das.
On the logical complexity of cyclic arithmetic.
Logical Methods in Computer Science, Volume 16, Issue 1, January 2020.

Anupam Das.
A circular version of Gödel's T and its abstraction complexity.
CoRR, abs/2012.14421, 2021.

Anupam Das.

On the logical strength of confluence and normalisation for cyclic proofs.

In Naoki Kobayashi, editor, *FSCD '21, July 17-24, 2021, Buenos Aires, Argentina (Virtual Conference)*, volume 195 of *LIPIcs*, pages 29:1–29:23. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.

Anupam Das and Abhishek De.

A proof theory of right-linear (omega-)grammars via cyclic proofs.

CoRR, abs/2401.13382, 2024.

Anupam Das and Marianna Girlando.

Cyclic proofs, hypersequents, and transitive closure logic.

In Jasmin Blanchette, Laura Kovács, and Dirk Pattinson, editors, *Automated Reasoning - 11th International Joint Conference, IJCAR 2022, Haifa, Israel, August 8-10, 2022, Proceedings*, volume 13385 of *Lecture Notes in Computer Science*, pages 509–528. Springer, 2022.

Anupam Das and Lukas Melgaard.

Cyclic proofs for arithmetical inductive definitions.

In Marco Gaboardi and Femke van Raamsdonk, editors, *8th International Conference on Formal Structures for Computation and Deduction, FSCD 2023, July 3-6, 2023, Rome, Italy*, volume 260 of *LIPIcs*, pages 27:1–27:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2023.

Anupam Das and Damien Pous.
A cut-free cyclic proof system for kleene algebra.
In Renate A. Schmidt and Cláudia Nalon, editors, *Automated Reasoning with Analytic Tableaux and Related Methods - 26th International Conference, TABLEAUX 2017, Brasília, Brazil, September 25-28, 2017, Proceedings*, volume 10501 of *Lecture Notes in Computer Science*, pages 261–277. Springer, 2017.

Anupam Das and Damien Pous.
Non-wellfounded proof theory for (kleene+action)(algebras+lattices).
In Dan R. Ghica and Achim Jung, editors, *27th EACSL Annual Conference on Computer Science Logic, CSL 2018, September 4-7, 2018, Birmingham, UK*, volume 119 of *LIPIcs*, pages 19:1–19:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018.

Jérôme Fortier and Luigi Santocanale.
Cuts for circular proofs: semantics and cut-elimination.
In Simona Ronchi Della Rocca, editor, *Computer Science Logic 2013 (CSL 2013), CSL 2013, September 2-5, 2013, Torino, Italy*, volume 23 of *LIPIcs*, pages 248–262. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2013.

Denis Kuperberg, Laureline Pinault, and Damien Pous.
Cyclic proofs, system T, and the power of contraction.
Proc. ACM Program. Lang., 5(POPL):1–28, 2021.

Martin Lange.
Games for modal and temporal logics.
PhD thesis, University of Edinburgh, UK, 2003.

Robert S. Lubarsky.
μ-definable sets of integers.
The Journal of Symbolic Logic, 58(1):291–313, 1993.

Michael Möllerfeld.
Generalized inductive definitions. The μ-calculus and Π^1_2-comprehension.
PhD thesis, 2002.
University of Münster,
https://nbn-resolving.de/urn:nbn:de:hbz:6-85659549572.

Damian Niwinski and Igor Walukiewicz.
Games for the mu-calculus.
Theor. Comput. Sci., 163(1&2):99–116, 1996.

Luigi Santocanale.
A calculus of circular proofs and its categorical semantics.
In Mogens Nielsen and Uffe Engberg, editors, *Foundations of Software Science and Computation Structures, 5th International Conference, FOSSACS 2002. Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2002 Grenoble, France, April 8-12, 2002, Proceedings*, volume 2303 of *Lecture Notes in Computer Science*, pages 357–371. Springer, 2002.

Daniyar S. Shamkanov.
Circular proofs for gödel-löb logic.
CoRR, abs/1401.4002, 2014.

Alex Simpson.
Cyclic arithmetic is equivalent to peano arithmetic.
In Javier Esparza and Andrzej S. Murawski, editors, *FOSSACS '17, Held as Part of ETAPS '17, Uppsala, Sweden, April 22-29, 2017, Proceedings*, volume 10203 of *Lecture Notes in Computer Science*, pages 283–300, 2017.

Thomas Studer.
On the proof theory of the modal mu-calculus.
Studia Logica: An International Journal for Symbolic Logic, 89(3):343–363, 2008.

References VIII

Sergei Tupailo.
On the intuitionistic strength of monotone inductive definitions.
J. Symb. Log., 69(3):790–798, 2004.