Cyclic Proofs for iGL via Corecursion

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Framework:

1. Cyclic/Non-wellfounded proof theory.
2. (Minimal) Intuitionistic modal logic
3. Categorical approach to corecursion

Our goals are:

1. Follow Shamkanov's Ideas for GL, in particular using a corecursive translation from a finitary to a non-wellfounded system to later obtain cyclic proofs.
2. Use G3i as our propositional logic base.

Outline

Coalgebra

$\mathcal{F}$-coalgebra

Let $\mathcal{F}$ be a endofunctor over a category $\mathcal{C}$. A $\mathcal{F}$-coalgebra consists in a pair (C, α) where C is an object of $\mathcal{C}$ and $\alpha : C \longrightarrow \mathcal{F}(C)$.

We can think of α as destructing objects of C into the structure described by $\mathcal{F}$.

$\mathcal{F}$-coalgebra morphism

If $(C, \alpha), (D, \beta)$ are $\mathcal{F}$-coalgebras, we say that $f \in \mathcal{C}(C, D)$ is a coalgebra morphism iff $\beta \circ f = \mathcal{F}(f) \circ \alpha$.

Coalgebras with coalgebra-morphisms (for a fixed endofunctor $\mathcal{F}$) forms a category. If this category has a final object we call it the final coalgebra, and the unique function from (C, α) to the final coalgebra the function defined by corecursion from α.

Tree endofunctor

Particularly, we will use the endofunctor $\mathcal{T} : \mathbf{Set} \longrightarrow \mathbf{Set}$ defined as:

$$\mathcal{T}(X) = (\mathbb{S} \times \mathcal{R}) \times X^*,$$
$$\mathcal{T}(f : X \longrightarrow Y) = \mathrm{id}_{\mathbb{S}\times\mathcal{R}} \times \mathrm{map}_f.$$

This endofunctor has a final coalgebra, which we can think as $\mathcal{S} \times \mathcal{R}$-labelled non-wellfounded (finitely-branching) trees, let us denote the collection of these trees as $\mathbb{T}^\infty$.
Finite trees are the special case of these trees where the number of nodes is finite, we denote them as $\mathbb{T}^{<\omega}$. Note that they form an initial algebra of $\mathcal{T}$.

Base: G3i

Sequents are ordered pairs of a multiset of formulas and a formula, usually denoted as $\Gamma \Rightarrow \phi$. The rules of G3i are the following:

$$\frac{}{\Gamma, p \Rightarrow p}\,\mathsf{Ax} \qquad \frac{}{\Gamma, \bot \Rightarrow \phi}\,\mathsf{L}\bot$$

$$\frac{\phi, \psi, \Gamma \Rightarrow \chi}{\phi \wedge \psi, \Gamma \Rightarrow \chi}\,\mathsf{L}\wedge \qquad \frac{\Gamma \Rightarrow \phi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \wedge \psi}\,\mathsf{R}\wedge$$

$$\frac{\phi, \Gamma \Rightarrow \chi \quad \psi, \Gamma \Rightarrow \chi}{\phi \vee \psi, \Gamma \Rightarrow \chi}\,\mathsf{L}\vee \qquad \frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \phi \vee \psi}\,\mathsf{R}\vee \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \vee \psi}\,\mathsf{R}\vee$$

$$\frac{\phi \to \psi, \Gamma \Rightarrow \phi \quad \psi, \Gamma \Rightarrow \chi}{\phi \to \psi, \Gamma \Rightarrow \chi}\,\mathsf{L}\to \qquad \frac{\phi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \to \psi}\,\mathsf{R}\to$$

Finitary System

We define the following sequent rule:

$$\frac{\Gamma, \Box\Gamma, \Box\phi \Rightarrow \phi}{\Pi, \Box\Gamma \Rightarrow \Box\phi}\,\Box_{\mathsf{GL}}$$

Then, a proof of iGL is just a finite tree generated by the rules of G3i $+\ \Box_{\mathsf{GL}}$.
We remark that the usual rules of weakening, contraction and cut are admissible in iGL.

Non-wellfounded/Cyclic Systems

We define the following sequent rule:

$$\frac{\Gamma, \Box\Gamma \Rightarrow \phi}{\Pi, \Box\Gamma \Rightarrow \Box\phi}\,\Box_{\mathsf{iK4}}$$

A non-wellfounded proof of iK4 is a non-wellfounded tree generated by the rules of $\mathsf{G3i} + \Box_{\mathsf{iK4}}$ such that in any infinite branch the rule $\Box_{\mathsf{iK4}}$ is applied infinitely many times. We denote this system as $\mathsf{iK4}^{\infty}$.
A cyclic proof of iK4 is a finite tree with backedges generated by the rules of $\mathsf{G3i} + \Box_{\mathsf{iK4}}$ such that the backedges go from a leaf to a node below it with the same sequent and in the cycle generated by the backedge there is an application of $\Box_{\mathsf{iK4}}$. We denote this system as $\mathsf{iK4}^{\circ}$.

Löb's rule in finitary system

The following rule is admissible in iGL:

$$\dfrac{\Gamma, \Box\Gamma, \Box\phi \Rightarrow \phi}{\Gamma, \Box\Gamma \Rightarrow \phi}\,\text{Löb}$$

Proof.

$$\dfrac{\dfrac{\Gamma, \Box\Gamma, \Box\phi \Rightarrow \phi}{\Gamma, \Box\Gamma \Rightarrow \Box\phi}\,\Box_{\mathsf{GL}} \qquad \Gamma, \Box\Gamma, \Box\phi \Rightarrow \phi}{\Gamma, \Box\Gamma \Rightarrow \phi}\,\text{cut}$$

□

Original Shamkanov's proof for GL

Shamkanov's original work has two differences with our setting: he works in a classical setting and he did not need any explicit branch/cyclic condition.

Then, he proceeds as follows:

1. Using admissibility of rules in the finitary system, particularly Löb's rule, he provides a corecursive translation from the finitary system to the non-wellfounded system.
2. Using admissibility of contraction in the non-wellfounded system he provides a corecursive translation from the non-wellfounded system to the non-wellfounded system where proofs has a special shape. Using this special shape he transforms the non-wellfounded proofs in cyclic proofs.
3. Using (a very smart) induction in the height of the cyclic proof he transforms the cyclic proof into Hilbert-style proofs, which is known to be an equivalent system to the finitary sequent system.

Iemhoff's Approach

Iemhoff studies a general setting, but again it is without explicit branch/cyclic condition. This will not work in our system since if we do not impose any condition the following is a valid derivation:

$$\dfrac{p \to q, q \to p \Rightarrow p \qquad \dfrac{\dfrac{}{q, q \to p \Rightarrow q}\,\mathsf{Ax} \qquad \dfrac{}{q, p \Rightarrow p}\,\mathsf{Ax}}{q, q \to p \Rightarrow p}\,{\to}\mathsf{L}}{p \to q, q \to p \Rightarrow p}\,{\to}\mathsf{L}$$

Our translation

Given a multiset Γ we divide it into a set Γ^s and a multiset Γ^m such that $\Gamma^s \cup \Gamma^m = \Gamma$.

We define two functions from proofs in iGL to proofs in iGL such that:

1. If $\pi \vdash \Gamma, \Box\Gamma \Rightarrow \phi$ then contract$(\pi) \vdash \Gamma^s, \Box\Gamma^s$.
2. If $\pi \vdash \Gamma, \Box\Gamma, \Box\phi \Rightarrow \phi$ then löb$(\pi) \vdash \Gamma, \Box\Gamma \Rightarrow \phi$.

Then, we define $\alpha : \mathbb{T}^{<\omega} \longrightarrow \mathcal{T}(\mathbb{T}^{<\omega})$ as:

$$\dfrac{\begin{array}{c}\pi\\ \Gamma, \Box\Gamma, \Box\phi \Rightarrow \phi\end{array}}{\Gamma, \Box\Gamma \Rightarrow \Box\phi}\,\Box_{\mathsf{GL}} \quad\longmapsto\quad \dfrac{\begin{array}{c}\text{contract}(\text{löb}(\pi))\\ \Gamma^s, \Box\Gamma^s \Rightarrow \phi\end{array}}{\Gamma, \Box\Gamma \Rightarrow \Box\phi}\,\Box_{\mathsf{K4}}$$

And for the rest of cases it keeps the same conclusion, rule and immediate subproofs. Let β be the function defined by corecursion from α.

That the result is a preproof of iK4$^\infty$ is easy to check.

Why branch condition is fulfilled?

Let us have τ proof in iGL and $\pi = \beta(\tau)$. Given $(b)_{i\in\mathbb{N}}$ we a branch of π we can find an infinite sequence $(\tau_i)_{i\in\mathbb{N}}$ such that:

1. π-subtree$(b{\restriction}i) = \beta(\tau_i)$.
2. $\tau_0 = \tau$ and $\tau_{n+1} = \tau_n^{b_i}$ where $\alpha(\tau_n) = ((S_n, R_n), [\tau_n^0, \dots, \tau_n^{k-1}])$.

We also notice that the height of $\tau_n^0, \dots, \tau_n^{k-1}$ is stricly lower than the height of τ_n unless $R_n = \Box_{\mathsf{K4}}$.
We can conclude that it is impossible that a branch of π does not progress infinitely often.

From non-wellfounded to cyclic

To transform a non-wellfounded proof of shape $\beta(\tau)$ to a cyclic proof we use two observations:

1. Thanks to the shape of the rules of $\mathrm{iK4}^\infty$ we have the subformula property.
2. The premise of a $\Box_{K4}$ rule is determined by a set of formulas and a formula.

These two conditions guarantees that if we cut the non-wellfounded tree whenever we found a repetition of a premise of a $\Box_K$ rule we obtain a finite tree. Clearly, all the internal nodes are still instances of the rules and the leaves which are not instances of the rules must appear below with the $\Box_{K4}$ rule applied between the repetitions (since this leaf must be a premise of a $\Box_K$ application).

Recursive approach

Why not use standard recursion? You can! It only changes where the admissibility of rules is needed. We decide to go this way since we know the admissibility of more rules in the finitary system than in the non-wellfounded.

Current/Future work

1. Study if it is possible to handle harder branch/cyclic conditions when doing a corecursion from finite system (using admissible rules of finite system) to non-wellfounded, or/and
2. Study if it is possible to provide general methods to make branch/cyclic conditions easier.
3. (Done) Use pure corecursive arguments to provide an alternative proof of cut-elimination for non-wellfounded version of Grz (and possibly also wGrz).
4. (In progress) Try pure corecursive arguments to provide an alternative proof of cut-elimintion for a trace-based system.

3 and 4 is joint work with Lukas Zenger, University of Bern.