Cut-elimination for the circular modal μ-calculus: the benefits of linearity

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BACKGROUND

Formulas and semantic for the modal μ-calculus

Formulas of the modal μ-calculus

$$A, B ::= A \vee B \mid A \wedge B \mid A \to B \mid T \mid F \mid \Box A \mid \Diamond A \mid X \in \mathcal{V} \mid \mu X.A \mid \nu X.A.$$

Formulas and semantic for the modal μ-calculus

Formulas of the modal μ-calculus

$$A, B ::= A \vee B \mid A \wedge B \mid A \rightarrow B \mid T \mid F \mid \square A \mid \lozenge A \mid X \in \mathcal{V} \mid \mu X.A \mid \nu X.A.$$

Semantic of the modal μ-calculus

Semantics comes from a transition system. Truth-value is defined for states:

$$[\![A \vee B]\!]_s = [\![A]\!]_s \vee [\![B]\!]_s \quad [\![A \wedge B]\!]_s = [\![A]\!]_s \wedge [\![B]\!]_s$$

$$[\![A \rightarrow B]\!]_s = [\![A]\!]_s \rightarrow [\![B]\!]_s \qquad [\![\top]\!] = \top \qquad [\![F]\!] = \bot$$

$[\![\square A]\!]_s$ holds for s if A holds in each states attainable from s.

$[\![\lozenge A]\!]_s$ holds for s if A holds in one of the states attainable from s.

$[\![\nu X.A]\!]_S$ is the greatest fixed-point of $S \rightarrow [\![A]\!]_{[X \text{ is true on each } S]}$

$[\![\mu X.A]\!]_S$ is the least fixed-point of $S \rightarrow [\![A]\!]_{[X \text{ is true on each } S]}$

Proof theory of modal μ-calculus

Modal μ-calculus proof theory has been studied...

Finitary systems

Fixed-point rule à la Park, completeness from Kozen '83.

Systems with ω-rule

Works from Kozen '88, Jäger, Kretz & Stüder '08. Mints '12, Mints & Studer '12, Brünnler & Studer '12

Non-wellfounded systems

Works from Stirling '14 and restriction to regular proofs studied by Afshari & Leigh '16.

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... None of them have a proper syntactic cut-elimination theorem on the full modal μ-calculus.

Fixed-point and the non-wellfounded system μLK^∞

Rules of μLK^∞

We add to LK two fixed-point rules:

$$\frac{\vdash A[X := \mu X.A], \Gamma}{\vdash \mu X.A, \Gamma}\ \mu \qquad \frac{\vdash A[X := \nu X.A], \Gamma}{\vdash \nu X.A, \Gamma}\ \nu$$

Proofs of μLK^∞

Proofs are the trees co-inductively generated by the rules of μLK^∞ satisfying a validity criterion.

Some example of infinite proofs

$$\mathsf{Nat} := \mu X.\top \vee X$$

Inhabitant of natural number type

$$\pi_0 := \begin{prooftree}\AxiomC{}\RightLabel{$\top$}\UnaryInfC{$\vdash \top$}\RightLabel{$\vee_1$}\UnaryInfC{$\vdash \top \vee \mathsf{Nat}$}\RightLabel{$\mu$}\UnaryInfC{$\vdash \mathsf{Nat}$}\end{prooftree} \qquad \pi_{n+1} := \begin{prooftree}\AxiomC{π_n}\noLine\UnaryInfC{$\vdash \mathsf{Nat}$}\RightLabel{$\vee_2$}\UnaryInfC{$\vdash \top \vee \mathsf{Nat}$}\RightLabel{$\mu$}\UnaryInfC{$\vdash \mathsf{Nat}$}\end{prooftree}$$

$$\pi_{\mathsf{succ}} :=$$

$$\begin{prooftree}\AxiomC{}\RightLabel{ax}\UnaryInfC{$\top \vdash \top$}\RightLabel{$\vee_r^1$}\UnaryInfC{$\top \vdash \top \vee \mathsf{Nat}$}\RightLabel{$\mu_r$}\UnaryInfC{$\top \vdash \mathsf{Nat}$}\RightLabel{$\vee_r^2$}\UnaryInfC{$\top \vdash \top \vee \mathsf{Nat}$}\RightLabel{$\mu_r$}\UnaryInfC{$\top \vdash \mathsf{Nat}$}\AxiomC{$\mathsf{Nat} \vdash \mathsf{Nat}$}\RightLabel{$\vee_r^2$}\UnaryInfC{$\mathsf{Nat} \vdash \top \vee \mathsf{Nat}$}\RightLabel{$\mu_r$}\UnaryInfC{$\mathsf{Nat} \vdash \mathsf{Nat}$}\RightLabel{$\vee_l$}\BinaryInfC{$\top \vee \mathsf{Nat} \vdash \mathsf{Nat}$}\RightLabel{$\mu_l$}\UnaryInfC{$\mathsf{Nat} \vdash \mathsf{Nat}$}\end{prooftree}$$

Some example of infinite proofs

$$\mathrm{Nat} := \mu X.\top \vee X$$

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$$\pi_0 := \cfrac{\cfrac{\cfrac{}{\vdash \top}\,\top}{\vdash \top \vee \mathrm{Nat}}\,\vee_1}{\vdash \mathrm{Nat}}\,\mu \qquad \pi_{n+1} := \cfrac{\cfrac{\cfrac{\pi_n}{\vdash \mathrm{Nat}}}{\vdash \top \vee \mathrm{Nat}}\,\vee_2}{\vdash \mathrm{Nat}}\,\mu$$

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$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\top \vdash \top}\,\mathrm{ax}}{\top \vdash \top \vee \mathrm{Nat}}\,\vee_r^1}{\top \vdash \mathrm{Nat}}\,\mu_r}{\top \vdash \top \vee \mathrm{Nat}}\,\vee_r^2}{\top \vdash \mathrm{Nat}}\,\mu_r \qquad \cfrac{\cfrac{\mathrm{Nat} \vdash \mathrm{Nat}}{\mathrm{Nat} \vdash \top \vee \mathrm{Nat}}\,\vee_r^2}{\mathrm{Nat} \vdash \mathrm{Nat}}\,\mu_r}{\top \vee \mathrm{Nat} \vdash \mathrm{Nat}}\,\vee_l}{\vdash \mathrm{Nat}^{\perp}, \mathrm{Nat}}\,\mu_l$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 0, \top}\,\mathsf{ax}}{\vdash 0, \top \vee \mathsf{Nat}}\,\vee^1}{\vdash 0, \mathsf{Nat}}\,\mu}{\vdash 0, \top \vee \mathsf{Nat}}\,\vee^2}{\vdash 0, \mathsf{Nat}}\,\mu \qquad \dfrac{\dfrac{\vdash \mathsf{Nat}^\perp, \mathsf{Nat}}{\vdash \mathsf{Nat}^\perp, \top \vee \mathsf{Nat}}\,\vee^2}{\vdash \mathsf{Nat}^\perp, \mathsf{Nat}}\,\mu}{\vdash 0 \wedge \mathsf{Nat}^\perp, \mathsf{Nat}}\,\wedge}{\vdash \mathsf{Nat}^\perp, \mathsf{Nat}}\,\nu$$

Modal fragment and the $\mu LK^\infty_\Box$ system

Derivation rules of $\mu LK^\infty_\Box$

Derivation rules of $\mu LK^\infty_\Box$ are the rules of μLK^∞, together with:

$$\frac{\vdash A, \Gamma}{\vdash \Box A, \Diamond \Gamma}\ \Box$$

Example

Taking $F := \nu X. \Diamond X$:

$$\cfrac{\cfrac{\vdash F^\perp, F}{\vdash \Box F^\perp, \Diamond F}\ \Box}{\vdash F^\perp, F}\ \mu, \nu$$

Formulas and derivation rules of $\mu\mathsf{MALL}^\infty$

$$F, G ::= F \parr G \mid F \otimes G \mid F \mathbin{\&} G \mid F \oplus G \mid \bot \mid 1 \mid \top \mid 0 \mid X \in \mathcal{V} \mid \mu X.F \mid \nu X.F.$$

$$\frac{}{\vdash A, A^\perp}\,\mathsf{ax} \qquad \frac{\vdash A, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,\mathsf{cut} \qquad \frac{\vdash \Gamma, B, A, \Delta}{\vdash \Gamma, A, B, \Delta}\,\mathsf{ex}$$

$$\frac{\vdash A, \Delta_1 \qquad \vdash B, \Delta_2}{\vdash A \otimes B, \Delta_1, \Delta_2}\,\otimes \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\,\parr$$

$$\frac{\vdash A_1, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\,\oplus^1 \qquad \frac{\vdash A_2, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\,\oplus^2 \qquad \frac{\vdash A_1, \Gamma \qquad \vdash A_2, \Gamma}{\vdash A_1 \mathbin{\&} A_2, \Gamma}\,\&$$

$$\frac{\vdash A[X := \mu X.A], \Gamma}{\vdash \mu X.A, \Gamma}\,\mu \qquad \frac{\vdash A[X := \nu X.A], \Gamma}{\vdash \nu X.A, \Gamma}\,\nu$$

Examples of infinite proofs in $\mu\mathsf{MALL}^\infty$

Stream of natural numbers

$$\mathsf{Stream\ Nat} := \nu X.\mathsf{Nat} \otimes X$$

$$\dfrac{\dfrac{\overset{\pi_0}{\vdash \mathsf{Nat}} \qquad \vdash \mathsf{Stream\ Nat}}{\vdash \mathsf{Nat} \otimes \mathsf{Stream\ Nat}}\,\otimes}{\vdash \mathsf{Stream\ Nat}}\,\nu$$

$$\dfrac{\dfrac{\overset{\pi_0}{\vdash \mathsf{Nat}} \qquad \dfrac{\dfrac{\overset{\pi_1}{\vdash \mathsf{Nat}} \qquad \dfrac{\dfrac{\overset{\pi_2}{\vdash \mathsf{Nat}} \qquad \overset{\vdots}{\vdash \mathsf{Stream\ Nat}}}{\vdash \mathsf{Nat} \otimes \mathsf{Stream\ Nat}}\,\otimes}{\vdash \mathsf{Stream\ Nat}}\,\nu}{\vdash \mathsf{Nat} \otimes \mathsf{Stream\ Nat}}\,\otimes}{\vdash \mathsf{Stream\ Nat}}\,\nu}{\vdash \mathsf{Nat} \otimes \mathsf{Stream\ Nat}}\,\otimes}{\vdash \mathsf{Stream\ Nat}}\,\nu$$

Exponentials

We add exponentials to $\mu\mathsf{MALL}^\infty$:

$$?A \quad \text{and} \quad !A$$

As well as the corresponding rules:

$$\frac{\vdash \Gamma}{\vdash ?A, \Gamma}\ ?_w \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\ ?_c \qquad \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ ?_d \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ !_p$$

This system is called $\mu\mathsf{LL}^\infty$

Cut-elimination steps of μLL^∞

$$
\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1}}{\vdash ?A, \Gamma_1}\,?_c \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2}}{\vdash !A^\perp, ?\Gamma_2}\,!_p}{\vdash \Gamma_1, ?\Gamma_2}\,\mathrm{cut}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1} \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2}}{\vdash !A^\perp, ?\Gamma_2}\,!_p}{\vdash ?A, \Gamma_1, ?\Gamma_2}\,\mathrm{cut} \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2}}{\vdash !A^\perp, ?\Gamma_2}\,!_p}{\vdash \Gamma_1, ?\Gamma_2, ?\Gamma_2}\,\mathrm{cut}}{\vdash \Gamma_1, ?\Gamma_2}\,?_c^{\#\Gamma_2}
$$

$$
\dfrac{\dfrac{\overset{\pi_1}{\vdash \Gamma_1}}{\vdash ?A, \Gamma_1}\,?_w \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2}}{\vdash !A^\perp, ?\Gamma_2}\,!_p}{\vdash \Gamma_1, ?\Gamma_2}\,\mathrm{cut}
\quad\rightsquigarrow\quad
\dfrac{\overset{\pi_1}{\vdash \Gamma_1}}{\vdash \Gamma_1, ?\Gamma_2}\,?_w^{\#\Gamma_2}
$$

Cut-elimination theorems

Cut-elimination of μALL$^\infty$ (Fortier & Santocanale 2013)

The cut-rule is admissible for Additive Linear Logic with fixpoints.

Cut-elimination of μMALL$^\infty$ (Baelde et al. 2016)

Each fair multi-cut reduction sequences of μMALL$^\infty$ are converging to a μMALL$^\infty$-cut-free proof.

Cut-elimination of μLL$^\infty$ (Saurin 2023)

Each fair multi-cut reduction sequences of μLL$^\infty$ are converging to a μLL$^\infty$-cut-free proof.

Goal

We want to prove cut-elimination in a syntactic way

$$
\begin{tikzcd}
\mu LK^\infty_\square \arrow[rr, "(-)^\bullet"] & & \mu LL^\infty_\square \arrow[d, "(-)^\circ"] \\
\mu MALL^\infty & & \mu LL^\infty \arrow[ll, "\text{TABLEAUX '23}"']
\end{tikzcd}
$$

$$\begin{tikzcd}
\mu LK^\infty_\square \arrow[rr, "(-)^\bullet"] & & \mu LL^\infty_\square \arrow[d, "(-)^\bullet"] \\
\mu MALL^\infty & & \mu LL^\infty \arrow[ll, "\text{TABLEAUX '23}"']
\end{tikzcd}$$

Translation of μLK$^\infty$ in μLL$^\infty$

$$
\begin{array}{rclcrcl}
(\mu X.A)^\bullet & := & !\mu X.?A^\bullet & \qquad & (\nu X.A)^\bullet & := & !\nu X.?A^\bullet \\
(A_1 \vee A_2)^\bullet & := & !(?A_1{}^\bullet \oplus ?A_2{}^\bullet) & & (A_1 \wedge A_2)^\bullet & := & !(?A_1{}^\bullet \,\&\, ?A_2{}^\bullet) \\
F^\bullet & := & !0 & & \top^\bullet & := & !\top \\
X^\bullet & := & !X & & a^\bullet & := & !a \\
(A_1 \to A_2)^\bullet & := & !(?A_1{}^\bullet \multimap ?A_2{}^\bullet) & & (\Gamma \vdash \Delta)^\bullet & := & \Gamma^\bullet \vdash ?\Delta^\bullet
\end{array}
$$

$$
\dfrac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\,\wedge_r \;\rightsquigarrow\; \dfrac{\dfrac{\Gamma^\bullet \vdash ?A^\bullet, ?\Delta^\bullet \qquad \Gamma^\bullet \vdash ?B^\bullet, ?\Delta^\bullet}{\Gamma^\bullet \vdash ?A^\bullet \,\&\, ?B^\bullet, ?\Delta^\bullet}\,\&_r}{\Gamma^\bullet \vdash ?!(?A^\bullet \,\&\, ?B^\bullet), ?\Delta^\bullet}\,?_d, !_p
$$

Cut-elimination for μLK$^\infty$

Cut-elimination for μLK$^\infty$ Saurin 2023

The cut-elimination system of μLK$^\infty$ is weakly normalizing.

$$\begin{tikzcd}
\mu LK^\infty & \mu LL^\infty \\
\pi = \pi_0 \arrow[r] \arrow[d, "*"] & \pi^\bullet = \theta = \theta_0 \arrow[d, "*"] \\
U(\theta_n) \arrow[d, "\infty"] & \theta_n \arrow[l] \arrow[d, "\infty"] \\
U(\theta_\infty) & \theta_\infty \arrow[l]
\end{tikzcd}$$

$$
\begin{tikzcd}
\mu LK^{\infty}_{\Box} \arrow[r, "(-)^{\bullet}"] & \mu LL^{\infty}_{\Box} \arrow[d, "(-)^{\circ}"] \\
\mu MALL^{\infty} & \mu LL^{\infty} \arrow[l]
\end{tikzcd}
$$

TABLEAUX '23

Naïve extension of the translation and issue with it

Let's consider the two-sided system μLL^∞ together with the two modal rules:

$$\frac{\Gamma, A \vdash \Delta}{\Box\Gamma, \Diamond A \vdash \Diamond\Delta}\ \Diamond \qquad \frac{\Gamma \vdash A, \Delta}{\Box\Gamma \vdash \Box A, \Diamond\Delta}\ \Box$$

We extend the translation, to get a translation from $\mu LK^\infty_\Box$ to $\mu LL^\infty_\Box$:

$$(\Box A)^\bullet := !\Box ?A^\bullet \qquad (\Diamond A)^\bullet := !\Diamond ?A^\bullet$$

Problem

Promotion rule

$$\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ !$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \Box A, \Diamond B}\ \Box$$

We start with the sequent:

$$\vdash ?!\Box ?A^\bullet, ?!\Diamond ?B^\bullet$$

Problem

Promotion rule

$$\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\;!$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \square A, \lozenge B}\;\square$$

We apply a dereliction a promotion:

$$\frac{\vdash ?!\square ?A^\bullet, \lozenge ?B^\bullet}{\vdash ?!\square ?A^\bullet, ?!\lozenge ?B^\bullet}\;?_{\mathrm{d}}, !$$

Problem

Promotion rule

$$\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ !$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \square A, \Diamond B}\ \square$$

We apply a dereliction again:

$$\dfrac{\dfrac{\vdash !\,\square\,?A^{\bullet}, \Diamond ?B^{\bullet}}{\vdash ?!\,\square\,?A^{\bullet}, \Diamond ?B^{\bullet}}\ ?_{\mathsf{d}}}{\vdash ?!\,\square\,?A^{\bullet}, ?!\Diamond ?B^{\bullet}}\ ?_{\mathsf{d}}, !$$

And we are blocked.

Solution

Promotion rule

$$\frac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\ !$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \square A, \Diamond B}\ \square$$

We apply a dereliction again:

$$\cfrac{\cfrac{\vdash !\,\square\,?A^{\bullet}, \Diamond ?B^{\bullet}}{\vdash ?!\,\square\,?A^{\bullet}, \Diamond ?B^{\bullet}}\ ?_{\mathsf{d}}}{\vdash ?!\,\square\,?A^{\bullet}, ?!\Diamond ?B^{\bullet}}\ ?_{\mathsf{d}}, !$$

Solution

Promotion rule

$$\frac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\ !$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \square A, \Diamond B}\ \square$$

Now, we can apply our promotion:

$$\cfrac{\cfrac{\cfrac{\vdash \square ?A^{\bullet}, \Diamond ?B^{\bullet}}{\vdash ! \square ?A^{\bullet}, \Diamond ?B^{\bullet}}\ !}{\vdash ?! \square ?A^{\bullet}, \Diamond ?B^{\bullet}}\ ?_{\mathsf{d}}}{\vdash ?! \square ?A^{\bullet}, ?!\Diamond ?B^{\bullet}}\ ?_{\mathsf{d}}, !$$

Solution

Promotion rule

$$\frac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\ !$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \square A, \Diamond B}\ \square$$

And finally our modal rule:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash ?A^\bullet, ?B^\bullet}{\vdash \square ?A^\bullet, \Diamond ?B^\bullet}\ \square}{\vdash !\square ?A^\bullet, \Diamond ?B^\bullet}\ !}{\vdash ?!\square ?A^\bullet, \Diamond ?B^\bullet}\ ?_\mathsf{d}}{\vdash ?!\square ?A^\bullet, ?!\Diamond ?B^\bullet}\ ?_\mathsf{d}, !$$

Solution

Promotion rule

$$\frac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\ !$$

But what does it imply for this system?

Solution

Promotion rule

$$\dfrac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\;!$$

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1}}{\vdash ?A, \Gamma_1}\,?_c \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2}}{\vdash !A^\perp, ?\Gamma_2}\,!_p}{\vdash \Gamma_1, ?\Gamma_2}\,\text{cut} \quad\rightsquigarrow\quad \cfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1} \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2}}{\vdash !A^\perp, ?\Gamma_2}\,!_p}{\vdash ?A, \Gamma_1, ?\Gamma_2}\,\text{cut} \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2}}{\vdash !A^\perp, ?\Gamma_2}\,!_p}{\vdash \Gamma_1, ?\Gamma_2, ?\Gamma_2}\,\text{cut}}{\vdash \Gamma_1, ?\Gamma_2}\,?_c^{\#\Gamma_2}$$

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash \Gamma_1}}{\vdash ?A, \Gamma_1}\,?_w \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2}}{\vdash !A^\perp, ?\Gamma_2}\,!_p}{\vdash \Gamma_1, ?\Gamma_2}\,\text{cut} \quad\rightsquigarrow\quad \cfrac{\overset{\pi_1}{\vdash \Gamma_1}}{\vdash \Gamma_1, ?\Gamma_2}\,?_w^{\#\Gamma_2}$$

Solution

Promotion rule

$$\dfrac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\ !$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash ?A, ?A, \Gamma_1\end{array}}{\vdash ?A, \Gamma_1}\,?_c \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdash A^\perp, ?\Gamma_2, \Diamond\Delta\end{array}}{\vdash !A^\perp, ?\Gamma_2, \Diamond\Delta}\,!_p}{\vdash \Gamma_1, ?\Gamma_2, \Diamond\Delta}\,\mathrm{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash ?A, ?A, \Gamma_1\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdash A^\perp, ?\Gamma_2\end{array}}{\vdash !A^\perp, ?\Gamma_2}\,!_p}{\vdash ?A, \Gamma_1, ?\Gamma_2}\,\mathrm{cut} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdash A^\perp, ?\Gamma_2\end{array}}{\vdash !A^\perp, ?\Gamma_2}\,!_p}{\vdash \Gamma_1, ?\Gamma_2, ?\Gamma_2}\,\mathrm{cut}}{\vdash \Gamma_1, ?\Gamma_2, \Diamond\Delta}\,?_c^{\#\Gamma_2}$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash \Gamma_1\end{array}}{\vdash ?A, \Gamma_1}\,?_w \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdash A^\perp, ?\Gamma_2\end{array}}{\vdash !A^\perp, ?\Gamma_2}\,!_p}{\vdash \Gamma_1, ?\Gamma_2}\,\mathrm{cut} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\pi_1\\ \vdash \Gamma_1\end{array}}{\vdash \Gamma_1, ?\Gamma_2}\,?_w^{\#\Gamma_2}$$

Solution

Promotion rule

$$\dfrac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\;!$$

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1}}{\vdash ?A, \Gamma_1}\,?_c \qquad \dfrac{\overset{\pi_2}{\vdash A^{\perp}, ?\Gamma_2, \Diamond\Delta}}{\vdash !A^{\perp}, ?\Gamma_2, \Diamond\Delta}\,!_p}{\vdash \Gamma_1, ?\Gamma_2, \Diamond\Delta}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1} \qquad \dfrac{\overset{\pi_2}{\vdash A^{\perp}, ?\Gamma_2}}{\vdash !A^{\perp}, ?\Gamma_2}\,!_p}{\vdash ?A, \Gamma_1, ?\Gamma_2}\,\mathsf{cut} \qquad \dfrac{\overset{\pi_2}{\vdash A^{\perp}, ?\Gamma_2}}{\vdash !A^{\perp}, ?\Gamma_2}\,!_p}{\vdash \Gamma_1, ?\Gamma_2, ?\Gamma_2, \Diamond\Delta, \Diamond\Delta}\,\mathsf{cut}}{\vdash \Gamma_1, ?\Gamma_2, \Diamond\Delta}\,?_c^{\#\Gamma_2}, \Diamond_c^{\#\Delta}$$

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash \Gamma_1}}{\vdash ?A, \Gamma_1}\,?_w \qquad \dfrac{\overset{\pi_2}{\vdash A^{\perp}, ?\Gamma_2}}{\vdash !A^{\perp}, ?\Gamma_2}\,!_p}{\vdash \Gamma_1, ?\Gamma_2}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\pi_1}{\vdash \Gamma_1}}{\vdash \Gamma_1, ?\Gamma_2}\,?_w^{\#\Gamma_2}$$

Solution

Promotion rule

$$\frac{\vdash A, ?\Gamma, \lozenge\Delta}{\vdash !A, ?\Gamma, \lozenge\Delta}\ !$$

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1}}{\vdash ?A, \Gamma_1}\ ?_c \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2, \lozenge\Delta}}{\vdash !A^\perp, ?\Gamma_2, \lozenge\Delta}\ !_p}{\vdash \Gamma_1, ?\Gamma_2, \lozenge\Delta}\ \mathrm{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1} \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2, \lozenge\Delta}}{\vdash !A^\perp, ?\Gamma_2, \lozenge\Delta}\ !_p}{\vdash ?A, \Gamma_1, ?\Gamma_2, \lozenge\Delta}\ \mathrm{cut} \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2, \lozenge\Delta}}{\vdash !A^\perp, ?\Gamma_2, \lozenge\Delta}\ !_p}{\vdash \Gamma_1, ?\Gamma_2, ?\Gamma_2, \lozenge\Delta, \lozenge\Delta}\ \mathrm{cut}}{\vdash \Gamma_1, ?\Gamma_2, \lozenge\Delta}\ ?_c^{\#\Gamma_2}, \lozenge_c^{\#\Delta}$$

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash \Gamma_1}}{\vdash ?A, \Gamma_1}\ ?_w \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2}}{\vdash !A^\perp, ?\Gamma_2}\ !_p}{\vdash \Gamma_1, ?\Gamma_2}\ \mathrm{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\pi_1}{\vdash \Gamma_1}}{\vdash \Gamma_1, ?\Gamma_2}\ ?_w^{\#\Gamma_2}$$

Solution

Promotion rule

$$\dfrac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\;!$$

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1}}{\vdash ?A, \Gamma_1}\,?_c \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2, \Diamond\Delta}}{\vdash !A^\perp, ?\Gamma_2, \Diamond\Delta}\,!_p}{\vdash \Gamma_1, ?\Gamma_2, \Diamond\Delta}\,\mathrm{cut} \;\rightsquigarrow\; \dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1} \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2, \Diamond\Delta}}{\vdash !A^\perp, ?\Gamma_2, \Diamond\Delta}\,!_p}{\vdash ?A, \Gamma_1, ?\Gamma_2, \Diamond\Delta}\,\mathrm{cut} \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2, \Diamond\Delta}}{\vdash !A^\perp, ?\Gamma_2, \Diamond\Delta}\,!_p}{\vdash \Gamma_1, ?\Gamma_2, ?\Gamma_2, \Diamond\Delta, \Diamond\Delta}\,\mathrm{cut}}{\vdash \Gamma_1, ?\Gamma_2, \Diamond\Delta}\,?_c^{\#\Gamma_2}, \Diamond_c^{\#\Delta}$$

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash \Gamma_1}}{\vdash ?A, \Gamma_1}\,?_w \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2, \Diamond\Delta}}{\vdash !A^\perp, ?\Gamma_2, \Diamond\Delta}\,!_p}{\vdash \Gamma_1, ?\Gamma_2, \Diamond\Delta}\,\mathrm{cut} \;\rightsquigarrow\; \dfrac{\overset{\pi_1}{\vdash \Gamma_1}}{\vdash \Gamma_1, ?\Gamma_2}\,?_w^{\#\Gamma_2}$$

Solution

Promotion rule

$$\dfrac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\,!$$

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1}}{\vdash ?A, \Gamma_1}\,?_c \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2, \Diamond\Delta}}{\vdash !A^\perp, ?\Gamma_2, \Diamond\Delta}\,!_p}{\vdash \Gamma_1, ?\Gamma_2, \Diamond\Delta}\,\text{cut} \;\rightsquigarrow\; \dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1} \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2, \Diamond\Delta}}{\vdash !A^\perp, ?\Gamma_2, \Diamond\Delta}\,!_p}{\vdash ?A, \Gamma_1, ?\Gamma_2, \Diamond\Delta}\,\text{cut} \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2, \Diamond\Delta}}{\vdash !A^\perp, ?\Gamma_2, \Diamond\Delta}\,!_p}{\vdash \Gamma_1, ?\Gamma_2, ?\Gamma_2, \Diamond\Delta, \Diamond\Delta}\,\text{cut}}{\vdash \Gamma_1, ?\Gamma_2, \Diamond\Delta}\,?_c^{\#\Gamma_2}, \Diamond_c^{\#\Delta}$$

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The $\mu LL^{\infty}_{\square}$ system

Modification of the promotion rule

$$\frac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\ !$$

Rules on modality

$$\frac{\vdash A, \Gamma}{\vdash \square A, \Diamond\Gamma}\ \square \qquad \frac{\vdash \Diamond A, \Diamond A, \Delta}{\vdash \Diamond A, \Delta}\ \Diamond_c \qquad \frac{\vdash \Delta}{\vdash \Diamond A, \Delta}\ \Diamond_w$$

$$\begin{tikzcd}
\mu LK^\infty_\Box \arrow[r, "(-)^\bullet"] & \mu LL^\infty_\Box \arrow[d, "(-)^\circ"] \\
\mu MALL^\infty & \mu LL^\infty \arrow[l]
\end{tikzcd}$$

Translation from $\mu LL^\infty_\Box$ to μLL^∞

Translation from $\mu LL^\infty_\Box$ to μLL^∞ and cut-elimination for $\mu LL^\infty_\Box$

We define:

$$(\Diamond A)^\circ \to ?A^\circ \text{ and } (\Box A)^\circ \to !A^\circ.$$

We easily get weakening and contractions of $\Diamond$ with weakening and contraction of ?.
For the modality rule, we have:

$$\dfrac{\vdash A, \Gamma}{\vdash \Box A, \Diamond\Gamma}\,\Box \rightsquigarrow \dfrac{\dfrac{\vdash A^\circ, \Gamma^\circ}{\vdash A^\circ, ?\Gamma^\circ}\,?_{\mathsf{d}}}{\vdash !A^\circ, ?\Gamma^\circ}\,!_{\mathsf{p}}$$

Translation from $\mu\mathsf{LL}^\infty_\Box$ to $\mu\mathsf{LL}^\infty$

Translation from $\mu\mathsf{LL}^\infty_\Box$ to $\mu\mathsf{LL}^\infty$ and cut-elimination for $\mu\mathsf{LL}^\infty_\Box$

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$$\dfrac{\vdash A, \Gamma}{\vdash \Box A, \Diamond\Gamma}\ \Box \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash A^\circ, \Gamma^\circ}{\vdash A^\circ, ?\Gamma^\circ}\ ?_{\mathsf{d}}}{\vdash !A^\circ, ?\Gamma^\circ}\ !_{\mathsf{p}}$$

Cut-elimination theorems

Using $(-)^\circ$, we obtain cut-elimination for $\mu\mathsf{LL}^\infty_\Box$.

Using $(-)^\bullet$, we get a proof of $\mu\mathsf{LL}^\infty_\Box$, from which we can eliminate cuts, we then can come back to $\mu\mathsf{LK}^\infty_\Box$ and get a cut-free proof of $\mu\mathsf{LL}^\infty_\Box$.

Sub-exponentials

B. & Laurent TLLA '20

The previous work actually works with a sub-exponential system inspired from the work of Nigam & Miller '09. With a promotion rule on signed exponentials:

$$\frac{\vdash A, ?_{e_1}A_1, \dots, ?_{e_n}A_n \qquad e \leq_g e_i}{\vdash !_e A, ?_{e_1}A_1, \dots, ?_{e_n}A_n}\ ! \qquad \frac{\vdash A, A_1, \dots, A_n \qquad e \leq_f e_i}{\vdash !_e A, ?_{e_1}A_1, \dots, ?_{e_n}A_n}\ !_f$$

and structural rules authorized only on some signed exponentials:

$$\frac{\vdash \Gamma \qquad e \in \mathcal{W}}{\vdash ?_e A, \Gamma}\ w \qquad \frac{\vdash ?_e A, ?_e A, \Gamma \qquad e \in \mathcal{C}}{\vdash ?_e A, \Gamma}\ c \qquad \frac{\vdash A, \Gamma \qquad e \in \mathcal{D}}{\vdash ?_e A, \Gamma}\ d$$

Modal logic is an instance of this sub-exponential system:

With two signatures e and e', with $! := !_e$ and $\Box = !_{e'}$.

$e' \leq_g e, \quad e \leq_g e, \quad e' \leq_f e', \quad e \not\leq_f e', \quad e' \not\leq_g e, e'$

$e, e' \in \mathcal{W},\ e, e' \in \mathcal{C}$ and $e \in \mathcal{D}$

Conclusion & future works

What did we prove?

We proved a syntactic cut-elimination theorem for the modal μ-calculus.

Conclusion & future works

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- We proved a syntactic cut-elimination theorem for the modal μ-calculus.
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Conclusion & future works

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Future works

- Integrate the digging rule (axiom S4) or the co-dereliction rule (axiom T)

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Future works

- Integrate the digging rule (axiom S4) or the co-dereliction rule (axiom T)
- Extension of the validity criterion

Super exponential system (TLLA '21)

Exponential signatures

An exponential signature is a boolean function on the set of rule names:

$$\{?_{\mathsf{m}_i}, ?_{\mathsf{c}_i} \mid i \in \mathbb{N}\}.$$

Formulas

Let $\mathcal{E}$ be a set of exponential signatures, formulas of μsuperLL$^\infty(\mathcal{E})$ (with $\sigma \in \mathcal{E}$) are:

$$A, B ::= A \parr B \mid A \otimes B \mid A \,\&\, B \mid A \oplus B \mid \bot \mid 1 \mid \top \mid 0 \mid X \in \mathcal{V} \mid \mu X.A \mid \nu X.A \mid$$

$$?_\sigma A \mid !_\sigma A.$$

Multiplexing:

$$\dfrac{\vdash \overbrace{A, \ldots, A}^{i}, \Gamma \qquad \sigma(?_{\mathsf{m}_i})}{\vdash ?_\sigma A, \Gamma}\ ?_{\mathsf{m}_i}$$

Generalized contraction:

$$\dfrac{\vdash \overbrace{?_\sigma A, \ldots, ?_\sigma A}^{i}, \Gamma \qquad \sigma(?_{\mathsf{c}_i})}{\vdash ?_\sigma A, \Gamma}\ ?_{\mathsf{c}_i}$$

Promotion rules

Given three relations $\leq_{\mathrm{g}}, \leq_{\mathrm{f}}$ and $\leq_{\mathrm{u}}$ on $\mathcal{E}$, we have the promotion rules of $\mu\mathrm{superLL}^{\infty}(\mathcal{E}, \leq_{\mathrm{g}}, \leq_{\mathrm{f}}, \leq_{\mathrm{u}})$:

$$\frac{\vdash A, ?_{\vec{\sigma'}}\Delta \qquad \sigma \leq_{\mathrm{g}} \vec{\sigma'}}{\vdash !_{\sigma}A, ?_{\vec{\sigma'}}\Delta}\ !_{\mathrm{g}} \qquad \frac{\vdash A, \Delta \qquad \sigma \leq_{\mathrm{f}} \vec{\sigma'}}{\vdash !_{\sigma}A, ?_{\vec{\sigma'}}\Delta}\ !_{\mathrm{f}} \qquad \frac{\vdash A, B \qquad \sigma_1 \leq_{\mathrm{u}} \sigma_2}{\vdash !_{\sigma_1}A, ?_{\sigma_2}B}\ !_{\mathrm{u}}$$

Instances of superLL

ELL

Elementary Linear Logic (ELL) is a variant of LL where we remove $(?_d)$ and $(!_g)$ and add the functorial promotion:

$$\frac{\vdash A, \Gamma}{\vdash !A, ?\Gamma}\ !_f$$

superLL$(\mathcal{E}, \leq_g, \leq_f, \leq_u)$ is ELL where:

- $\mathcal{E} = \{\bullet\}$;
- $\bullet(?_{c_2}) = \bullet(?_{m_0}) = \text{true}$ (and $(\bullet)(r) = \text{false}$ otherwise);
- $\leq_g = \leq_u = \varnothing$ and $\bullet \leq_f \bullet$.

superLL subsume many existing system of linear logic such as LL, LL with shifts, ELL, LLL, SLL or seLL.

Cut-elimination axioms

$\sigma \leq_{\mathbf{g}} \sigma'$	$\Rightarrow$	$\sigma(?_{\mathbf{m}_i})$	$\Rightarrow$	$\bar{\sigma}'(?_{\mathbf{c}_i})$	$i \geq \mathbf{0}$	(**axgmpx**)
$\sigma \leq_s \sigma'$	$\Rightarrow$	$\sigma(?_{\mathbf{m}_i})$	$\Rightarrow$	$\bar{\sigma}'(?_{\mathbf{m}_i})$	$i \geq \mathbf{0}$ **and** $s \neq g$	(**axfumpx**)
$\sigma \leq_s \sigma'$	$\Rightarrow$	$\sigma(?_{\mathbf{c}_i})$	$\Rightarrow$	$\bar{\sigma}'(?_{\mathbf{c}_i})$	$i \geq \mathbf{2}$	(**axcontr**)
$\sigma \leq_s \sigma'$	$\Rightarrow$	$\sigma' \leq_s \sigma''$	$\Rightarrow$	$\sigma \leq_s \sigma''$		(**axTrans**)
$\sigma \leq_{\mathbf{g}} \sigma'$	$\Rightarrow$	$\sigma' \leq_s \sigma''$	$\Rightarrow$	$\sigma \leq_{\mathbf{g}} \sigma''$		(**axleqgs**)
$\sigma \leq_{\mathbf{f}} \sigma'$	$\Rightarrow$	$\sigma' \leq_{\mathbf{u}} \sigma''$	$\Rightarrow$	$\sigma \leq_{\mathbf{f}} \sigma''$		(**axleqfu**)
$\sigma \leq_{\mathbf{f}} \sigma'$	$\Rightarrow$	$\sigma' \leq_{\mathbf{g}} \sigma''$	$\Rightarrow$	$(\sigma \leq_{\mathbf{g}} \sigma'' \wedge (\sigma \leq_{\mathbf{f}} \sigma''' \Rightarrow (\sigma \leq_{\mathbf{g}} \sigma''' \wedge \sigma'''(?_{\mathbf{m_1}})))$		(**axleqfg**)
$\sigma \leq_{\mathbf{u}} \sigma'$	$\Rightarrow$	$\sigma' \leq_s \sigma''$	$\Rightarrow$	$\sigma \leq_s \sigma''$		(**axlequs**)

with $s \in \{g, f, u\}$, all the axioms are universally quantified.

Instances of superLL satisfies cut-elimination axioms

LL, LL with shifts, ELL, LLL, SLL, seLL satisfy cut-elimination axioms.

Cut-elimination for superLL

Let's consider the following axiom:

$$\sigma \leq_g \sigma' \quad \Rightarrow \quad \sigma' \leq_f \sigma'' \quad \Rightarrow \quad \sigma \leq_g \sigma'' \qquad \text{(axleqgs)}$$

We use it for the following cut-elimination step:

$$\dfrac{\dfrac{\vdash A, ?_{\vec{\tau}}\Gamma \qquad \sigma \leq_{g} \vec{\tau}, \tau}{\vdash !_{\sigma} A, ?_{\vec{\tau}}\Gamma, ?_{\tau} C}\,!_{g} \qquad \dfrac{\vdash C, \Delta \qquad \tau \leq_{f} \vec{\rho}}{\vdash !_{\tau} C^{\perp}, ?_{\vec{\rho}}\Delta}\,!_{f}}{\vdash !_{\sigma} A, ?_{\vec{\tau}}\Gamma, ?_{\vec{\rho}}\Delta}\,\mathrm{cut} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\vdash A, ?_{\vec{\tau}}\Gamma \qquad \dfrac{\vdash C, \Delta \qquad \tau \leq_{f} \vec{\rho}}{\vdash !_{\tau} C^{\perp}, ?_{\vec{\rho}}\Delta}\,!_{f}}{\vdash A, ?_{\vec{\tau}}\Gamma, ?_{\vec{\rho}}\Delta}\,\mathrm{cut} \qquad \sigma \leq_{g} \vec{\tau}, \vec{\rho}}{\vdash !_{\sigma} A, ?_{\vec{\tau}}\Gamma, ?_{\vec{\rho}}\Delta}\,!_{g}$$

superLL eliminates cuts

Cut-eliminations (B. & Laurent '21)

As soon as the 8 cut-elimination axioms are satisfied, cut elimination holds for superLL$(\mathcal{E}, \leq_{\mathsf{g}}, \leq_{\mathsf{f}}, \leq_{\mathsf{u}})$.

Cut-elimination steps

Example

If

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A, ?_{\vec{\tau}}\Delta\end{array} \qquad \sigma \leq_{\mathsf{g}} \vec{\tau}}{\vdash !_{\sigma} A, ?_{\vec{\tau}}\Delta}\;!_{\mathsf{g}} \qquad \mathcal{C}^{!}}{\vdash !_{\sigma} A, ?_{\vec{\rho}}\Gamma}\;\mathsf{mcut}(\iota, \perp\!\!\!\perp)$$

is a μsuperLL$^\infty(\mathcal{E}, \leq_{\mathsf{g}}, \leq_{\mathsf{f}}, \leq_{\mathsf{u}})$-proof then

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A, ?_{\vec{\tau}}\Delta\end{array} \qquad \mathcal{C}^{!}}{\vdash A, ?_{\vec{\rho}}\Gamma}\;\mathsf{mcut}(\iota, \perp\!\!\!\perp) \qquad \sigma \leq_{\mathsf{g}} \vec{\rho}}{\vdash !_{\sigma} A, ?_{\vec{\rho}}\Gamma}\;!_{\mathsf{g}}$$

is also a μsuperLL$^\infty(\mathcal{E}, \leq_{\mathsf{g}}, \leq_{\mathsf{f}}, \leq_{\mathsf{u}})$-proof.

Translation of μsuperLL$^\infty$ into μLL$^\infty$

Translation of formulas

We translate formulas by induction using:

$$(!_\sigma A)^\circ := \,!A^\circ \qquad\qquad (?_\sigma A)^\circ := \,?A^\circ$$

$$\dfrac{\vdash \overbrace{A, \dots, A}^{n}, \Gamma \qquad \sigma(?_{\mathsf{m}_i})}{\vdash ?_\sigma A, \Gamma}\; ?_{\mathsf{m}_i} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \overbrace{A^\circ, \dots, A^\circ}^{i}, \Gamma^\circ}{\vdash ?A^\circ, \dots, ?A^\circ, \Gamma^\circ}\; ?_{\mathsf{d}} \times i}{\vdash ?A^\circ, \Gamma^\circ}\; ?_{\mathsf{c}} \times i$$

$$\dfrac{\vdash \overbrace{?_\sigma A, \dots, ?_\sigma A}^{n}, \Gamma \qquad \sigma(?_{\mathsf{c}_i})}{\vdash ?_\sigma A, \Gamma}\; ?_{\mathsf{c}_i} \quad \rightsquigarrow \quad \dfrac{\vdash \overbrace{?A^\circ, \dots, ?A^\circ}^{i}, \Gamma^\circ}{\vdash ?A^\circ, \Gamma^\circ}\; ?_{\mathsf{c}} \times i$$

Translation of μsuperLL$^\infty$ into μLL$^\infty$

$$\dfrac{\vdash ?_{\sigma_1}A_1,\ldots,?_{\sigma_n}A_n, A \qquad \begin{array}{c} i \in [\![1,n]\!] \\ \sigma \leq_{\mathrm{g}} \sigma_i \end{array}}{\vdash ?_{\sigma_1}A_1,\ldots,?_{\sigma_n}A_n, !_\sigma A}\,!_{\mathrm{g}} \quad \rightsquigarrow \quad \dfrac{\vdash ?A_1^\circ,\ldots,?A_n^\circ, A^\circ}{\vdash ?A_1^\circ,\ldots,?A_n^\circ, !A^\circ}\,!_{\mathrm{p}}$$

$$\dfrac{\vdash A_1,\ldots,A_n, A \qquad \begin{array}{c} i \in [\![1,n]\!] \\ \sigma \leq_{\mathrm{f}} \sigma_i \end{array}}{\vdash ?_{\sigma_1}A_1,\ldots,?_{\sigma_n}A_n, !_\sigma A}\,!_{\mathrm{f}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash A_1^\circ,\ldots,A_n^\circ, A^\circ}{\vdash ?A_1^\circ,\ldots,?A_n^\circ, A^\circ}\,?_{\mathrm{d}}}{\vdash ?A_1^\circ,\ldots,?A_n^\circ, !A^\circ}\,!_{\mathrm{p}}$$

$$\dfrac{\vdash B, A \qquad \sigma_1 \leq_{\mathrm{u}} \sigma_2}{\vdash ?_{\sigma_2}B, !_{\sigma_1}A}\,!_{\mathrm{u}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash B^\circ, A^\circ}{\vdash ?B^\circ, A^\circ}\,?_{\mathrm{d}}}{\vdash ?B^\circ, !A^\circ}\,!_{\mathrm{p}}$$

Cut-elimination for μsuperLL$^\infty$

Cut-elimination reduction system correctness

For every μsuperLL$^\infty(\mathcal{E}, \leq_{\mathsf{g}}, \leq_{\mathsf{f}}, \leq_{\mathsf{u}})$ reduction sequences $(\pi_i)_{i \in \mathbb{N}}$, there exists a μLL$^\infty$ reduction sequence $(\theta_i)_{i \in \mathbb{N}}$ such that for each i, there exists j such that π_i° is equal to θ_j up to rule-permutations.

Cut-elimination reduction system completeness

If there is a μLL$^\infty$-redex $\mathcal{R}$ sending π° to π'° then there is also a μsuperLL$^\infty(\mathcal{E}, \leq_{\mathsf{g}}, \leq_{\mathsf{f}}, \leq_{\mathsf{u}})$-redex $\mathcal{R}'$ sending π to a proof π'', such that in the translation of $\mathcal{R}'$, $\mathcal{R}$ is reduced.

Cut-elimination theorem for μsuperLL$^\infty$

Every fair (mcut)-reduction sequence of μsuperLL$^\infty(\mathcal{E}, \leq_{\mathsf{g}}, \leq_{\mathsf{f}}, \leq_{\mathsf{u}})$ converges to a μsuperLL$^\infty(\mathcal{E}, \leq_{\mathsf{g}}, \leq_{\mathsf{f}}, \leq_{\mathsf{u}})$ cut-free proof.