A fibrational characterization for unicity of solutions to generalized context-free systems

FICS 2024

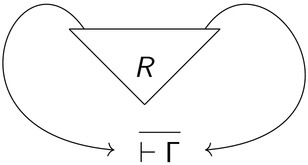
Farzad Jafarrahmani, Noam Zeilberger

Sorbonne Université, École Polytechnique

Introduction

What are cyclic proofs, categorically?

The starting point of this work was trying to develop a fibrational semantics of cyclic proofs.



One of the main difficulties is how to capture the validity criterion categorically.

Recursive definitions

More generally, we want to model recursive definitions.
What makes a recursive function valid?

```
add Z     n = n
add (S m) n = S (add m n)
```

```
bad n = bad (S n)
```

```
foo 0     = 0
foo (S m) = S (foo m)
foo (S m) = foo m
```

It seem that if a recursive definition has a unique solution, then it is a valid definition.

Context free grammars

This led us to consider the question of unicity for CFGs.

$$G_1 = \begin{array}{lll} S & \to^a & \epsilon \\ S & \to^b & [S] \\ S & \to^c & SS \end{array} \qquad G_2 = \begin{array}{lll} S & \to^d & \epsilon \\ S & \to^e & [S]S \end{array} \tag{1}$$

$$\begin{aligned} \mathcal{L}_{G_1} = \mathcal{L}_{G_2} &= \text{the Dyck language of balanced brackets} \\ &= \text{the minimal solution of the following equations} \end{aligned}$$

$$L = \epsilon + [L] + LL \tag{2}$$

$$L = \epsilon + [L]L \tag{3}$$

$L = \Sigma^*$ is another solution of the equation (2), while equation (3) has a *unique* solution.

Context free language

There are at least two different ways of interpreting a context-free language as the solution to a recursive system of constraints.

- Traditionally, as the smallest language closed under the production rule.

  For example, $L_1 = \mu(F_1)$ and $L_2 = \mu(F_2)$ where the operators $F_1, F_2 : \mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^*)$ are defined by

$$F_1(X) = \epsilon + [X] + XX \qquad F_2(X) = \epsilon + [X]X$$

- Considering the recursive equations literally (rather than as inclusions) and find their solution.

This talk

- We want to formulate the question of unicity of solutions to equations arising from CFGs in a very general fibrational framework.
- The problem of characterizing unicity of solution to systems of polynomial equations induced by context-free grammars has been considered in early work of Courcelle [1], and our work can be seen as a categorical revisiting of Courcelle's work (although it did not start out that way)[2].

[1] Bruno Courcelle (1986): Equivalences and transformations of regular systems–applications to recursive program schemes and grammars

[2] We gratefully thank Sylvain Salvati for pointing us to this work

gCFGs

Following [3], we define a generalized CFG as a functor of operads $p : \text{Free}(\mathcal{S}) \to \mathcal{O}$ where $\text{Free}(\mathcal{S})$ is the freely generated operad from a finite species $\mathcal{S}$.
This encompasses ordinary CFGs by taking $\mathcal{O} = \mathcal{W}[\Sigma]$ to be the "operad of spliced words" whose n-ary operations are sequences $w_0 - w_1 - \cdots - w_n$ of $n+1$ words over Σ.

$$S \to^a \epsilon \quad S \to^b [S] \quad S \to^c SS$$

$$\mathcal{S} = \{S, a : . \to S, b : S \to S, c : S, S \to S\}$$

$$p(a) = \epsilon$$

$$p(b) = [-\epsilon-]$$

$$p(c) = \epsilon - \epsilon - \epsilon$$

[3] Paul-André Melliès, Noam Zeilberger: The categorical contours of the Chomsky-Schützenberger representation theorem.

gCFGs

Following [3], we define a generalized CFG as a functor of operads $p : \text{Free}(\mathcal{S}) \to \mathcal{O}$ where $\text{Free}(\mathcal{S})$ is the freely generated operad from a finite species $\mathcal{S}$.
This encompasses ordinary CFGs by taking $\mathcal{O} = \mathcal{W}[\Sigma]$ to be the "operad of spliced words" whose n-ary operations are sequences $w_0 - w_1 - \cdots - w_n$ of $n+1$ words over Σ.

$$S \to^a \epsilon \quad S \to^b [S] \quad S \to^c SS$$
$$\mathcal{S} = \{S, a : . \to S, b : S \to S, c : S, S \to S\}$$
$$p(a) = \epsilon$$
$$p(b) = [-\epsilon-]$$
$$p(c) = \epsilon - \epsilon - \epsilon$$

[3] Paul-André Melliès, Noam Zeilberger: The categorical contours of the Chomsky-Schützenberger representation theorem.

Models of gCFGs

A model of p is given by a square of the following form:

$$\begin{array}{ccc} \mathsf{Free}(\mathcal{S}) & \xrightarrow{\tilde{M}} & \mathcal{E} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{O} & \xrightarrow{M} & \mathcal{B} \end{array}$$

the only requirements on q are that it admits pushforwards and fiberwise coproducts.

We mainly consider two models:

- $q : \mathrm{Subset} \to \mathrm{Set}$.
    - proof-irrelevant: $\mathcal{L}_S \subseteq \Sigma^*$ (subset of words)
- $q : \mathrm{Set}^{\to} \to \mathrm{Set}$
    - proof-relevant: $\mathcal{L}_S \longrightarrow \Sigma^*$ (set of derivations equipped with underlying word)

Question

Let M be the initial functor $\mathcal{O} \to \mathrm{Set}$. We want to find a sufficient and necessary condition for unicity of $\tilde{M}$.

$$\begin{array}{ccc} \mathsf{Free}(\mathcal{S}) & \xrightarrow{\quad \tilde{M} \quad} & \mathrm{Set}^{\to} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ \mathcal{O} & \xrightarrow{\quad M \quad} & \mathrm{Set} \end{array}$$

We focus on the question of unicity of solutions in the proof-relevant model, since it implies unicity of solutions in the proof-irrelevant model.

A non-unital suboperad

The starting point is to consider the base operad $\mathcal{O}$ as being equipped with a *non-unital suboperad* $\mathcal{O}^+ \subset \mathcal{O}$, whose operations induce a well-founded ordering on the constants of $\mathcal{O}$:

$$\frac{f : A_1, \cdots, A_n \to B \qquad (a_i : A_i)_{1 \le i \le n}}{f(a_1, \cdots, a_n) > a_i}$$

In the case of CFGs, $\mathcal{W}[\Sigma]^+$ is operad of spliced words n-ary operations are sequences of $n+1$ words containing at least one non-empty word.

$$\frac{f = w_0 - w_1 - \cdots - w_n \qquad (u_i : \Sigma^*)_{1 \le i \le n}}{w_0 u_1 w_1 u_2 \cdots u_n w_n > u_i}$$

Composition of grammars

$$S \to^d T \quad T \to^e [T]T \quad T \to^f \epsilon$$

$$\mathcal{S} = \{S, T, d : T \to S, e : T, T \to T, f : . \to T\}$$

$$p(d) = \epsilon - \epsilon$$

$$p(e) = [-\epsilon-]$$

$$p(f) = \epsilon$$

$$p(\mathcal{S}^-) \not\subset \mathcal{W}[\Sigma]^+$$

$\mathcal{S}^- \circ \mathcal{S}$:

Composition of grammars

$$S \to^d T \quad T \to^e [T]T \quad T \to^f \epsilon$$

$$\mathcal{S} = \{S, T, d : T \to S, e : T, T \to T, f : . \to T\}$$

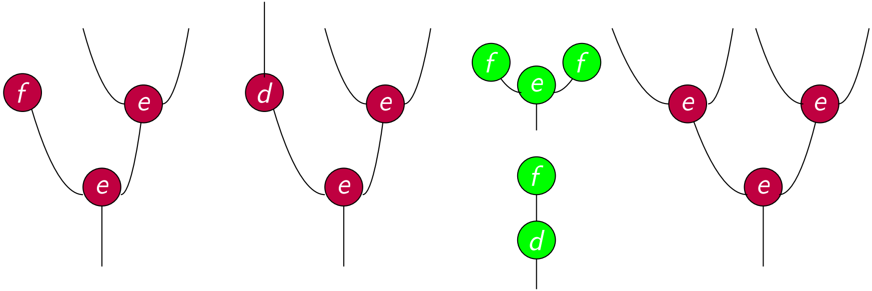
$$p(d) = \epsilon - \epsilon$$

$$p(e) = [-\epsilon-]$$

$$p(f) = \epsilon$$

$$p(\mathcal{S}^-) \not\subset \mathcal{W}[\Sigma]^+$$

$\mathcal{S}^- \circ \mathcal{S}$:



Composition of grammars

$$S \to^d T \quad T \to^e [T]T \quad T \to^f \epsilon$$

$$\mathcal{S} = \{S, T, d : T \to S, e : T, T \to T, f : . \to T\}$$

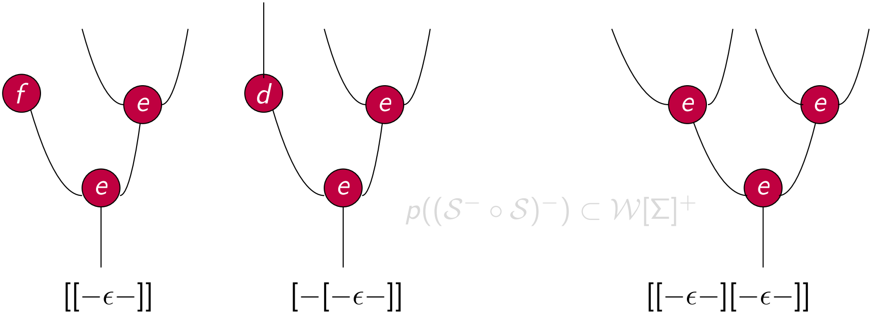
$$p(d) = \epsilon - \epsilon$$

$$p(e) = [-\epsilon-]$$

$$p(f) = \epsilon$$

$$p(\mathcal{S}^-) \not\subset \mathcal{W}[\Sigma]^+$$

$(\mathcal{S}^- \circ \mathcal{S})^-$:



$$p((\mathcal{S}^- \circ \mathcal{S})^-) \subset \mathcal{W}[\Sigma]^+$$

Composition of grammars

$$S \to^{d} T \quad T \to^{e} [T]T \quad T \to^{f} \epsilon$$

$$\mathcal{S} = \{S, T, d : T \to S, e : T, T \to T, f : . \to T\}$$

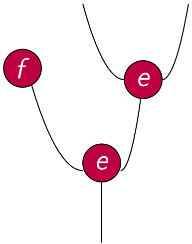
$$p(d) = \epsilon - \epsilon$$

$$p(e) = [-\epsilon-]$$

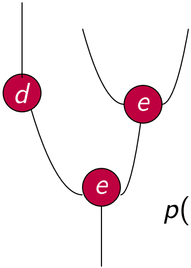
$$p(f) = \epsilon$$

$$p(\mathcal{S}^-) \not\subset \mathcal{W}[\Sigma]^+$$

$(\mathcal{S}^- \circ \mathcal{S})^-$:

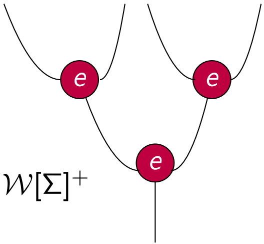


$[[-\epsilon-]]$



$[-[-\epsilon-]]$

$$p((\mathcal{S}^- \circ \mathcal{S})^-) \subset \mathcal{W}[\Sigma]^+$$



$[[-\epsilon-][-\epsilon-]]$

Some notations

- We use the *composition product* from the theory of species [4].
- The *unit* of the composition is given by the species $\mathbb{I}$ with a single unary node $*_R : R \to R$ for every color R.
- We denote by $\mathcal{R}^-$ the species $\mathcal{R}^- := \mathcal{R} - \mathcal{R}(0)$ obtained by removing all nullary nodes from any species $\mathcal{R}$.
- We write $\Delta_\mathcal{S}$ for the endofunctor $\Delta_\mathcal{S} : \mathrm{Spec}_X \to \mathrm{Spec}_X$ defined by $\Delta_\mathcal{S} := \mathcal{R} \mapsto (\mathcal{R} \circ \mathcal{S})^-$.

[4] André Joyal (1981): Une théorie combinatoire des séries formelles

Relative nilpotency

We say a gCFG $p : \text{Free}(\mathcal{S}) \to \mathcal{O}$ has the **relative nilpotency** if

there exists a k such that $p(\Delta^k_{\mathcal{S}}\,\mathbb{I}) \subset \mathcal{O}^+$.

$$G_1 = \begin{array}{lll} S & \to^a & \epsilon \\ S & \to^b & [S] \\ S & \to^c & SS \end{array} \qquad G_2' = \begin{array}{lll} S & \to^d & T \\ T & \to^e & [T]T \\ T & \to^f & \epsilon \end{array}$$

- the grammar G_2 from slide 5 satisfies the relative nilpotency condition with k=1.
- As we saw, G_2' satisfies the relative nilpotency condition with $k = 2$.
- G_1 does not satisfy the relative nilpotency condition.

Relative nilpotency

We say a gCFG $p : \mathrm{Free}(\mathcal{S}) \to \mathcal{O}$ has the **relative nilpotency** if

$$\textit{there exists a } k \textit{ such that } p(\Delta_{\mathcal{S}}^{k}\,\mathbb{I}) \subset \mathcal{O}^{+}.$$

$$G_1 = \begin{array}{lll} S & \to^{a} & \epsilon \\ S & \to^{b} & [S] \\ S & \to^{c} & SS \end{array} \qquad G_2' = \begin{array}{lll} S & \to^{d} & T \\ T & \to^{e} & [T]\,T \\ T & \to^{f} & \epsilon \end{array}$$

- the grammar G_2 from slide 5 satisfies the relative nilpotency condition with k=1.
- As we saw, G_2' satisfies the relative nilpotency condition with $k = 2$.
- G_1 does not satisfy the relative nilpotency condition.

Result

Let $\mathcal{O}$ be an operad which is equipped with a *non-unital suboperad* $\mathcal{O}^+ \subset \mathcal{O}$, whose operations induce a well-founded ordering on the constants of $\mathcal{O}$. Then $p : \mathsf{Free}(\mathcal{S}) \to \mathcal{O}$ has a unique model in $\mathrm{Set}^{\to} \to \mathrm{Set}$ iff p satisfies the relative nilpotency condition.

$$\begin{array}{ccc} \mathsf{Free}(\mathcal{S}) & \xrightarrow{\tilde{M}} & \mathrm{Set}^{\to} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{O} & \xrightarrow{M} & \mathrm{Set} \end{array}$$

Future work

- We still need to write up!
- What does the standard proof-relevant results on CFGs say in relation with models and unicity? For example, Greibach Normal Form grammars satisfy the relative nilpotency condition.
- We eventually want to deal with other examples including cyclic proofs as well as recursive definitions in type theory and functional programming.
- We also have a fibrational setting for inductively defined predicate. Does the comparison of these two settings give us a hint to better understand the relationship between cyclic and inductive proofs?