Characterizing the Exponential-Space Hierarchy via Partial Fixpoints

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Descriptive Complexity

characterize complexity classes via logical ressources. $\mathcal{L}$ captures $\mathcal{C}$ if:

1) queries defined by formulas in $\mathcal{L}$ can be computed in $\mathcal{C}$
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only (boolean) queries on finite structures of interest

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Some extensions/generalizations:

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- $\mathrm{HO}^{k+1} + \mathrm{PFP} = k\mathrm{-EXPSPACE}$ (this talk)

$*$: for $k = 0$

Partial Fixpoints

Knaster-Tarski/Kleene guarantee well-definedness of fixpoints of monotone functions:

- $f\colon X \mapsto f(X)$ (in powerset lattice) yields sequence $\emptyset \subseteq f(\emptyset) \subseteq f^2(\emptyset) \subseteq \cdots$
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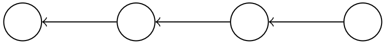

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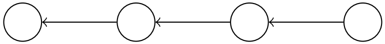

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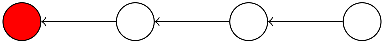

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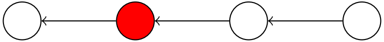

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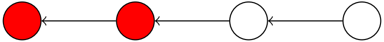

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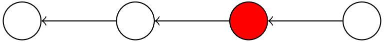

binary incrementation not monotone $\to$ need partial fixpoint:

- $f\colon X \mapsto f(X)$ yields sequence $\emptyset, f(\emptyset), f^2(\emptyset), \ldots$
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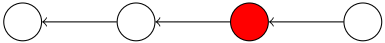

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Obs.: sequence either stabilizes after at most exponentially many steps, or it cycles

Higher-Order Logic (+ PFP)

Types:

$$\tau ::= \bullet \text{ (individuals)} \mid \tau_1, \ldots, \tau_n \text{ (cross product)} \mid (\tau) \text{ sets}$$

- $\mathrm{ord}(\bullet) = 1$,
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Formulas of HO :

$$\varphi ::= p(X) \mid E(X, Y) \mid X(\vec{Y}) \mid \varphi \vee \varphi \mid \neg\varphi \mid \exists X^{\tau}.\ \varphi$$

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Ex.: $<$ (written inline) is strict total order in ψ

$$\exists <^{(\bullet,\bullet)} .\, \psi(<) \wedge \forall x^\bullet, y^\bullet, z^\bullet.\, (x < y \wedge y < z \rightarrow x < z)$$
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Obs.: w.l.o.g. structures ordered from now on

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Ex.: $\Big(\mathrm{PFP}\ X^{(\tau)}.\ (X = \emptyset \land \varphi_{\mathsf{init}}(x)) \lor (\exists y^{\tau}.\varphi_{\mathsf{step}}(x, y) \land X(y))\Big)\ \varphi_{\mathsf{end}}$

with τ as type of x (reachability of some end position from start)

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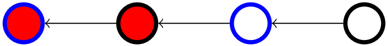

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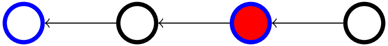

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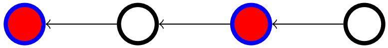

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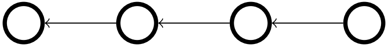

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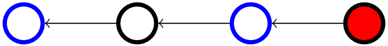

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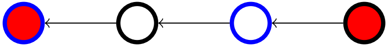

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can continue this:
Ex.: $\varphi_<^{((\bullet,\dots,\bullet))}(X^{((\bullet,\dots,\bullet))},Y^{((\bullet,\dots,\bullet))}) =$
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Lemma: All types can be considered ordered from now on

also have formulas to find e.g., first, last element of order

Characterizing k-EXPSPACE

HO^{k+1} = formulas with types of order at most $k+1$ (PFP of order at most $k+2$)

for $\mathsf{HO}^{k+1} + \mathsf{PFP} = k$-EXPSPACE we need:

1) queries def. by formulas in $\mathsf{HO}^{k+1} + \mathsf{PFP}$ can be computed in k-EXPSPACE
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- query result for (encoding of) accepting conf.

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configuration of $\mathcal{M}$ on input w is (q, h, t) with

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- yields variable of type $Q \times \tau \times (\tau \times \Gamma) \to$ order $k+2$

Encoding Configurations of Space-Bounded DTM, cont'd.

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- $I \in ([\![\tau]\!] \times \mathcal{T})$ encodes tape
- ex. exactly one s' with $(i, s') \in I$ f.a. $i \in [\![\tau]\!]$

NB: A bit more challenging if cross product type not available standalone

Encoding Runs

type (τ) from last slide can be chosen of order $k+2$ (form: $((\cdots(\bullet,\ldots,\bullet)\cdots)))$

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- requires some basic arithmetic (but no uniform notion of addition)

Theorem 1

$\mathrm{HO}^{k+1}{+}\mathrm{PFP}$ captures k-EXPSPACE for $k \geq 0$.

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Thm. ([B./Kronenberger/Lange'22]): $\sim$-inv. k-EXPTIME is capt. by order-k poly. HFL

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