The Limit of Recursion
in State-based Systems

Giacomo Barlucchi

j.w.w. B. Afshari & G.E. Leigh

Göteborgs Universitet

FICS 2024

μ-calculus

Modal logic with fixed points

$$\mu x\, \Box\bot \vee (q \wedge \Diamond x) \qquad\qquad \nu x\, p \wedge \Diamond x \wedge \mu y\, (\overline{p} \vee y)$$

Frame

A frame $\mathcal{S} = (S, R, \Lambda)$ consisting of: a set S of states; a binary relation $R \subseteq S \times S$; a labelling function Λ.

$$f_\varphi \colon \mathcal{P}(S) \to \mathcal{P}(S)$$

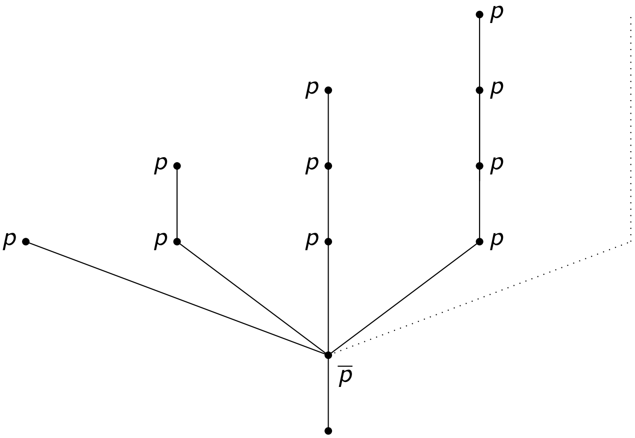
$$\|\mu x\, \varphi\|^{\mathcal{S}}_{\mathcal{V}} := \bigcap \{\, U \subseteq S \mid \|\varphi\|^{\mathcal{S}}_{\mathcal{V}[x \mapsto U]} \subseteq U \,\}$$

$$\emptyset \;\subseteq\; f_\varphi(\emptyset) \;\subseteq\; f_\varphi(f_\varphi(\emptyset)) \;\subseteq \cdots \subseteq\; f^{\alpha}_\varphi(\emptyset) \;\subseteq\; f^{\alpha+1}_\varphi(\emptyset) \;\subseteq \cdots$$

$$\|\mu x\, \varphi\|^{\mathcal{S}}_{\mathcal{V}} := \bigcup_{\alpha} f^{\alpha}_\varphi(\emptyset)$$

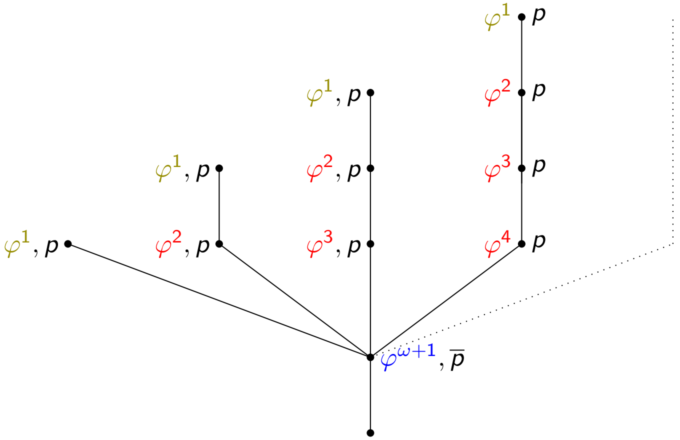
Approximation

$$\varphi \equiv \mu x.\Box\bot \vee (p \wedge \Box p \wedge \Diamond x) \vee (\overline{p} \wedge \Box p \wedge \Box x)$$



Approximation

$$\varphi \equiv \mu x.\Box\bot \vee (p \wedge \Box p \wedge \Diamond x) \vee (\overline{p} \wedge \Box p \wedge \Box x)$$



Closure Ordinal

Approximations

$$\varphi^{\alpha} := (\varphi, \alpha)$$

$$\begin{aligned} \|\mu x\, \varphi^{\alpha}\|^{\mathcal{S}}_{\mathcal{V}} &:= \bigcup_{\beta<\alpha} \|\varphi\|^{\mathcal{S}}_{\mathcal{V}[x \mapsto \|\mu x\, \varphi^{\beta}\|^{\mathcal{S}}_{\mathcal{V}}]} \\ \|(\psi_0 \vee \psi_1)^{\alpha}\| &:= \|\psi_0^{\alpha}\| \cup \|\psi_1^{\alpha}\| \\ \|(\psi_0 \wedge \psi_1)^{\alpha}\| &:= \|\psi_0^{\alpha}\| \cap \|\psi_1^{\alpha}\| \end{aligned}$$

Closure

Given a frame $\mathcal{S}$ and formula φ, $\mathsf{CO}_{\mathcal{S}}(\varphi)$ is the least κ s.t.

$$\|\varphi^{\kappa}\|^{\mathcal{S}} = \|\varphi\|^{\mathcal{S}}$$

Closure Ordinal

$\mathsf{CO}(\varphi)$ is the least ordinal κ such that for all countable frames $\mathcal{S}$

$$\mathsf{CO}_{\mathcal{S}}(\varphi) \leq \kappa$$

Goal

Theorem

A countable ordinal α is the closure ordinal of a formula in the Σ-fragment iff $\alpha < \omega^2$

Σ-fragment

closure under logical connectives, modalities and μ-quantification of

- variables and literals,
- closed formulas in the language of μ-calculus.

$$\mu x. \Diamond x \wedge (\nu z. \Diamond(z \wedge p) \vee \Box \overline{p}) \qquad \mu x. \Diamond x \wedge (\nu z. \Diamond(z \wedge p) \vee \Box x)$$

M.Czarnecki - 2010

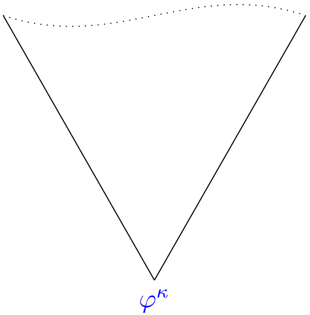
For every $\alpha < \omega^2$ there exists a Σ-formula φ such that $\mathrm{CO}(\varphi) = \alpha$.

The argument

Theorem

If $\alpha < \omega_1$ is the closure ordinal of a Σ-formula φ then $\alpha < \omega^2$.

The argument for $\mathsf{CO}(\varphi) > \kappa$



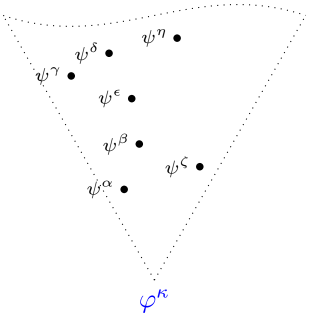
A assume $\mathcal{S}$ maximal for φ^κ

The argument

Theorem

If $\alpha < \omega_1$ is the closure ordinal of a Σ-formula φ then $\alpha < \omega^2$.

The argument for $\mathrm{CO}(\varphi) > \kappa$



A assume $\mathcal{S}$ maximal for φ^κ

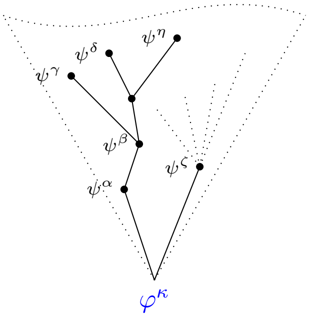
B annotate $\mathrm{CL}(\varphi)$

The argument

Theorem

If $\alpha < \omega_1$ is the closure ordinal of a Σ-formula φ then $\alpha < \omega^2$.

The argument for $\mathsf{CO}(\varphi) > \kappa$



A assume $\mathcal{S}$ maximal for φ^κ

B annotate $\mathsf{CL}(\varphi)$

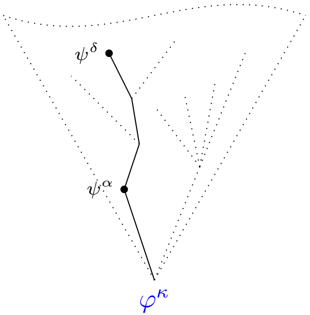
C trace the ordinal progression

The argument

Theorem

If $\alpha < \omega_1$ is the closure ordinal of a Σ-formula φ then $\alpha < \omega^2$.

The argument for $\mathsf{CO}(\varphi) > \kappa$



A assume $\mathcal{S}$ maximal for φ^κ

B annotate $\mathsf{CL}(\varphi)$

C trace the ordinal progression

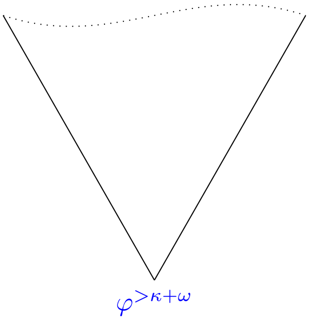
D find repetitions

The argument

Theorem

If $\alpha < \omega_1$ is the closure ordinal of a Σ-formula φ then $\alpha < \omega^2$.

The argument for $\mathsf{CO}(\varphi) > \kappa$



- A assume $\mathcal{S}$ maximal for φ^{κ}
- B annotate $\mathsf{CL}(\varphi)$
- C trace the ordinal progression
- D find repetitions
- E $\mathcal{S}$ is not maximal

Syntax

$$\varphi \coloneqq p \mid \overline{p} \mid x \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \nabla \Gamma \mid \mu x\, \varphi \mid \nu x\, \varphi$$

$$\nabla\{\varphi_0, \ldots \varphi_n\} \coloneqq \bigvee_i^n \Box \varphi_i \vee \Diamond \bigwedge_i^n \varphi_i$$

$$\Box\, \varphi \coloneqq \nabla\{\, \varphi, \bot\,\} \qquad \Diamond\, \varphi \coloneqq \nabla\{\, \varphi\,\} \wedge \nabla\emptyset$$

Conjunctive form φ
For every $\mu x\, \psi \in \mathsf{CL}(\varphi)$, ψ is of the form

$$\bigwedge_{i<k}(\bigvee Z_i \vee \nabla \Gamma_i)$$

with Z_i closed formulas, and every formula in Γ_i is a variable, or is of the form $\mu y\, \psi'$.

Conservative Well-annotation

Well-annotation (Kozen -1988)
Given φ^κ, a *well-annotation* of $\mathcal{S}$ is a function associating to each state $s \in S$ a set Θ_s of annotated formulas from $\mathrm{CL}(\varphi)$ such that

1. if $\mu x^\alpha\, \psi \in \Theta_s$ then $\psi[\mu x\, \psi/x]^\beta \in \Theta_s$ for $\beta < \alpha$;
2. if $(\varphi \wedge \psi)^\alpha \in \Theta_s$ then $\varphi^\alpha, \psi^\alpha \in \Theta_s$;
3. ...

Conservative well-annotation
A well-annotation that is also minimal

1' if $\mu x^\alpha\, \psi \in \Theta_s$ then $\psi[\mu x\, \psi/x]^\beta \in \Theta_s$ for $\alpha = \beta + 1$;

2' ...

$\boldsymbol{O(\varphi, \Gamma)}$
Given φ, Γ, $O(\varphi, \Gamma)$ is the supremum of $\kappa < \omega_1$ for which there exists a conservative Θ s.t. $\Gamma = \Theta_\rho$ and $\varphi^\kappa \in \Theta_\rho$.

Relevant part (1)

Relevant part

We call Φ a *relevant part* of Θ if for every $s \in S$,

1. $\Phi_s \subseteq \Theta_s$;
2. if $\mu x^{\alpha+1}.\psi \in \Phi_s$ then $\psi[\mu x.\psi/x]^{\alpha} \in \Phi_s$;
3. if $(\varphi_0 \wedge \varphi_1)^{\alpha} \in \Phi_s$ then exactly one $\varphi_i^{\alpha} \in \Phi_s$;
4. if $(\varphi_0 \vee \varphi_1)^{\alpha} \in \Phi_s$ then exactly one $\varphi_i^{\alpha} \in \Phi_s$;

$$\mu x.\bigwedge_i(\bigvee Z_i \vee \nabla\Gamma_i)^{\alpha+1}$$
$$\bigwedge_i(\bigvee Z_i \vee \nabla\Gamma_i)^{\alpha}$$
$$(\bigvee Z \vee \nabla\Gamma)^{\alpha}$$
$$\nabla\Gamma^{\alpha}$$

Relevant part (2)

Relevant part

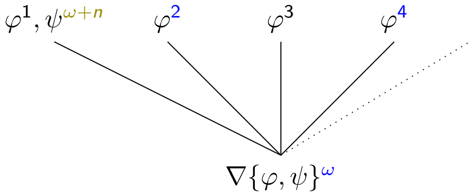
4 if $\nabla\Gamma^{\alpha} \in \Phi_s$ then:

4.1 one formula on every state where $\Diamond$

4.2 one/ω formula/s for every $\Box$

Traces are:

- at most one ∇ per Φ_s
- possibly increasing
- exhaustive



$$\nabla\{\varphi, \psi\}^{\omega} = \Box\varphi^{\omega} \vee \Diamond(\varphi \wedge \psi)^{\omega+n}$$

Relevant part (2)

Relevant part

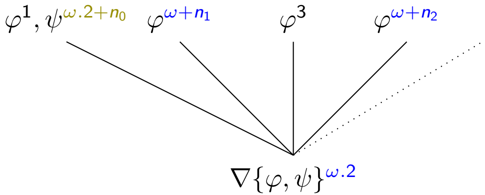
4 if $\nabla\Gamma^{\alpha} \in \Phi_s$ then:

4.1 one formula on every state where $\Diamond$

4.2 one/ω formula/s for every $\Box$

Traces are:

- at most one ∇ per Φ_s
- possibly increasing
- exhaustive



$$\nabla\{\varphi, \psi\}^{\omega.2} = \Box\varphi^{\omega.2} \vee \Diamond(\varphi \wedge \psi)^{\omega.2+n_0}$$

Repetition and Optimality

Repetition pair

A pair of states (s, t) in $\mathcal{S}$ is a *repetition pair* if:

1. there is a path $s = s_0 \ldots s_k = t$,
2. $(\Theta_s^-, \Phi_s^-) = (\Theta_t^-, \Phi_t^-)$,
3. the unique relevant $\nabla\Gamma$ is s.t. $\nabla\Gamma^\alpha \in \Phi_s$ and $\nabla\Gamma^\beta \in \Phi_t$ for $\alpha > \beta$ limit ordinals.

Optimal state

Given a state s in a conservative well annotation Θ, Θ_s is *optimal* with respect to a formula $\varphi^\alpha \in \Theta_s$ if $O(\varphi, \Theta_s) < \alpha + \omega$.

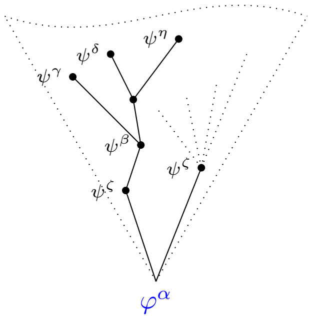
Lemma

On every optimal path there are no repetition pairs.

Up to ω^2

First pumping lemma
For every φ and Γ there exists $N < \omega$ such that, if $O(\varphi, \Gamma) < \omega^2$ then $O(\varphi, \Gamma) \leq \omega.N$.

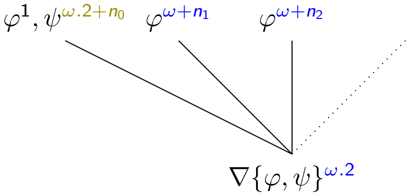
- $N = 2^{2|\mathsf{CL}(\varphi)|} + 1$ and
  $\omega^2 > \alpha > \omega.N$
- Θ optimal for φ^α:
  $O(\varphi, \Theta_\rho) < \alpha + \omega$



Up to ω^2

First pumping lemma
For every φ and Γ there exists $N < \omega$ such that, if $O(\varphi, \Gamma) < \omega^2$ then $O(\varphi, \Gamma) \leq \omega.N$.

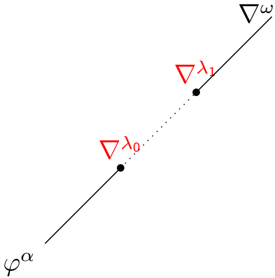
- $N = 2^{2^{|\mathsf{CL}(\varphi)|}} + 1$ and $\omega^2 > \alpha > \omega.N$
- Θ optimal for φ^α: $O(\varphi, \Theta_\rho) < \alpha + \omega$
- an optimal relevant trace is strictly decreasing



Up to ω^2

First pumping lemma
For every φ and Γ there exists $N < \omega$ such that, if $O(\varphi, \Gamma) < \omega^2$ then $O(\varphi, \Gamma) \leq \omega.N$.

- $N = 2^{2|\mathsf{CL}(\varphi)|} + 1$ and $\omega^2 > \alpha > \omega.N$
- Θ optimal for φ^α: $O(\varphi, \Theta_\rho) < \alpha + \omega$
- an optimal relevant trace is strictly decreasing
- N limit points entail a repetition pair on the path
- contradiction by Lemma



Limits on CO

First pumping lemma
For every φ and Γ there exists $N < \omega$ such that, if $O(\varphi, \Gamma) < \omega^2$ then $O(\varphi, \Gamma) \leq \omega.N$.

Second pumping lemma
For every φ and Γ, if $O(x, \Gamma) \leq \kappa$ then $O(x, \Gamma) < \omega^2$.

$$O(\varphi, \Gamma) = \kappa \implies CO(\varphi) \geq \kappa$$
$$CO(\varphi) = \kappa \implies O(\varphi, \Gamma') = \kappa$$

Theorem

A countable ordinal α is the closure ordinal of a formula in the Σ-fragment iff $\alpha < \omega^2$.

Thank you for your attention!

Definitions

Definition (Well-annotation)

An *annotation* of a frame $\mathcal{S}$ for an equation system (X, E) is a function $\Theta\colon \mathcal{S} \to \mathcal{P}(\mathcal{L}_\mu \times \omega_1)$ associating to each state $s \in \mathcal{S}$ a set Θ_s of annotated formulas from $\mathsf{CL}(X, E)$. A *well-annotation* of $\mathcal{S}$ for (X, E) is an annotation Θ such that for all $s \in \mathcal{S}$, φ, Γ and $\alpha < \omega_1$:

1. if $\varphi^\alpha \in \Theta_s$ and $\varphi \in \mathcal{L}_\mu^-$ then $s \in \|\varphi\|^{\mathcal{S}}$;
2. if $x^\alpha \in \Theta_s$ for $x \in X$, then $E(x)^\beta \in \Theta_s$ for some $\beta < \alpha$;
3. if $\bigvee\Gamma^\alpha \in \Theta_s$ then $\Gamma^\beta \cap \Theta_s \neq \emptyset$ for some $\beta \leq \alpha$;
4. if $\bigwedge\Gamma^\alpha \in \Theta_s$ then $\Theta_s \preceq \Gamma^\alpha$;
5. if $\nabla\Gamma^\alpha \in \Theta_s$, then one of the following properties holds
   - 5.1 there exists $r \in R[s]$ such that $\Theta_r \preceq \Gamma^\alpha$,
   - 5.2 there exists $\varphi \in \Gamma$ such that $\Theta_r \preceq \{\, \varphi^\alpha \,\}$ for all $r \in R[s]$.

Definitions

Definition (Relevant part)

Let Θ be a conservative well-annotation of $\mathcal{S}$ and Φ an annotation of the same frame. We call Φ a *relevant part* of Θ if for every $s \in \mathcal{S}$,

1. $\Phi_s \subseteq \Theta_s$;
2. if $x^\alpha \in \Phi_s$ then $E(x)^\beta \in \Phi_s$ where $\alpha = \beta + 1$;
3. if $\bigvee \Gamma^\alpha \in \Phi_s$ and $\alpha > 0$ then $\Gamma^\alpha \cap \Theta_s \subseteq \Phi_s$;
4. if $\bigwedge \Gamma^\alpha \in \Phi_s$ then $\chi^\alpha \in \Phi_s$ for exactly one $\chi \in \Gamma$;
5. if $\nabla \Gamma^\alpha \in \Phi_s$ and $\alpha > 0$ then:
   - 5.1 for all $\eta < \alpha$ and $\varphi \in \Gamma^s_\Box$ there is a $r \in R[s]$ and $\beta > \eta$ s.t. $\varphi^\beta \in \Phi_r$,
   - 5.2 $\Gamma \cap \Phi^-_r \neq \emptyset$ for every $r \in \Gamma^s_\Diamond$, and
   - 5.3 for all $r \in R[s]$ if $\Phi_r \neq \emptyset$ then $y^\beta \in \Phi_r$ for some $y \in \Gamma$ and $\beta > \mathrm{p}(\alpha)$.

Related works

1. Mathis Kretz (2006) - Proof-theoretic aspects of modal logic with fixed points.
2. G. Fontaine (2008) - Continuous Fragment of the mu-Calculus
3. M.J.Gouveia, L.Santocanale (2019) - $\aleph_1$ and the modal μ-calculus
4. G.C. Milanese (2018) - An exploration of closure ordinals in the modal μ-calculus.
5. M.Czarnecki (2010) - How fast can the fixpoints in modal μ-calculus be reached.
6. B.Afshari, G.E.Leigh (2013) - On closure ordinals for the modal mu-calculus