Non-Uniform Polynomial Time and Non-Wellfounded Parsimonious Proofs

FICS

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What is this talk about?

- Parsimonious linear logic = subsystem of LL based on the principles of **parsimonious logic** [MT15, Maz15]

  linear logic modalities $!, ?$ as least and greatest fixed points

  $$!A \;\sim\; \textbf{streams} \qquad\qquad ?A \;\sim\; \textbf{lists}$$

- In [MT15] characterisation of complexity classes via parsimonious logic:

  $$\begin{array}{lcl} \mathbf{P} & = & \text{polynomial time decidability} \\ \mathbf{P}/\text{poly} & = & \textbf{non-uniform} \text{ version of } \mathbf{P} \end{array}$$

- **This talk:** non-wellfounded proof systems for parsimonious linear logic:

  $$\mathsf{wrPLL}_2^\infty \text{ (non-uniform)} \quad \text{vs} \quad \mathsf{rPLL}_2^\infty \text{ (uniform)}$$

- **Our main result:** complexity-theoretic **characterisations**:

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Parsimonious linear logic

- From linear logic...

$$\textit{dig}\,\frac{\Gamma, ??A}{\Gamma, ?A} \qquad \textit{fp}\,\frac{\Gamma, A}{?\Gamma, !A} \qquad \textit{w}\,\frac{\Gamma}{\Gamma, ?A} \qquad \textit{abs}\,\frac{\Gamma, A, ?A}{\Gamma, ?A}$$

digging — functorial promotion — weakening — absorption

- **Key idea:** $!A$ as a type of (very special) **streams**

$$\textit{cut}\,\dfrac{\textit{fp}\,\dfrac{\overset{\mathcal{D}}{A}}{!A} \qquad \textit{abs}\,\dfrac{\otimes\,\dfrac{\textit{ax}\,\dfrac{}{A^\perp, A} \qquad \textit{ax}\,\dfrac{}{?A^\perp, ?A}}{A^\perp, ?A^\perp, A \otimes ?A}}{?A^\perp, A \otimes ?A}}{A \otimes !A} \quad \rightsquigarrow^* \quad \otimes\,\dfrac{\overset{\mathcal{D}}{A} \qquad \textit{fp}\,\dfrac{\overset{\mathcal{D}}{A}}{!A}}{A \otimes !A}$$

$$\mathsf{pop}\,\langle \mathcal{D}, \mathcal{D}, \dots, \mathcal{D}, \dots \rangle \qquad\qquad \mathcal{D} \otimes \langle \mathcal{D}, \mathcal{D}, \dots, \mathcal{D}, \dots \rangle$$

Parsimonious linear logic

- ... to parsimonious linear logic (PLL_2)

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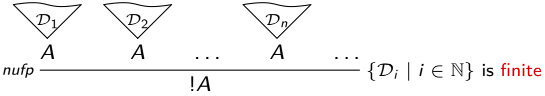
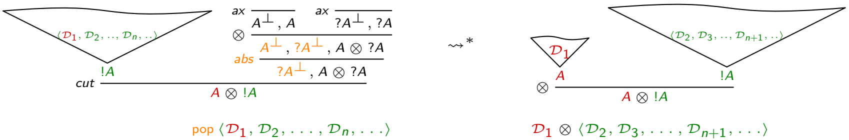
Non-uniform parsimonious linear logic

- Non-uniform PLL (nuPLL) = replace *fp* with the following:



$\sim \quad \langle \mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n, \dots \rangle$

- **Improvement:** $!A$ as a type of streams over finite data of type A



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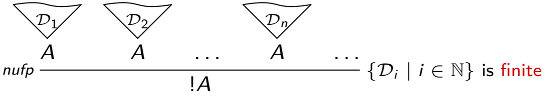

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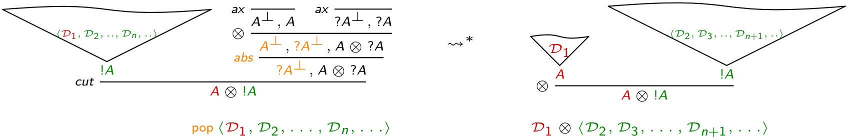

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Non-uniformity via non-wellfoundedness

- **Goal:** non-wellfounded formulations of nuPLL and PLL
- (Partial) **recipe**:

  (1) Replace *fp* and *nufp* with conditional promotion (*cp*): $\mathsf{cp}\dfrac{\Gamma, A \quad ?\Gamma, !A}{?\Gamma, !A}$

  (2) From (wellfounded) **infinite branching** to **non-wellfounded** (finite branching):

$=$ non-wellfounded box

  (3) Progressing condition = logical consistency

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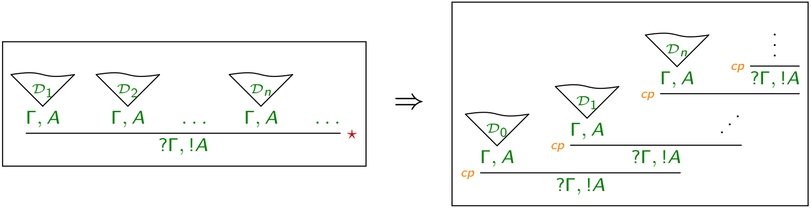

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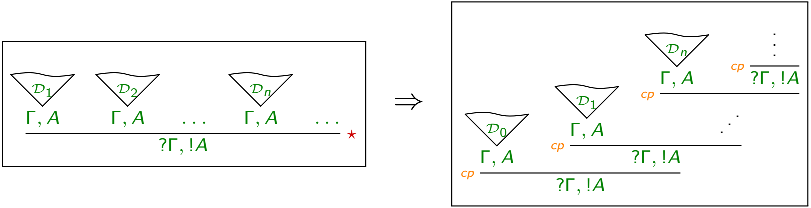

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Regularity vs weak regularity

- Weakly regular = finitely many distinct subproofs whose conclusions are left premises of *cp* rules

  Idea: streams have finite support ...

- Regular proof = finitely many distinct subproofs

  Idea: streams are periodic, so they only represent computable real numbers

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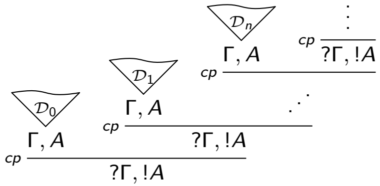

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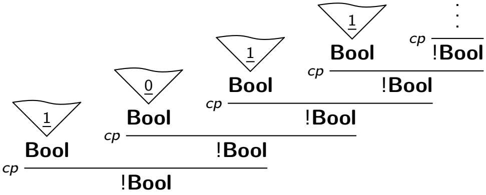

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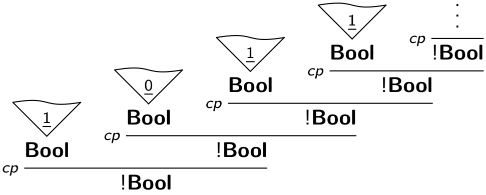

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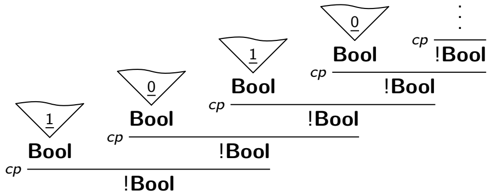

Two non-wellfounded proof systems

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  wrPLL^{∞} = weakly regular progressing [...] proofs

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- Relating inductive and non-wellfounded systems:

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Cyclic implicit complexity

- Implicit computational complexity (ICC) = characterise complexity classes by means of languages/calculi **without** explicit reference to machine models or external resource bounds.
  - Originates with the Bellantoni and Cook's paper on **safe recursion** [BC92].
  - Borrows techniques from recursion theory, proof/type theory, model theory ...
  - Pervasive notion of **stratification**: data are organized into *strata*
  - **Example: light linear logics** = weaker versions of linear logic modality ! that induce a bound on cut-elimination [Gir87, DJ03, Laf04]
- Cyclic Implicit Complexity (CIC)
  - introduced in joint works with Anupam Das [CD22, CD23b], aiming at studying the principles of ICC using the tools of non-wellfounded and cyclic proof-theory
  - characterisation of **P** and **ELEMENTARY** [CD22], and **P**/poly [CD23b]
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Computational meaning of regularity conditions

- Regular proofs = **finitely** many distinct subproofs

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- Weakly regular proofs = relaxation of regularity to represent real numbers

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Our characterisation results in a nutshell

Theorem:

	inductive	**non-wellfounded**
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Idea of the proof.

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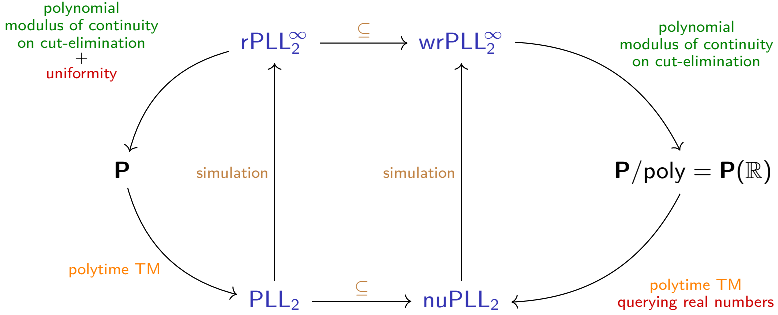

Conclusion and future work

- **To sum up:**
  - We introduced two non-wellfounded proof systems rPLL^∞ and wrPLL^∞
  - We showed that the second-order extensions of rPLL^∞ and wrPLL^∞ characterise, respectively, **P** and **P**/poly

- **Ongoing work:**

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...i.e., we restate in a non-wellfounded setting the results in [Maz15, MT15].

Thank you! Questions?

References:

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Appendix

Finite expandability

- Finitely expandable proof = any branch has finitely many *cut* and *abs* rules

  Example:

$$\mathit{cut}\ \dfrac{\mathit{ax}\ \dfrac{}{A^{\perp}, A} \qquad \mathit{cut}\ \dfrac{\mathit{ax}\ \dfrac{}{A^{\perp}, A} \qquad \mathit{cut}\ \dfrac{\vdots}{\Gamma, A}}{\Gamma, A}}{\Gamma, A} \qquad\qquad \mathit{abs}\ \dfrac{\mathit{abs}\ \dfrac{\mathit{abs}\ \dfrac{\vdots}{A, A, ?A}}{A, ?A}}{?A}$$

- **Theorem:** **decomposition** for finitely expandable and progressing proofs

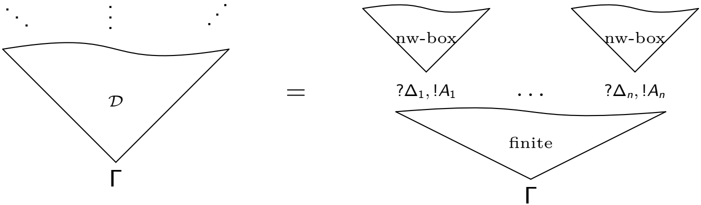
Finite expandability

- Finitely expandable proof = any branch has finitely many *cut* and *abs* rules

  Example:

$$\mathit{cut}\,\dfrac{\mathit{ax}\,\dfrac{}{A^{\perp}, A} \qquad \mathit{cut}\,\dfrac{\mathit{ax}\,\dfrac{}{A^{\perp}, A} \quad \mathit{cut}\,\dfrac{\vdots}{\Gamma, A}}{\Gamma, A}}{\Gamma, A} \qquad\qquad \mathit{abs}\,\dfrac{\mathit{abs}\,\dfrac{\mathit{abs}\,\dfrac{\vdots}{A, A, ?A}}{A, ?A}}{?A}$$

- **Theorem: decomposition** for finitely expandable and progressing proofs



Cut-elimination rules for non-wellfounded parsimonious logic

- **Cut-elimination rules** for the exponential modalities ! and ?:

$$
\dfrac{\dfrac{\Gamma, A \quad ?\Gamma, !A}{?\Gamma, !A}\,\mathsf{cp} \qquad \dfrac{A^\perp, \Delta, B \quad ?A^\perp, ?\Delta, !B}{?A^\perp, ?\Delta, !B}\,\mathsf{cp}}{?\Gamma, ?\Delta, !B}\,\mathsf{cut}
\rightsquigarrow
\dfrac{\dfrac{\Gamma, A \quad A^\perp, \Delta, B}{\Gamma, \Delta, B}\,\mathsf{cut} \qquad \dfrac{?\Gamma, !A \quad ?A^\perp, ?\Delta, !B}{?\Gamma, ?\Delta, !B}\,\mathsf{cut}}{?\Gamma, ?\Delta, !B}\,\mathsf{cp}
$$

$$
\dfrac{\dfrac{\Gamma, A \quad ?\Gamma, !A}{?\Gamma, !A}\,\mathsf{cp} \qquad \dfrac{\Delta}{\Delta, ?A^\perp}\,\mathsf{w}}{?\Gamma, \Delta}\,\mathsf{cut}
\rightsquigarrow
\dfrac{\Delta}{\overline{?\Gamma, \Delta}}\,\mathsf{w}
$$

$$
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$$

Cut-elimination rules preserve progressing, (weak) regularity, and finite expandability conditions

Our domain-theoretic approach

Starting from non-wellfonded proof $\mathcal{D}$:

- Special **infinitary rewriting strategies** σ that induce continuous functions over domains of (partially defined) non-wellfounded proofs
- **Productivity:** If $\mathcal{D}$ is progressing non-wellfounded proof then $f_\sigma(\mathcal{D})$ is (cut-free and) totally defined
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Approximating non-wellfounded proofs

(1) A new rule: the **hypothesis**

$$hyp \, \frac{}{\Gamma}$$

(2) Open proof = non-wellfounded proof that might contain *hyp*

(3) Normal open proof = proofs where *cut* rules are irreducible
(e.g., cut between hypotheses)

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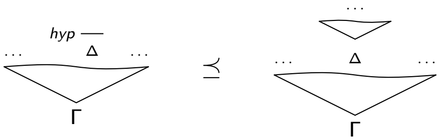

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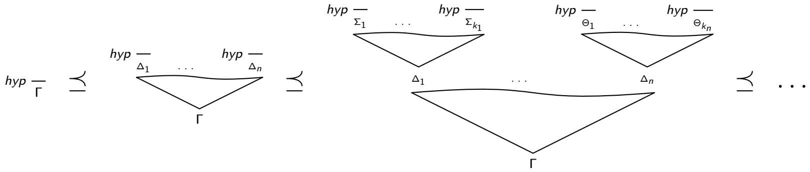

Infinitary cut-elimination strategies

- Infinitary cut-elimination strategy (ices) := family $\sigma = (\sigma_{\mathcal{D}})_{\mathcal{D} \in \mathsf{oPLL}^\infty}$ where each $\sigma_{\mathcal{D}}$ is a countable sequence of proofs such that:

$$\mathcal{D} = \sigma_{\mathcal{D}}(0) \rightsquigarrow \sigma_{\mathcal{D}}(1) \rightsquigarrow \ldots \rightsquigarrow \sigma_{\mathcal{D}}(n) \rightsquigarrow \ldots$$

- Given an ices σ we define $f_\sigma : \mathsf{oPLL}^\infty(\Gamma) \to \mathsf{oPLL}^\infty(\Gamma)$ as

$$f_\sigma(\mathcal{D}) := \bigsqcup_{i=0}^{\ell(\sigma_{\mathcal{D}})} \mathsf{cf}(\sigma_{\mathcal{D}}(i))$$

where $\mathsf{cf}(\mathcal{D}_i)$ is the greatest cut-free approximation of $\mathcal{D}_i$ (w.r.t. $\preceq$)

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Continuous cut-elimination theorem

- **Existence of** mc-ices**:** intuitively, always apply a cut-elimination step to the leftmost reducible *cut* rule with minimal height.
- **Confluence:** if σ and σ' are mc-ices, then $f_\sigma = f_{\sigma'}$
- **Theorem (Continuous cut-elimination):** Given σ a mc-ices:
  (1) $\mathcal{D}$ is progressing then $f_\sigma(\mathcal{D})$ is *hyp*-free (*productivity*)
  (2) f_σ preserves progressing and finite expandability
  (3) If $\mathcal{D} \in \mathsf{wrPLL}^\infty$ then $f_\mathcal{D}(\mathcal{D}) \in \mathsf{wrPLL}^\infty$
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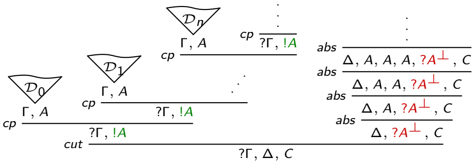

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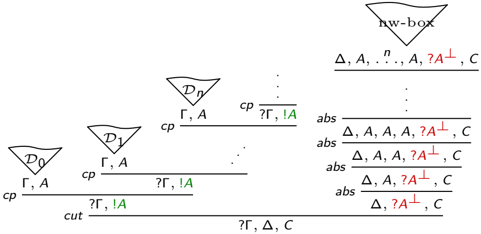

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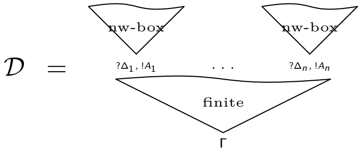

We define a transfinite cut-elimination sequence preserving (weak) regularity by induction on the "nesting" of non-wellfounded boxes:

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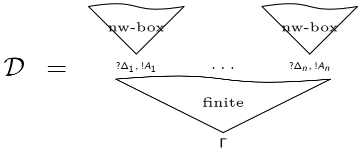

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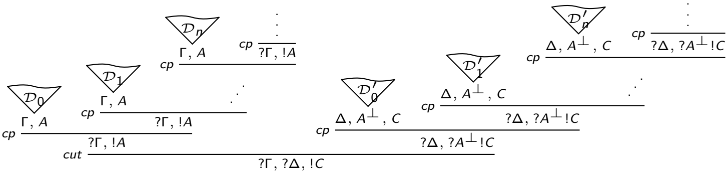

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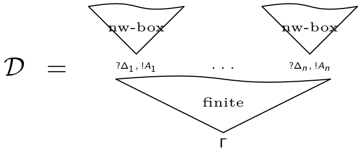

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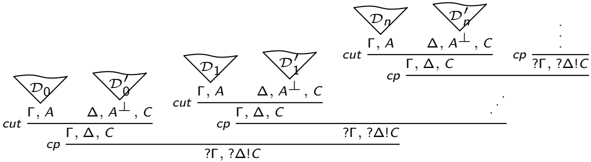

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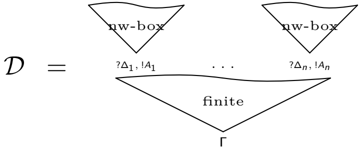

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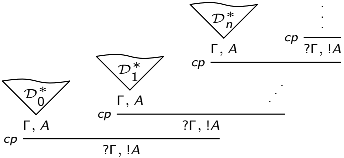

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5 Non-uniform complexity classes

Non-uniform complexity classes

- **FP** $=$ class of functions computable in polynomial time on a Turing machine.
- **FP**/poly is an extension of **FP** that intuitively has access to a 'small' amount of *advice*, determined only by the length of the input.
- **FP**/poly $=$ class of functions $f(\vec{x})$ for which there exists some strings $\alpha_{\vec{n}} \in \{0,1\}^*$ and a function $f'(x, \vec{x}) \in$ **FP** with:
  - $|\alpha_{\vec{n}}|$ is polynomial in $\vec{n}$.
  - $f(\vec{x}) = f'(\alpha_{|\vec{x}|}, \vec{x})$.
- Note, in particular, that **FP**/poly admits undecidable problems. E.g. the function $f(x) = 1$ just if $|x|$ is the code of a halting Turing machine (and 0 otherwise) is in **FP**/poly. Indeed, the point of the class **FP**/poly is to rather characterise a more non-uniform notion of computation.
- **Theorem**: $f(\vec{x}) \in$ **FP**/poly iff there are poly-size circuits computing $f(\vec{x})$.

- The class $\mathbf{FP}(\mathbb{R})$ consists of just the functions computable in polynomial time by a Turing machine with access to oracles from:

$$\mathbb{R} := \{f(x) : \mathbb{N} \to \{0, 1\} \mid |x| = |y| \implies f(x) = f(y)\}$$

- Note that the notation $\mathbb{R}$ is suggestive here, since its elements are essentially maps from lengths/positions to Booleans, and so may be identified with Boolean streams.

- **Theorem** [Folklore]: $\mathbf{FP}/\mathrm{poly} = \mathbf{FP}(\mathbb{R})$.