Lifting final coalgebras and initial algebras, a reconstruction

Luigi Santocanale, Gregory Chichery

LIS, CNRS UMR 7020, Aix-Marseille Université, France

FICS@Napoli, 19/02/2024

Context

For a category C and a functor $Q : C \to \text{Pos}$, the Grothendieck construction $\int Q$ is so described:

- An object is a pair (X, α) with $X \in C$ and $\alpha \in Q(X)$.

Context

For a category C and a functor $Q : C \to \text{Pos}$, the Grothendieck construction $\int Q$ is so described:

- An object is a pair (X, α) with $X \in C$ and $\alpha \in Q(X)$.
- An arrow $f : (X, \alpha) \to (Y, \beta)$ is an arrow $f : X \to Y$ of $\mathcal{C}$ such that $Q(f)(\alpha) \leq \beta$.

Context

For a category C and a functor $Q : C \to \text{Pos}$, the Grothendieck construction $\int Q$ is so described:

- An object is a pair (X, α) with $X \in C$ and $\alpha \in Q(X)$.
- An arrow $f : (X, \alpha) \to (Y, \beta)$ is an arrow $f : X \to Y$ of $\mathcal{C}$ such that $Q(f)(\alpha) \leq \beta$.

The first projection $\pi : \int Q \to C$ is an op-fibration.

Lifting star-autonomous structures [2]

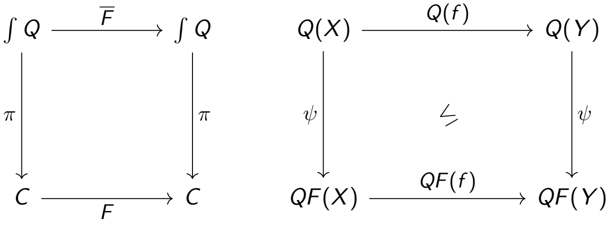
The aim of this preprint paper is lifting many structures (functors, star autonomous structures) to the Grothendieck construction $\int Q$ of a functor $Q : C \to \mathrm{Pos}$.

Lifting star-autonomous structures [2]

The aim of this preprint paper is lifting many structures (functors, star autonomous structures) to the Grothendieck construction $\int Q$ of a functor $Q : C \to \mathrm{Pos}$.

Theorem

There is a bijection between liftings of a functor $F : C \to C$ *to* $\overline{F} : \int Q \to \int Q$ *and lax natural transformations* $\psi : Q \to QF$.



Lifting star-autonomous structures [2]

The aim of this preprint paper is lifting many structures (functors, star autonomous structures) to the Grothendieck construction $\int Q$ of a functor $Q : C \to \text{Pos}$.

Theorem

There is a bijection between liftings of a functor $F : C \to C$ to $\overline{F} : \int Q \to \int Q$ and lax natural transformations $\psi : Q \to QF$.

$$
\begin{array}{ccc}
\int Q & \xrightarrow{\overline{F}} & \int Q \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
C & \xrightarrow[F]{} & C
\end{array}
\qquad
\begin{array}{ccc}
Q(X) & \xrightarrow{Q(f)} & Q(Y) \\
\downarrow{\scriptstyle \psi} & \leq & \downarrow{\scriptstyle \psi} \\
QF(X) & \xrightarrow{QF(f)} & QF(Y)
\end{array}
$$

Remark

For the rest, let's fix $\overline{F}$ a lifting with ψ the correspondant lax natural transormations.

Lifting coalgebras

For a coalgebra $(X, \gamma : X \to F(X))$, define

$$Q^{\nu}(X,\gamma) := \{\alpha \in Q(X) \quad | \quad Q(\gamma)(\alpha) \leq \psi(\alpha)\}.$$

SLatt is the category of posets with joins and maps that preserve them.

Lifting coalgebras

For a coalgebra $(X, \gamma : X \to F(X))$, define

$$Q^{\nu}(X, \gamma) := \{\alpha \in Q(X) \quad | \quad Q(\gamma)(\alpha) \leq \psi(\alpha)\}.$$

Proposition

([4, 5, 1, 6], Folklore ?)
Q^{ν} extends (in an obvious way) to a functor

$$Q^{\nu} : \mathsf{CoAlg}_C(F) \to \mathsf{Pos}$$

SLatt is the category of posets with joins and maps that preserve them.

Lifting coalgebras

For a coalgebra $(X, \gamma : X \to F(X))$, define

$$Q^\nu(X, \gamma) := \{\alpha \in Q(X) \quad | \quad Q(\gamma)(\alpha) \leq \psi(\alpha)\}.$$

Proposition

([4, 5, 1, 6], Folklore ?)
Q^ν extends (in an obvious way) to a functor

$$Q^\nu : \mathsf{CoAlg}_C(F) \to \mathsf{Pos}$$

and we have an isomorphism

$$\mathsf{CoAlg}_{\int Q}(\overline{F}) \simeq \int Q^\nu.$$

SLatt is the category of posets with joins and maps that preserve them.

Lifting coalgebras

For a coalgebra $(X, \gamma : X \to F(X))$, define

$$Q^\nu(X,\gamma) := \{\alpha \in Q(X) \quad | \quad Q(\gamma)(\alpha) \leq \psi(\alpha)\}.$$

Proposition

([4, 5, 1, 6], Folklore ?)
Q^ν extends (in an obvious way) to a functor

$$Q^\nu : \mathsf{CoAlg}_C(F) \to \mathsf{Pos}$$

and we have an isomorphism

$$\mathsf{CoAlg}_{\int Q}(\overline{F}) \simeq \int Q^\nu.$$

Remark

If $Q : C \to \mathsf{SLatt}$, then so $Q^\nu : C \to \mathsf{SLatt}$.

SLatt is the category of posets with joins and maps that preserve them.

Final coalgebras

We can therefore restrict to the question of lifting terminal objects to the total category.

Final coalgebras

We can therefore restrict to the question of lifting terminal objects to the total category.

Lemma

Consider a functor $G : D \to \mathrm{Pos}$*. If* 1 *is a terminal object of* D *and* $\top \in G(1)$ *is the greatest element of this poset, then* $(1, \top)$ *is a terminal object of* $\int G$*.*

Final coalgebras

We can therefore restrict to the question of lifting terminal objects to the total category.

Lemma

Consider a functor $G : D \to \mathrm{Pos}$. If 1 is a terminal object of D and $\top \in G(1)$ is the greatest element of this poset, then $(1, \top)$ is a terminal object of $\int G$.

Proposition

(c.f. [3, Theorem 2.6] and [4, Corollary 4.3])
Given a final coalgebra $(\nu.F, \xi)$. If the greatest fixed point $\nu.f$ of

$$f := Q(\xi^{-1}) \circ \psi_{\nu.F} : Q(\nu.F) \to Q(F(\nu.F)) \to Q(\nu.F)$$

exists, then $(\nu.F, \nu.f, \xi)$ is a final coalgebra of $\overline{F}$.

Lifting algebras: natural case

Because of lax naturality of ψ, we do not have, in general, a similar representation for $\mathsf{Alg}_{\int Q}(\overline{F})$.

Lifting algebras: natural case

Because of lax naturality of ψ, we do not have, in general, a similar representation for $\mathsf{Alg}_{\int Q}(\overline{F})$.

Proposition

Suppose that ψ is natural. Define

$$Q^{\mu}(X,\gamma) := \{\alpha \in Q(X) \quad | \quad Q(\gamma)(\psi(\alpha)) \leq \alpha\}.$$

Then Q^{μ} extends (in an obvious way) to a functor

$$Q^{\mu} : \mathsf{Alg}_{C}(F) \to \mathsf{Pos}$$

Lifting algebras: natural case

Because of lax naturality of ψ, we do not have, in general, a similar representation for $\mathsf{Alg}_{\int Q}(\overline{F})$.

Proposition

Suppose that ψ *is natural. Define*

$$Q^{\mu}(X, \gamma) := \{\alpha \in Q(X) \quad | \quad Q(\gamma)(\psi(\alpha)) \leq \alpha\}.$$

Then Q^{μ} *extends (in an obvious way) to a functor*

$$Q^{\mu} : \mathsf{Alg}_{C}(F) \to \mathsf{Pos}$$

and we have an isomorphism $\mathsf{Alg}_{\int Q}(\overline{F}) \simeq \int Q^{\mu}$.

Lifting algebras: natural case

Because of lax naturality of ψ, we do not have, in general, a similar representation for $\mathsf{Alg}_{\int Q}(\overline{F})$.

Proposition

Suppose that ψ is natural. Define

$$Q^{\mu}(X,\gamma) := \{\alpha \in Q(X) \quad | \quad Q(\gamma)(\psi(\alpha)) \leq \alpha\}.$$

Then Q^{μ} extends (in an obvious way) to a functor

$$Q^{\mu} : \mathsf{Alg}_{C}(F) \to \mathsf{Pos}$$

and we have an isomorphism $\mathsf{Alg}_{\int Q}(\overline{F}) \simeq \int Q^{\mu}$.

Remark

In litterature, $\overline{F}$ is often required to preserve cartesian arrows. It is equivalent that ψ required to be natural.

Lifting algebras: with duality (1)

Even if ψ is not natural, we can say something when $Q : C \to \mathsf{SLatt}$, exploiting the internal duality of SLatt.

Lifting algebras: with duality (1)

Even if ψ is not natural, we can say something when $Q : C \to \mathsf{SLatt}$, exploiting the internal duality of SLatt.

Let $Q^* : C^{\mathsf{op}} \to \mathsf{SLatt}$ be defined by $Q^*(X) := Q(X)^{\mathsf{op}}$ and $Q^*(f) = Q(f)^*$, the right adjoint to $Q(f)$.

$$C^{\mathsf{op}} \xrightarrow{Q} \mathsf{SLatt}^{\mathsf{op}} \xrightarrow{(-)^*} \mathsf{SLatt}$$

Lifting algebras: with duality (1)

Even if ψ is not natural, we can say something when $Q : C \to \mathsf{SLatt}$, exploiting the internal duality of SLatt.

Let $Q^* : C^{op} \to \mathsf{SLatt}$ be defined by $Q^*(X) := Q(X)^{op}$ and $Q^*(f) = Q(f)^*$, the right adjoint to $Q(f)$.

$$C^{op} \xrightarrow{Q} \mathsf{SLatt}^{op} \xrightarrow{(-)^*} \mathsf{SLatt}$$

Lemma

We have $(\int Q)^{op} = \int Q^*$ and, moreover, $\overline{F}^{op}$ is the lifting of $F^{op} : C^{op} \to C^{op}$ to $\int Q^*$ via $\psi^{op} : Q(X)^{op} \to Q(F(X))^{op}$.

Lifting algebras: with duality (2)

Theorem

If $Q : C \to \mathsf{SLatt}$, *then*

$$\mathsf{Alg}_{\int Q}(\overline{F}) = [\mathsf{CoAlg}_{\int Q^*}(\overline{F^{\mathsf{op}}})]^{\mathsf{op}} \simeq [\int Q^{*\nu}]^{\mathsf{op}} = \int Q^{*\nu *}. \qquad (1)$$

Remark

This isomorphism allows us to use the terminal coalgebras results.

Lifting algebras: with duality (2)

Theorem

If $Q : C \to \mathsf{SLatt}$, *then*

$$\mathsf{Alg}_{\int Q}(\overline{F}) = [\mathsf{CoAlg}_{\int Q^*}(\overline{F^{\mathsf{op}}})]^{\mathsf{op}} \simeq [\int Q^{*\nu}]^{\mathsf{op}} = \int Q^{*\nu *}. \qquad (1)$$

Remark

This isomorphism allows us to use the terminal coalgebras results.

Moreover, put $Q^{\mu} := Q^{*\nu *} : \mathsf{Alg}_C(F) \to \mathsf{SLatt}$, we have an explicit description of this functor: for $f : (X, \gamma) \to (Y, \delta) \in Q^{\mu}(X, \gamma)$,

$$Q^{\mu}(X, \gamma) = \{\alpha \in Q(X) \mid Q(\gamma)(\psi_X(\alpha)) \leq \alpha\}\,, \qquad (2)$$

$$Q^{\mu}(f)(\alpha) = \text{least } \beta \in Q^{\mu}(Y, \delta) \text{ such that } Q(f)(\alpha) \leq \beta\,. \qquad (3)$$

Lifting algebras: with duality (2)

Theorem

If $Q : C \to \mathsf{SLatt}$, *then*

$$\mathsf{Alg}_{\int Q}(\overline{F}) = [\mathsf{CoAlg}_{\int Q^*}(\overline{F^{\mathsf{op}}})]^{\mathsf{op}} \simeq [\int Q^{*\nu}]^{\mathsf{op}} = \int Q^{*\nu *}. \quad (1)$$

Remark

This isomorphism allows us to use the terminal coalgebras results.

Moreover, put $Q^\mu := Q^{*\nu *} : \mathsf{Alg}_C(F) \to \mathsf{SLatt}$, we have an explicit description of this functor: for $f : (X,\gamma) \to (Y,\delta) \in Q^\mu(X,\gamma)$,

$$Q^\mu(X,\gamma) = \{\alpha \in Q(X) \mid Q(\gamma)(\psi_X(\alpha)) \leq \alpha\}, \quad (2)$$

$$Q^\mu(f)(\alpha) = \text{least } \beta \in Q^\mu(Y,\delta) \text{ such that } Q(f)(\alpha) \leq \beta\ . \quad (3)$$

Remark

If ψ *is natural, this definition of* Q^μ *coincide with the previous.*

Lifting algebras: more general

Suppose that $Q : C \to \mathsf{SLatt}^{-}$, where SLatt^{-} is the category of complete lattice with morphisms of posets.

Lifting algebras: more general

Suppose that $Q : C \to \mathsf{SLatt}^-$, where SLatt^- is the category of complete lattice with morphisms of posets.

Then with the previous definition $Q^\mu : \mathsf{Alg}_C(F) \to \mathsf{SLatt}^-$ is an oplax functor, that is, it satisfies $Q^\mu(gf) \leq Q^\mu(g)Q^\mu(f)$. So the general Grothendieck construction allows us to define the category $\int Q^\mu$.

Lifting algebras: more general

Suppose that $Q : C \to \mathsf{SLatt}^-$, where SLatt^- is the category of complete lattice with morphisms of posets.

Then with the previous definition $Q^\mu : \mathsf{Alg}_C(F) \to \mathsf{SLatt}^-$ is an oplax functor, that is, it satisfies $Q^\mu(gf) \le Q^\mu(g)Q^\mu(f)$. So the general Grothendieck construction allows us to define the category $\int Q^\mu$.

Proposition

Again, we have $\mathsf{Alg}_{\int Q}(\overline{F}) \simeq \int Q^\mu$.

Initial algebras (1)

We can therefore restrict to the question of lifting initial objects to the total category.

Initial algebras (1)

We can therefore restrict to the question of lifting initial objects to the total category.

Lemma

If 0 *is an initial object of* D*,* $\bot \in G(0)$ *is the least element of this poset, and, for each object* X *of* D*, the unique arrow* $?_X : 0 \to X$ *is such that* $G(?_X)(\bot)$ *is the least element of* $G(X)$*, then* $(0, \bot)$ *is an initial object of* $\int G$*.*

Initial algebras (1)

We can therefore restrict to the question of lifting initial objects to the total category.

Lemma

If 0 *is an initial object of* D*,* $\bot \in G(0)$ *is the least element of this poset, and, for each object* X *of* D*, the unique arrow* $?_X : 0 \to X$ *is such that* $G(?_X)(\bot)$ *is the least element of* $G(X)$*, then* $(0, \bot)$ *is an initial object of* $\int G$*.*

Proposition

Using one of the three definition of Q^μ*, given an initial algebra* $(\mu.F, \xi)$*, if the least fixed point* $\mu.f$ *of*

$$f := Q(\xi) \circ \psi_{\mu.F} : Q(\mu.F) \to Q(F(\mu.F)) \to Q(\mu.F)$$

exists, and the $Q(?_X)$ *preserves least fixed point of* $Q(\gamma)\psi_X$*, then* $(\mu.F, \mu.f, \xi)$ *is an initial algebra of* $\overline{F}$*.*

Initial algebras (2)

Corollary

If we have $Q : C \to \mathsf{SLatt}$*, then so is* Q^μ*, morever an initial algebra* $(\mu.F, \xi)$ *of* F *give us an initial algebra* $(\mu.F, \mu.f, \xi)$ *of* $\overline{F}$*.*

Initial algebras (2)

Corollary

If we have $Q : C \to \mathsf{SLatt}$, *then so is* Q^{μ}, *morever an initial algebra* $(\mu.F, \xi)$ *of* F *give us an initial algebra* $(\mu.F, \mu.f, \xi)$ *of* $\overline{F}$.

Proposition

Suppose that the $Q(X)$ *are complete lattice and the* $Q(f)$ *preserves suprema of (possibly empty) chains. Then an initial algebra* $(\mu.F, \xi)$ *of* F *gives us an initial algebra* $(\mu.F, \mu.f, \xi)$ *of* $\overline{F}$.

Initial algebras (2)

Corollary

If we have $Q : C \to \mathsf{SLatt}$, then so is Q^{μ}, morever an initial algebra $(\mu.F, \xi)$ of F give us an initial algebra $(\mu.F, \mu.f, \xi)$ of $\overline{F}$.

Proposition

Suppose that the $Q(X)$ are complete lattice and the $Q(f)$ preserves suprema of (possibly empty) chains. Then an initial algebra $(\mu.F, \xi)$ of F gives us an initial algebra $(\mu.F, \mu.f, \xi)$ of $\overline{F}$.

The proposition relies on and establishes a link with [3], where a constructive setting is being considered: ipos+Pataraya's least fixed point theorem.

Initial algebras: a lemma

This is a consequence of the next lemma applied to

$$
\begin{array}{ccccc}
Q(\mu.F) & \xrightarrow{\psi_{\mu.F}} & Q(F(\mu.F)) & \xrightarrow{Q(\xi)} & Q(\mu.F) \\
{\scriptstyle Q(?_X)}\downarrow & \lightning & \downarrow{\scriptstyle Q(F(?_X))} & & \downarrow{\scriptstyle Q(?_X)} \\
Q(X) & \xrightarrow[\psi_X]{} & Q(F(X)) & \xrightarrow[Q(\gamma)]{} & Q(X)
\end{array}
$$

Initial algebras: a lemma

This is a consequence of the next lemma applied to

$$
\begin{array}{ccccc}
Q(\mu.F) & \xrightarrow{\psi_{\mu.F}} & Q(F(\mu.F)) & \xrightarrow{Q(\xi)} & Q(\mu.F) \\
\downarrow{\scriptstyle Q(?_X)} & \leq & \downarrow{\scriptstyle Q(F(?_X))} & & \downarrow{\scriptstyle Q(?_X)} \\
Q(X) & \xrightarrow[\psi_X]{} & Q(F(X)) & \xrightarrow[Q(\gamma)]{} & Q(X)
\end{array}
$$

Lemma

Consider a half-commuting diagram of posets as the one below. If A and B are complete lattices and f preserves suprema of (possibly empty) chains, then $f(\mu.g^A) \leq \mu.g^B$*.*

$$
\begin{array}{ccc}
A & \xrightarrow{g^A} & A \\
\downarrow{\scriptstyle f} & \leq & \downarrow{\scriptstyle f} \\
B & \xrightarrow[g^B]{} & B
\end{array}
$$

THANKS

(Part of) Related literature

[1] F. Bonchi, D. Petrisan, D. Pous, and J. Rot.
Coinduction up-to in a fibrational setting.
In T. A. Henzinger and D. Miller, editors, *CSL-LICS '14, Vienna, Austria, July 14 - 18, 2014*, pages 20:1–20:9. ACM, 2014.

[2] C. de Lacroix, G. Chichery, and L. Santocanale.
Lifting star-autonomous structures.
In *CSL 2024*, July 2023.

[3] M. Fiore, Z. Galal, and F. Jafarrahmani.
Fixpoint constructions in focused orthogonality models of linear logic.
To appear in the proceeedings of the conference MFPS 2023, Electronic Notes in Theoretical Informatics and Computer Science, 2023.

[4] I. Hasuo, K. Cho, T. Kataoka, and B. Jacobs.
Coinductive predicates and final sequences in a fibration.
Electronic Notes in Theoretical Computer Science, 298:197–214, 2013.
Proceedings of the Twenty-ninth Conference on the Mathematical Foundations of Programming Semantics, MFPS XXIX.

[5] B. Jacobs.
Invariants and Assertions, page 334–439.
Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2016.

[6] D. Sprunger, S. Katsumata, J. Dubut, and I. Hasuo.
Fibrational bisimulations and quantitative reasoning: Extended version.
J. Log. Comput., 31(6):1526–1559, 2021.