The μ-calculus' Alternation Hierarchy is Strict over Non-Trivial Fusion Logics

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20 February 2024

Available at: leonardopacheco.xyz/slides/fics2024.pdf

THE μ-CALCULUS

modal μ-calculus = modal logic + fixed-point operators

- μ: least fixed-point operator
- ν: greatest fixed-point operator

ALTERNATION DEPTH

The valuation of νX and μY depend on each other:

$$\nu X.\, \underbrace{\mu Y.\, \overbrace{(P \wedge \Diamond X) \vee (\neg P \wedge \Diamond Y)}^{\text{scope of } \mu Y}}_{\text{scope of } \nu X}$$

Alternation depth of φ

Maximum number of codependent alternating μ and ν operators in φ.

Alternation hierarchy

Classifies μ-formulas with respect to their alternation depth.

SOME RESULTS ON THE UNIMODAL μ-CALCULUS

Theorem (Bradfield [2])

The μ-calculus alternation hierarchy is strict over all frames.

Theorem (Alberucci–Facchini [1])

The μ-calculus alternation hierarchy collapses to the alternation-free fragment over transitive frames.

Theorem (Alberucci–Facchini [1])

The μ-calculus alternation hierarchy collapses to modal logic over equivalence relations.

For a survey, see [4].

Our result — simplified

The fusion $\mathsf{S5} \otimes \mathsf{S5}$ contains two independent pairs of modalities $\Box_0/\Diamond_0$ and $\Box_1/\Diamond_1$, each satisfying $\mathsf{S5}$.

Theorem

The μ-calculus' alternation hierarchy is strict over $\mathsf{S5} \otimes \mathsf{S5}$.

This holds for the fusion of any two non-trivial logics.

DEFINITIONS

The μ-formulas are defined by the following grammar:

$$\varphi := P \mid \neg P \mid X \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box_i \varphi_i \mid \Diamond_i \varphi \mid \mu X.\varphi \mid \nu X.\varphi,$$

Let $M = \langle W, R_0, R_1, V \rangle$ be a Kripke model. Then:

- $M, w \models \Box_i \varphi$ iff, for all v, if wR_iv then $M, u \models \varphi$;
- $M, w \models \Diamond_i \varphi$ iff there is v such that wR_iv and $M, u \models \varphi$.

Given a μ-formula φ, define:

$$\Gamma_{\varphi(X)}(A) \to \|\varphi(A)\|^M.$$

Then:

- $M, w \models \mu X.\varphi$ iff w is in the least fixed point of $\Gamma_{\varphi(X)}$;
- $M, w \models \nu X.\varphi$ iff w is in the greatest fixed point of $\Gamma_{\varphi(X)}$.

ALTERNATION HIERARCHY

- $\Sigma_0^\mu (= \Pi_0^\mu)$:= set of all formulas with no fixed-point operators.
- Σ_{n+1}^μ is the closure of $\Sigma_n^\mu \cup \Pi_n^\mu$ under:
  - propositional operators;
  - modal operators;
  - μX;
  - and the substitution: if $\varphi(X) \in \Sigma_{n+1}^\mu$ and $\psi \in \Sigma_{n+1}^\mu$ are such that no free variable of ψ becomes bound in $\varphi(\psi)$, then $\varphi(\psi) \in \Sigma_{n+1}^\mu$.
- Π_{n+1}^μ is the dual of Σ_{n+1}^μ.

Game Semantics

We define an evaluation game for $M, w \models \varphi$.

- Two players: Verifier and Refuter.
- Examples of moves:
  - At $\langle \psi \vee \theta, w \rangle$, Verifier moves to one of $\langle \psi, w \rangle$ and $\langle \theta, w \rangle$.
  - At $\langle \Diamond_i \psi, w \rangle$, Refuter picks v such that wR_iv and moves to $\langle \psi, v \rangle$.
  - At $\langle X, w \rangle$, go to $\langle \mu X.\psi, w \rangle$.
  - At $\langle P, w \rangle$, Verifier wins iff $w \in V(P)$.
- $M, w \models \varphi$ iff Verifier wins the evaluation game.

On an infinite run, if the variable with biggest scope which repeats infinitely often is ν, then Verifier wins.

Proposition

Kripke semantics and game semantics are equivalent.

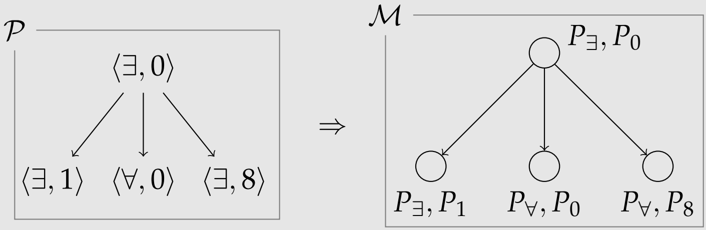
Parity games

- $\mathcal{P} = \langle V_\exists, V_\forall, v_0, E, \Omega \rangle$
- Two players $\exists$ and $\forall$ move a token in the graph $\langle V_\exists \cup V_\forall, E \rangle$ starting at v_0.
- $\exists$ wins $\rho = v_0, v_1, v_2, \ldots$ iff the greatest priority $\Omega(v_i)$ which appears infinitely often in ρ is even.

Proposition

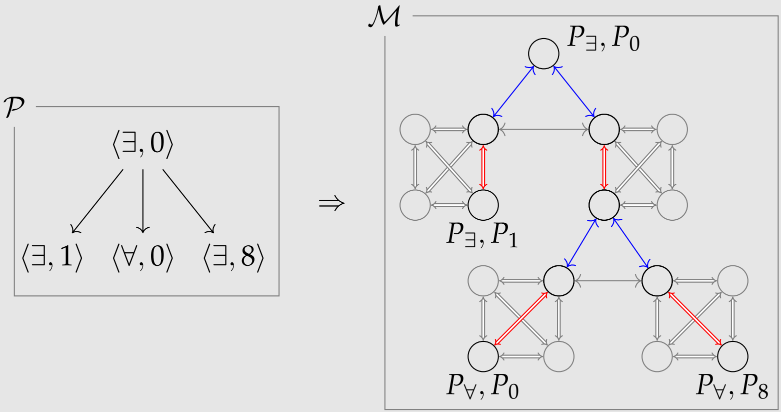
Evaluation games are parity games.

Parity games as unimodal Kripke frames



$$W_n := \eta X_n \ldots \nu X_0. \bigvee_{0 \leq j \leq n} [(P_j \wedge P_\exists \wedge \Diamond X_j) \vee (P_j \wedge P_\forall \wedge \Box X_j)].$$

PARITY GAMES AS S5 ⊗ S5 FRAMES



BIMODAL WINNING REGION FORMULAS

$$W'_n := \eta X_n \ldots \nu X_0. \bigvee_{0 \leq j \leq n} [(P_j \wedge P_\exists \wedge \blacklozenge X_j) \vee (P_j \wedge P_\forall \wedge \blacksquare X_j)].$$

BIMODAL WINNING REGION FORMULAS

$$W'_n := \eta X_n \dots \nu X_0. \bigvee_{0 \le j \le n} [(P_j \wedge P_\exists \wedge \blacklozenge X_j) \vee (P_j \wedge P_\forall \wedge \blacksquare X_j)].$$

Where

- $\blacklozenge\varphi := \nu Y.\mathrm{pre}_0 \wedge \mathrm{bd} \wedge \Diamond_0(\mathrm{nxt}_0 \wedge \mathrm{pre}_1 \wedge \mathrm{bd} \wedge \Diamond_1(\mathrm{nxt}_1 \wedge \mathrm{bd} \wedge ((Y \wedge \neg\mathrm{st}) \vee (\varphi \wedge \mathrm{st}))))$; and
- $\blacksquare\varphi := \nu Y.\mathrm{pre}_0 \wedge \mathrm{bd} \rightarrow \Box_0(\mathrm{nxt}_0 \wedge \mathrm{pre}_1 \wedge \mathrm{bd} \rightarrow \Box_1(\mathrm{nxt}_1 \wedge \mathrm{bd} \rightarrow ((Y \wedge \neg\mathrm{st}) \wedge (\varphi \wedge \mathrm{st}))))$,

PROOF SKETCH

- Let n be even. Then $W_n \in \Pi^{\mu}_{n+1}$.
- Suppose that W_n is equivalent to some formula in Π^{μ}_{n}. Let $\varphi \in \Sigma^{\mu}_{n}$ be equivalent to $\neg W_n$.
- $f_{\varphi \wedge \varphi}$ takes (M, w) to the evaluation game of $M, w \models \varphi \wedge \varphi$ (as a Kripke model).
- Let (M, w) be a fixed-point of $f_{\varphi \wedge \varphi}$. Then

$$\begin{aligned} M, w \models \neg W_n &\iff M, w \models \varphi \wedge \varphi \\ &\iff f_{\varphi \wedge \varphi}(M, w) \models W_n \\ &\iff M, w \models W_n. \end{aligned}$$

- This is a contradiction.

Our result

Theorem

Let F_0, F_1, *and* F_2 *be classes of unimodal Kripke frames closed under isomorphic copies and disjoint unions. If*

1. $\circ \leftarrow \circ \rightarrow \circ$ *is a subframe of* F_0 *and* $\circ \rightarrow \circ$ *a subframe of* F_1*; or*
2. $\circ \rightarrow \circ \rightarrow \circ$ *is a subframe of* F_0 *and* $\circ \rightarrow \circ$ *a subframe of* F_1*;*

then the μ*-calculus' alternation hierarchy is strict over* $\mathsf{F}_0 \otimes \mathsf{F}_1$*. If*

3. $\circ \rightarrow \circ$ *is a subframe of* F_0, F_1, *and* F_2*;*

then the μ*-calculus' alternation hierarchy is strict over* $\mathsf{F}_0 \otimes \mathsf{F}_1 \otimes \mathsf{F}_2$*.*

Conjecture

Suppose $\circ \rightarrow \circ$ *is a subframe of* F_0 *and* F_1*. We can only show that each* μ*-formula is equivalent to an alternation-free formula over* $\mathsf{F}_0 \otimes \mathsf{F}_1$*.*

COLLAPSE ON MULTIMODAL LOGICS

GLP is a provability logic which contains countably many modal operators.

Theorem (Ignatiev [3])

GLP *has the fixed-point property.*

IS5 is an intuitionistic version of S5 which can be treated as a bimodal logic.

Theorem (P. [5])

The μ-calculus collapses to modal logic over IS5.

References

[1] L. Alberucci, A. Facchini, “The modal μ-calculus hierarchy over restricted classes of transition systems”, 2009.

[2] J.C. Bradfield, “Simplifying the modal mu-calculus alternation hierarchy”, 1998.

[3] K.N. Ignatiev, “On Strong Provability Predicates and the Associated Modal Logics”, 1993.

[4] L. Pacheco, “Exploring the difference hierarchies on μ-calculus and arithmetic—from the point of view of Gale–Stewart games”, PhD Thesis, 2023.

[5] L. Pacheco, “Game Semantics for the Constructive μ-Calculus”, arXiv:2308.16697, 2023.