Combining fixpoint and differentiation theory

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In math and computer science

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- various notions of recursion (fixpoint operators, trace, ...)

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Goal: general account of the interactions between fixpoints, trace and differentiation

Cartesian differential categories

Blute, R. F. and Cockett, J. R. B. and Seely, R. A. G. "Cartesian Differential Categories." *Theory and Applications of Categories*, 22(23):622–672 (2009).

A Cartesian differential category is a Cartesian left additive category $\mathbb{C}$ equipped with an operator $\mathbf{D}$

$$\begin{array}{rcl} \mathbf{D} : \mathbb{C}(A, B) & \longrightarrow & \mathbb{C}(A \times A, B) \\ f : A \to B & \longmapsto & \mathbf{D}[f] : A \times A \to B \end{array}$$

satisfying seven axioms.

There is an associated term calculus

$$\mathbf{D}[f](a, b) := \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b$$

Chain rule for Cartesian differential categories

Chain rule axiom: for $f : A \to B$ and $g : B \to C$,

$$\frac{\mathrm{d}g\left(f(x)\right)}{\mathrm{d}x}(a) \cdot b \quad = \quad \frac{\mathrm{d}g(y)}{\mathrm{d}y}(f(a)) \cdot \left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b\right)$$

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$$A \times A \xrightarrow{\mathbf{D}[gf]} C \quad = \quad A \times A \xrightarrow{\mathbf{T}(f)} B \times B \xrightarrow{\mathbf{D}[g]} C$$

where $\mathbf{T}$ is the tangent bundle

$$\mathbf{T} : \mathbb{C} \longrightarrow \mathbb{C}$$

$$\begin{array}{ccc} A & \longmapsto & A \times A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \langle f\pi_1, \mathbf{D}[f] \rangle} \\ B & \longmapsto & B \times B \end{array} \qquad \begin{array}{c} (a, b) \\ \downarrow \\ \left(f(a), \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b\right) \end{array}$$

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$\mathbf{T}$ is a functor $\quad\Leftrightarrow\quad$ the chain rule holds

Examples

- Smooth functions between Euclidean spaces $\mathbb{R}^n$, for $f : \mathbb{R}^n \to \mathbb{R}^m$,

$$\text{Derivative } \mathbf{D}[f] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m \quad \overset{\text{curry}}{\rightsquigarrow} \quad \text{Jacobian } \mathbf{J}[f] : \mathbb{R}^n \to \mathbf{Lin}(\mathbb{R}^n, \mathbb{R}^m)$$

- Relational and weighted relational models
- Reduced power series
- Convenient vector spaces
- Any model of the differential λ-calculus (Cartesian closed differential categories)
- And many more...

Fixpoint operators

- Parametrized fixpoint operators are fixpoints for morphisms in context:

$$f : \Gamma \times X \longrightarrow X \qquad \rightsquigarrow \qquad \mu x.f(-,x) : \Gamma \to X$$

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- Example: in PCF, we have an operator **fix**

$$\frac{\Gamma, X \vdash M : X}{\Gamma \vdash \ \mathbf{fix}\, M : X}\ \textsc{fix}$$

with reduction $\mathbf{fix}\, M \to M(\mathbf{fix}\, M)$

Categorical fixpoint operators

Bloom, S. L., Ésik Z. *Iteration theories*, 1993.

Simpson, A. and Plotkin, G. *Complete axioms for categorical fixed-point operators*, 2000.

For a Cartesian category $\mathbb{C}$, a **parametrized fixpoint operator** on $\mathbb{C}$ is family of functions

$$\begin{aligned} \mathbf{fix}_X^\Gamma : \mathbb{C}(\Gamma \times X, X) &\longrightarrow \mathbb{C}(\Gamma, X) \\ f &\longmapsto \mathbf{fix}_X^\Gamma(f) \end{aligned}$$

indexed by pairs of objects in $\mathbb{C}$ verifying:

- fixpoint axiom: for all $f : \Gamma \times X \to X$,
$$f \circ \langle \mathrm{id}_\Gamma, \mathbf{fix}(f) \rangle = \mathbf{fix}(f)$$
- naturality axiom: for all morphisms $g : \Gamma \to \Delta$ and $f : \Delta \times X \to X$,
$$\mathbf{fix}(f) \circ g = \mathbf{fix}(f \circ (g \times \mathrm{id}_X)).$$

Examples

- for domains Γ, X and a Scott-continuous map $f : \Gamma \times X \to X$,

$$f(a, \bigvee_{n \in \omega} f^n(a, \bot)) = \bigvee_{n \in \omega} f^n(a, \bot)$$

  is a least parametrized fixpoint for f.

- co-Kleisli category (weighted) relations with the finite multisets comonad

- special case: power series over continuous semi-rings

- any cpo-enriched cartesian closed category

- complete metric spaces and contractive maps

- and many more ...

Additional axioms

A fixpoint operator **fix** is a **Conway** operator if it satisfies

- Dinaturality axiom: for $f : \Gamma \times X \to Y$ and $g : Y \to X$

$$\mu x.g\,(f(a,x)) = g\,(\mu y.f(a,g(y)))$$

- Bekić axiom: for $f : \Gamma \times X \times Y \to X$ and $g : \Gamma \times X \times Y \to Y$

$$\mu(x,y).\,(f(a,x,y), g(a,x,y)) =$$
$$(\mu x.f\,(a,x,\mu y.g(a,x,y))\,, \mu y.g\,(a, \mu x.f\,(a,x,\mu y.g(a,x,y))\,, y))$$

Conway fixpoint operator $\Leftrightarrow$ Trace operator with $\times$ as tensor

Particular cases

Cartesian

$$\frac{f : \Gamma \times X \to X}{\mathbf{fix}(f) : \Gamma \to X}$$

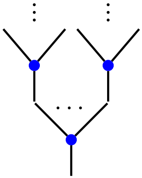
$$\mathbf{fix}\, f = f \langle \mathrm{id}, \mathbf{fix}\, f \rangle$$

general trees

Cartesian closed

$$\mathbf{Y} : X \Rightarrow X \to X$$

$$\mathbf{Y} = \mathrm{eval}\langle \mathrm{id}, \mathbf{Y} \rangle$$



linear trees

Finite biproducts

$$\frac{f : X \to X}{f^* : X \to X}$$

$$f^* = \mathrm{id} \oplus f^* f$$

How should derivatives and fixpoints interact?

The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has a parametrized fixpoint $g : \mathbb{R} \to \mathbb{R}$

$(x, y) \mapsto 1 - x^2 y$ $\qquad$ $x \mapsto \frac{1}{1+x^2}$

$$g(x) = f(x, g(x)) \tag{1}$$

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By the chain rule,

$$g'(x) = \frac{\partial f}{\partial x}(x, g(x)) + \frac{\partial f}{\partial y}(x, g(x)) \cdot g'(x) = \frac{-2x}{(1+x^2)^2} \tag{2}$$

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Another way of computing $g'(x)$:

- Compute the tangent of f, $\mathbf{T}(f) : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$:

$$\mathbf{T}(f)(x, y, a, b) = \left(f(x, y), \frac{\partial f}{\partial x}(x, y) \cdot a + \frac{\partial f}{\partial y}(x, y) \cdot b \right)$$

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- (1) and (2) imply $\mathbf{T}(g)$ is a fixpoint for $\mathbf{T}(f)$

$$\mathbf{T}(g)(x, a) = (g(x), g'(x) \cdot a) = \mathbf{T}(f)(x, g(x), a, g'(x) \cdot a)$$

Cartesian differential fixpoint categories

Computing the derivative of the fixpoint is equivalent to computing the fixpoint of the tangent.

- **Option 1** differentiating the fixpoint

$$\dfrac{\dfrac{\Gamma \times X \xrightarrow{f} X}{\Gamma \xrightarrow{\mathbf{fix}(f)} X}}{\Gamma \times \Gamma \xrightarrow{\mathbf{D}[\mathbf{fix}(f)]} X}$$

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- **Option 2** fixpoint of the tangent

$$\dfrac{\dfrac{\dfrac{\Gamma \times X \xrightarrow{f} X}{\Gamma \times X \times \Gamma \times X \xrightarrow{\mathbf{T}(f)} X \times X}}{\Gamma \times \Gamma \times X \times X \xrightarrow{c} \Gamma \times X \times \Gamma \times X \xrightarrow{\mathbf{T}(f)} X \times X}}{\Gamma \times \Gamma \xrightarrow{\mathbf{fix}(\mathbf{T}(f) \circ c)} X \times X \xrightarrow{\pi_2} X}$$

where $c := \mathrm{id}_\Gamma \times \langle \pi_2, \pi_1 \rangle \times \mathrm{id}_X : \Gamma \times \Gamma \times X \times X \xrightarrow{\cong} \Gamma \times X \times \Gamma \times X$

Cartesian differential fixpoint categories

For a Cartesian differential category $\mathbb{C}$, a parametrized fixpoint operator **fix** satisfies the

- **differential-fixpoint axiom** if for every $f : \Gamma \times X \to X$

$$\mathbf{D}[\mathbf{fix}(f)] = \pi_2 \, \mathbf{fix}(\mathbf{T}(f \circ c))$$

- **tangent-fixpoint axiom** if for every $f : \Gamma \times X \to X$

$$\mathbf{T}[\mathbf{fix}(f)] = \mathbf{fix}(\mathbf{T}(f \circ c))$$

Lemma

If the fixpoint operator **fix** *is Conway, the following are equivalent:*

- **fix** *satisfies the differential-fixpoint rule;*
- **fix** *satisfies the tangent-fixpoint rule.*

Examples

- Differential fixpoint axiom: any category with finite biproducts where $\mathbf{D}[f] = f \circ \pi_2$ provided that the fixpoint is uniform wrt projection maps
- Tangent fixpoint axiom: weighted relations
- Special case: formal power series over continuous semi-rings
- General case: quantale enriched profunctors with the free exponential
- More generally: any Cartesian differential category where the fixpoint operator is obtained from bifree algebras and is uniform wrt linear maps.

Cartesian closed differential fixpoint categories

Ehrhard, T. "A coherent differential PCF." *Logical Methods in Computer Science*, 19 (2023).

For a Cartesian closed differential category $\mathbb{C}$, a fixpoint combinator $\mathbf{Y} : X \Rightarrow X \to X$ satisfies the

- **differential-fixpoint axiom** if

$$\mathbf{D}[\mathbf{Y}] = \pi_2 \mathbf{Y} \lambda(\mathbf{T}(\mathrm{eval}) \circ c)$$

- **tangent-fixpoint axiom** if

$$\mathbf{T}[\mathbf{Y}] = \mathbf{Y} \lambda(\mathbf{T}(\mathrm{eval}) \circ c)$$

Examples cpo-enriched cartesian closed categories, categories with fixpoint objects

Fixpoints and linearity

In a Cartesian differential category $\mathbb{C}$, a morphism $f : X \to Y$ is linear if

$$\mathbf{D}[f] = X \times X \xrightarrow{\pi_2} X \xrightarrow{f} Y$$

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Lemma

If $\mathbb{C}$ is a Cartesian differential fixpoint category, then the fixpoint of a linear morphism is linear.

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Lemma

If $\mathbb{C}$ is a Cartesian differential fixpoint category, then the fixpoint of a linear morphism is linear.

- Differential fixpoint axiom $\Rightarrow$ $\mathbf{Lin}(\mathbb{C})$ has a fixpoint operator.
- Differential fixpoint axiom + Conway fixpoint $\Rightarrow$ $\mathbf{Lin}(\mathbb{C})$ has a repetition operator.

Newton-Raphson approximation

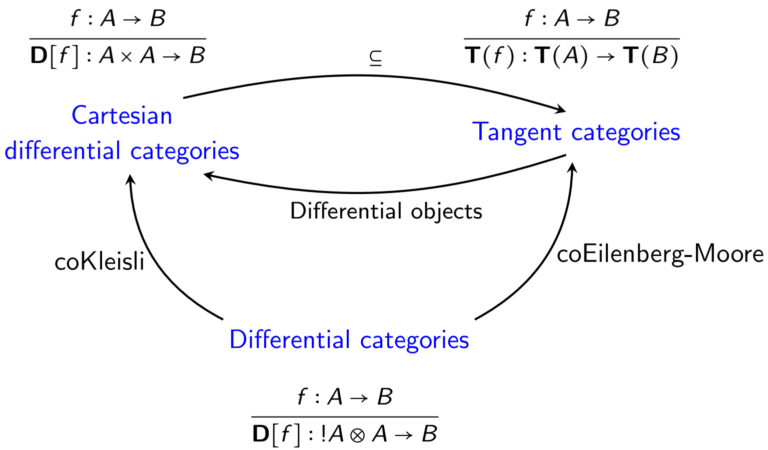
Inspired by:

- Combinatorics: Newton's method for combinatorial species (Decoste, Labelle, Leroux, Pivoteau, Salvy, Soria, ...)
- Formal languages and dataflow analysis: Newton's method for power series over ω-continuous semirings (Esparza, Kiefer, Luttenberger, Schlund,...)

Main idea: assuming the differential-fixpoint axiom

compute the non-linear fixpoint more efficiently	using	the derivative and the linear repetition operator

What about differential categories?



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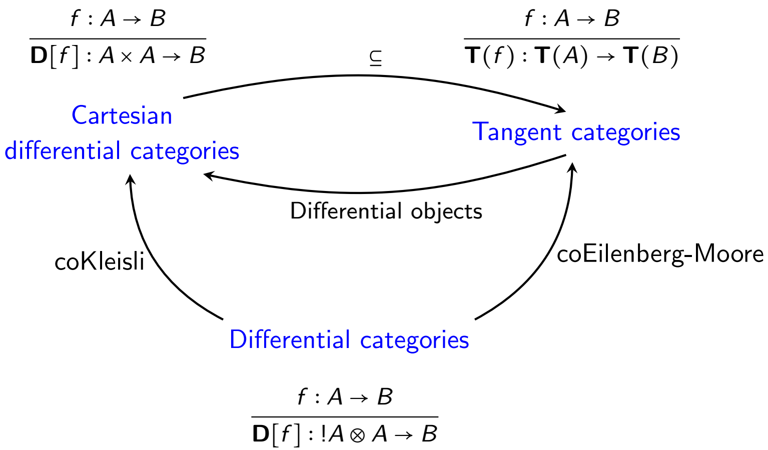

A natural notion of fixpoint for differential categories a trace for the monoidal product $\otimes$, how should it interact with differentiation?

Future work

- More general iterative schemes than Newton-Raphson
- Approximation of solutions of differential equations
- Reverse differential categories and fixpoints
- Restriction differential category for local implicit function theorems
- Guarded recursion and differentiation for fixpoints in metric models
- Syntactic counterpart (Taylor expansion)
- Differentiation in the coinductive world (e.g. stream calculus)

Thank you