LDP Seminar:
Interpolation as Cut-introduction

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Outline

1. Introduction: Background on Craig-Lyndon Interpolation
   - Statement
   - An easy proof in the proposition case
   - Proof-theoretic interpolation: Maheara 1960 and Prawitz 1965
2. Revisiting proof-theoretic interpolation: proof-relevant interpolation in linear logic
   - Proof-relevant Maheara's method for interpolation
   - Disclaimer: Not all of this is new: Čubrić for the λ-calculus
3. Interpolation as cut-introduction
4. What about proof-relevant interpolation for circular proofs?
5. Conclusion

Outline

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2. Revisiting proof-theoretic interpolation: proof-relevant interpolation in linear logic
3. Interpolation as cut-introduction
4. What about proof-relevant interpolation for circular proofs?
5. Conclusion

Statement of Craig's Interpolation Theorem

The interpolation property was first stated and proved by Craig and soon refined by Lyndon.

Definition (Interpolation property)

A logic L has the interpolation property if, for any formulas A, B such that $A \vdash_L B$, there is a formula C satisfying $\mathscr{L}(C) \subseteq \mathscr{L}(A) \cap \mathscr{L}(B)$ and such that $A \vdash_L C$ and $B \vdash_L B$.

$\mathscr{L}(C) \subseteq \mathscr{L}(A) \cap \mathscr{L}(B)$ means:

- in Craig's interpolation, that all predicate symbols occurring in C occur both in A and B.
- in Lyndon's interpolation, that all predicate symbols occurring positively (resp. negatively) in C occur both positively (resp. negatively) in A and B.

Remark that this amount to a cut-rule: $\dfrac{A \vdash_L C \qquad C \vdash_L B}{A \vdash_L B}$ (Cut)

An easy proof for classical propositional logic

By induction on the cardinality of $\mathscr{L}(A) \setminus \mathscr{L}(B)$.

- if $\mathscr{L}(A) \subseteq \mathscr{L}(B)$, then A is an interpolant.
- Otherwise, take some propositional variable p occurring in A and not in B and consider $A' = A[\top/p] \vee A[\bot/p]$. Clearly:
  (i) $A \vdash A'$ and $A' \vdash B$ (since p does not occur in B) and
  (ii) an interpolant of A' and B exists by induction hypothesis: it is an interpolant for A and B by transitivity of entailment.

Remark

- *Does not rely on a specific proof system (provability only);*
- *This is constructive;*
- *Crucial use of the logical constants $\bot, \top$;*
- *In fact, the interpolant does not really depend on B, only on the language of B: this is a uniform interpolant for all formulas with the same language as B.*

Proof(-theoretic) methods for interpolation

Maehara 1960, for Sequent calculus – Prawitz 1965, for Natural Deduction

Maehara:

- Induction on the structure of a cut-free derivation of $A \vdash B$,
- Strengthen the induction hypothesis, by showing that if $\Gamma \vdash \Delta$, then for any partitioning Γ', Γ'' of Γ and Δ', Δ'' of Δ there exists an interpolant C with $\mathscr{L}(C) \subseteq \mathscr{L}(\Gamma', \Delta') \cap \mathscr{L}(\Gamma'', \Delta'')$ such that $\Gamma' \vdash C, \Delta'$ and $\Gamma'', C \vdash \Delta''$.

Prawitz:

LEMMA. *Let* Π *be a normal deduction in* C' *of* A *depending on* Γ, *and let* Γ_1 *and* Γ_2 *be two disjoint sets such that* $\Gamma_1 \cup \Gamma_2 = \Gamma$. *Then there is a formula* F, *called an interpolation formula to* $\langle \Gamma_1, \langle \Gamma_2, A \rangle\rangle$, *such that* $\Gamma_1 \vdash F$ *and* $\{F\} \cup \Gamma_2 \vdash A$ *and such that every parameter that occurs positively [negatively] in* F *occurs positively [negatively] in some formula of* Γ_1 *and negatively [positively] in some formula of* $\Gamma_2 \cup \{\sim A\}$.

(for NK and NJ)

Proof(-theoretic) methods for interpolation

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- Induction on the structure of a cut-free derivation of $A \vdash B$,
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Method of wide applicability for sequent calculi with cut-admissibility:

- Maehara (1960) proved it for LK, soon extended by Schütte to LJ (1962)
- Roorda gives a proof for LL analyzing in which fragments interpolation actually holds (1994) *(actually the proof covers only the fragments of MALL...)*;
- Application to various modal logics.

Proof(-theoretic) methods for interpolation

Maehara 1960, for Sequent calculus – Prawitz 1965, for Natural Deduction

Example of $(\wedge_r)$, that is if $\Delta = A \wedge B, \Delta_1, \Delta_2$ and if π ends with

$$\frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B, \Delta_1, \Delta_2}\ (\wedge_r)$$

Consider a partitioning of Γ and Δ as Γ', Γ'' and Δ', Δ'' assuming that $A \wedge B$ is in Δ'. The partitionings of Γ and Δ induce partitionings of $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ as (Γ_1', Γ_1''), $(\Gamma_2', \Gamma_2''), (\Delta_1', \Delta_1'')$ and (Δ_2', Δ_2'') and the induction hypothesis ensures the existence of interpolants C_1, C_2 such that (i) $\Gamma_1' \vdash A, \Delta_1', C_1$, (ii) $\Gamma_1'', C_1 \vdash \Delta_1''$, (iii) $\Gamma_2' \vdash B, \Delta_2', C_2$ and (iv) $\Gamma_2'', C_2 \vdash \Delta_2''$ are provable, from which one can derive:

$$\dfrac{\dfrac{\dfrac{by\ (i)}{\Gamma_1' \vdash A, \Delta_1', C_1} \qquad \dfrac{by\ (iii)}{\Gamma_2' \vdash B, \Delta_2', C_2}}{\Gamma_1', \Gamma_2' \vdash A \wedge B, \Delta_1', \Delta_2', C_1, C_2}\ (\wedge_r)}{\Gamma_1', \Gamma_2' \vdash A \wedge B, \Delta_1', \Delta_2', C_1 \vee C_2}\ (\vee_r) \qquad \dfrac{\dfrac{by\ (i)}{\Gamma_1'', C_1 \vdash \Delta_1''} \qquad \dfrac{by\ (iii)}{\Gamma_2'', C_2 \vdash \Delta_2''}}{\Gamma_1'', \Gamma_2'', C1 \vee C2 \vdash \Delta_1'', \Delta_2''}\ (\wedge_l)$$

First-order muLL sequent calculus

(a)

$$\frac{}{\vdash F, F^{\perp}}\ (\mathrm{Ax}) \qquad \frac{\vdash \Gamma, F \qquad \vdash F^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (\mathrm{Cut}) \qquad \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ (\mathrm{X})$$

$$\frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ (\parr) \qquad \frac{\vdash F, \Gamma \qquad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{}{\vdash 1}\ (1)$$

$$\frac{\vdash F, \Gamma \qquad \vdash G, \Gamma}{\vdash F \mathbin{\&} G, \Gamma}\ (\&) \qquad \frac{\vdash A_i, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ (\oplus^{i}) \qquad \frac{}{\vdash \top, \Gamma}\ (\top)$$

(no rule for 0)

(b)

$$\frac{\vdash F, \Gamma}{\vdash ?F, \Gamma}\ (?\mathrm{d}) \qquad \frac{\vdash F, ?\Gamma}{\vdash !F, ?\Gamma}\ (!\mathrm{p}) \qquad \frac{\vdash \Gamma}{\vdash ?F, \Gamma}\ (?\mathrm{w}) \qquad \frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma}\ (?\mathrm{c})$$

(c)

$$\frac{\vdash G\{t/x\}, \Gamma}{\vdash \exists x G, \Gamma}\ (\nu) \qquad \frac{\vdash F, \Gamma}{\vdash \forall x F, \Gamma}\ (\forall)$$

(in $(\forall)$, $x \notin FV(\Gamma)$)

(d)

$$\frac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma}\ (\nu) \qquad \frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\ (\mu)$$

Outline

Proof-Relevant Interpolation

Theorem

Let Γ, Δ be lists of LL *formulas and $\pi \vdash \Gamma, \Delta$.*
There exists a LL *formula C such that $\mathscr{L}(C) \subseteq \mathscr{L}(\Gamma) \cap \mathscr{L}(\Delta)$ and two cut-free proofs π_1, π_2 of $\vdash \Gamma, C$ and $\vdash C^\perp, \Delta$ respectively such that*

$$\dfrac{\dfrac{\pi_1}{\vdash \Gamma, C} \quad \dfrac{\pi_2}{\vdash C^\perp, \Delta}}{\vdash \Gamma, \Delta}\ (\text{Cut}) \quad =_{cut} \pi.$$

Remark

- *We will see (some cases of) the proof next: the proof goes by induction on the structure of a cut-free derivation, no surprise...*
- *Similar result established by Čubrić in 1993 for simply typed λ-calculus with products and sums, adapting a proof of interpolation by Prawitz for natural deduction. We will discuss this later.*

Proof-Relevant Interpolation

Theorem

Let Γ, Δ be lists of LL *formulas and $\pi \vdash \Gamma, \Delta$ be cut-free. There exists a* LL *formula C such that $\mathscr{L}(C) \subseteq \mathscr{L}(\Gamma) \cap \mathscr{L}(\Delta)$ and two cut-free proofs π_1, π_2 of $\vdash \Gamma, C$ and $\vdash C^\perp, \Delta$ respectively such that*

$$\dfrac{\dfrac{\pi_1}{\vdash \Gamma, C} \qquad \dfrac{\pi_2}{\vdash C^\perp, \Delta}}{\vdash \Gamma, \Delta}\ (\mathrm{Cut}) \quad \longrightarrow^{\star}_{\mathrm{cut}} \ \pi.$$

Remark

- *We will see (some cases of) the proof next: the proof goes by induction on the structure of a cut-free derivation, no surprise...*
- *Similar result established by Čubrić in 1993 for simply typed λ-calculus with products and sums, adapting a proof of interpolation by Prawitz for natural deduction. We will discuss this later.*

Proof

$$\pi = \cfrac{}{\vdash F, F^\perp}\ (\mathrm{Ax}) :$$

- If $\Gamma = F$, one simply takes $C = F^\perp$ $\pi_1 = \pi_2 = \cfrac{}{\vdash F, F^\perp}\ (\mathrm{Ax})$.
  (the case when $\Gamma = F^\perp$ is symmetrical, taking $C = F$.)
- If $\Gamma = F, F^\perp$, one simply takes $C = \perp$, $\pi_1 = \cfrac{\pi}{\vdash \Gamma, \perp}\ (\perp)$ and $\pi_2 = \cfrac{}{\vdash 1}\ (1)$.

$$\cfrac{\cfrac{\pi}{\vdash \Gamma, \perp}\ (\perp) \qquad \cfrac{}{\vdash 1}\ (1)}{\vdash \Gamma}\ (\mathrm{Cut}) \longrightarrow_{\mathrm{cut}} \pi$$

(the case when Γ is empty is symmetrical, taking $C = 1$.)

Proof

Last rule is $(\otimes)$

$$\pi = \dfrac{\dfrac{\pi'}{\vdash F, \Gamma', \Delta'} \qquad \dfrac{\pi''}{\vdash G, \Gamma'', \Delta''}}{\vdash F \otimes G, \Gamma', \Gamma'', \Delta', \Delta''}\ (\otimes)$$

, assuming $\Gamma = F \otimes G, \Gamma', \Gamma''$.

By IH, there are interpolants C', C'' and interpolating proofs (i) $\pi_1' \vdash F, \Gamma', C'$, (ii) $\pi_2' \vdash C'^{\perp}, \Delta'$, (iii) $\pi_1'' \vdash G, \Gamma'', C''$ and (iv) $\pi_2'' \vdash C''^{\perp}, \Delta''$ st.

$$\dfrac{\pi_1' \qquad \pi_2'}{\vdash F, \Gamma', \Delta'}\ (\mathrm{Cut}) \longrightarrow^{\star}_{\mathrm{cut}} \pi' \qquad\qquad \dfrac{\pi_1'' \qquad \pi_2''}{\vdash G, \Gamma'', \Delta''}\ (\mathrm{Cut}) \longrightarrow^{\star}_{\mathrm{cut}} \pi'$$

Let $C = C' \parr C''$, $\pi_1 = \dfrac{\dfrac{\pi_1' \qquad \pi_1''}{\vdash F \otimes G, \Gamma', \Gamma'', C', C''}\ (\otimes)}{\vdash F \otimes G, \Gamma', \Gamma'', C' \parr C''}\ (\parr)$ $\quad \pi_2 = \dfrac{\pi_2' \qquad \pi_2''}{\vdash C'^{\perp} \otimes C''^{\perp}, \Delta', \Delta''}\ (\otimes)$

One observes that:

$$\dfrac{\dfrac{\pi_1}{\vdash \Gamma, C} \qquad \dfrac{\pi_2}{\vdash C^{\perp}, \Delta}}{\vdash \Gamma, \Delta}\ (\mathrm{Cut}) \quad \longrightarrow^{\star}_{\mathrm{cut}} \quad \dfrac{\dfrac{\dfrac{\pi_1' \qquad \pi_1''}{\vdash \Gamma, C', C''}\ (\otimes) \qquad \pi_2''}{\vdash \Gamma, C', \Delta''}\ (\mathrm{Cut}) \qquad \pi_2'}{\vdash \Gamma, \Delta}\ (\mathrm{Cut})$$

Proof

Last rule is $(\otimes)$

$$\pi = \cfrac{\cfrac{\pi'}{\vdash F, \Gamma', \Delta'} \qquad \cfrac{\pi''}{\vdash G, \Gamma'', \Delta''}}{\vdash F \otimes G, \Gamma', \Gamma'', \Delta', \Delta''} \ (\otimes)$$
, assuming $\Gamma = F \otimes G, \Gamma', \Gamma''$.

By IH, there are interpolants C', C'' and interpolating proofs (i) $\pi_1' \vdash F, \Gamma', C'$, (ii) $\pi_2' \vdash C'^{\perp}, \Delta'$, (iii) $\pi_1'' \vdash G, \Gamma'', C''$ and (iv) $\pi_2'' \vdash C''^{\perp}, \Delta''$ st.

$$\cfrac{\pi_1' \qquad \pi_2'}{\vdash F, \Gamma', \Delta'} \ (\mathrm{Cut}) \ \longrightarrow^{\star}_{\mathrm{cut}} \pi' \qquad \cfrac{\pi_1'' \qquad \pi_2''}{\vdash G, \Gamma'', \Delta''} \ (\mathrm{Cut}) \ \longrightarrow^{\star}_{\mathrm{cut}} \pi'$$

Let $C = C' \parr C''$, $\pi_1 = \cfrac{\cfrac{\pi_1' \qquad \pi_1''}{\vdash F \otimes G, \Gamma', \Gamma'', C', C''} \ (\otimes)}{\vdash F \otimes G, \Gamma', \Gamma'', C' \parr C''} \ (\parr)$ $\quad \pi_2 = \cfrac{\pi_2' \qquad \pi_2''}{\vdash C'^{\perp} \otimes C''^{\perp}, \Delta', \Delta''} \ (\otimes)$

One observes that:

$$\cfrac{\cfrac{\pi_1}{\vdash \Gamma, C} \qquad \cfrac{\pi_2}{\vdash C^{\perp}, \Delta}}{\vdash \Gamma, \Delta} \ (\mathrm{Cut}) \quad \longrightarrow^{\star}_{\mathrm{cut}} \quad \cfrac{\cfrac{\pi_1' \qquad \pi_2'}{\vdash F, \Gamma', \Delta'} \ (\mathrm{Cut}) \qquad \cfrac{\pi_1'' \qquad \pi_2''}{\vdash G, \Gamma'', \Delta''} \ (\mathrm{Cut})}{\vdash F \otimes G, \Gamma', \Gamma'', \Delta', \Delta''} \ (\otimes)$$

Proof

Last rule is $(\otimes)$

$$\pi = \dfrac{\dfrac{\pi'}{\vdash F, \Gamma', \Delta'} \qquad \dfrac{\pi''}{\vdash G, \Gamma'', \Delta''}}{\vdash F \otimes G, \Gamma', \Gamma'', \Delta', \Delta''} \ (\otimes)$$, assuming $\Gamma = F \otimes G, \Gamma', \Gamma''$.

By IH, there are interpolants C', C'' and interpolating proofs (i) $\pi'_1 \vdash F, \Gamma', C'$, (ii) $\pi'_2 \vdash C'^{\perp}, \Delta'$, (iii) $\pi''_1 \vdash G, \Gamma'', C''$ and (iv) $\pi''_2 \vdash C''^{\perp}, \Delta''$ st.

$$\dfrac{\pi'_1 \qquad \pi'_2}{\vdash F, \Gamma', \Delta'} \ (\text{Cut}) \longrightarrow^{\star}_{\text{cut}} \pi' \qquad \dfrac{\pi''_1 \qquad \pi''_2}{\vdash G, \Gamma'', \Delta''} \ (\text{Cut}) \longrightarrow^{\star}_{\text{cut}} \pi'$$

Let $C = C' \parr C''$, $\pi_1 = \dfrac{\dfrac{\pi'_1 \qquad \pi''_1}{\vdash F \otimes G, \Gamma', \Gamma'', C', C''} \ (\otimes)}{\vdash F \otimes G, \Gamma', \Gamma'', C' \parr C''} \ (\parr)$ $\quad \pi_2 = \dfrac{\pi'_2 \qquad \pi''_2}{\vdash C'^{\perp} \otimes C''^{\perp}, \Delta', \Delta''} \ (\otimes)$

One observes that:

$$\dfrac{\dfrac{\pi_1}{\vdash \Gamma, C} \qquad \dfrac{\pi_2}{\vdash C^{\perp}, \Delta}}{\vdash \Gamma, \Delta} \ (\text{Cut}) \quad \longrightarrow^{\star}_{\text{cut}} \quad \pi \qquad \text{by IH.}$$

Proof

Last rule is $(\parr)$

If $\pi = \dfrac{\dfrac{\pi'}{\vdash F, G, \Gamma', \Delta}}{\vdash F \parr G, \Gamma', \Delta}\ (\parr)$, assuming $\Gamma = F \parr G, \Gamma'$. By IH, there is an interpolant C' such that $\mathscr{L}(C') \subseteq \mathscr{L}(F, G, \Gamma') \cap \mathscr{L}(\Delta)$ as well as proofs $\pi_1' \vdash F, G, \Gamma', C'$ and $\pi_2' \vdash C'^{\perp}, \Delta$ such that

$$\dfrac{\dfrac{\pi_1'}{\vdash F, G, \Gamma', C'} \qquad \dfrac{\pi_2'}{\vdash C'^{\perp}, \Delta'}}{\vdash F, G, \Gamma', \Delta'}\ (\mathrm{Cut}) \quad \longrightarrow^{\star}_{\mathrm{cut}} \pi'.$$

Setting $C = C'$, $\pi_1 = \dfrac{\pi_1'}{\vdash F \parr G, \Gamma', C}\ (\parr)$ and $\pi_2 = \pi_2'$ we get:

$$\dfrac{\dfrac{\pi_1}{\vdash \Gamma, C} \qquad \dfrac{\pi_2}{\vdash C^{\perp}, \Delta}}{\vdash \Gamma, \Delta}\ (\mathrm{Cut}) \quad \longrightarrow_{\mathrm{cut}} \quad \dfrac{\dfrac{\dfrac{\pi_1'}{\vdash F, G, \Gamma', C'} \qquad \dfrac{\pi_2'}{\vdash C'^{\perp}, \Delta'}}{\vdash F, G, \Gamma', \Delta'}\ (\mathrm{Cut})}{\vdash F \parr G, \Gamma', \Delta'}\ (\parr) \quad \longrightarrow^{\star}_{\mathrm{cut}} \quad \pi.$$

Proof

Last rule is (?d)

If $\pi = \dfrac{\dfrac{\pi'}{\vdash F, \Gamma', \Delta}}{\vdash ?F, \Gamma', \Delta}\ (?\mathrm{d})$ assuming $\Gamma = ?F, \Gamma'$.

By IH, there is an interpolant C' (with $\mathscr{L}(C') \subseteq \mathscr{L}(F,\Gamma') \cap \mathscr{L}(\Delta)$) and proofs $\pi_1' \vdash F, \Gamma', C'$ and $\pi_2' \vdash C'^{\perp}, \Delta$ such that

$$\dfrac{\dfrac{\pi_1'}{\vdash F, \Gamma', C'} \qquad \dfrac{\pi_2'}{\vdash C'^{\perp}, \Delta}}{\vdash F, \Gamma', \Delta}\ (\mathrm{Cut}) \quad \longrightarrow^{\star}_{\mathrm{cut}} \pi'.$$

By setting $C = C'$, $\pi_1 = \dfrac{\pi_1'}{\vdash ?F, \Gamma', C'}\ (?\mathrm{d})$, one gets:

$$\dfrac{\dfrac{\pi_1}{\vdash ?F, \Gamma', C} \qquad \dfrac{\pi_2}{\vdash C^{\perp}, \Delta}}{\vdash ?F, \Gamma', \Delta}\ (\mathrm{Cut}) \quad \longrightarrow_{\mathrm{cut}} \quad \dfrac{\dfrac{\dfrac{\pi_1'}{\vdash F, \Gamma', C'} \qquad \dfrac{\pi_2'}{\vdash C'^{\perp}, \Delta}}{\vdash F, \Gamma', \Delta}\ (\mathrm{Cut})}{\vdash ?F, \Gamma', \Delta}\ (?\mathrm{d}) \quad \longrightarrow^{\star}_{\mathrm{cut}} \pi.$$

Proof

Last rule is (!p)

If $\pi = \dfrac{\dfrac{\pi'}{\vdash F, ?\Gamma', ?\Delta'}}{\vdash !F, ?\Gamma', ?\Delta'}$ (!p) assuming $\Gamma = !F, ?\Gamma'$ and $\Delta = ?\Delta'$.

By IH, there is an interpolant C' (with $\mathscr{L}(C') \subseteq \mathscr{L}(F, ?\Gamma') \cap \mathscr{L}(\Delta)$) and proofs $\pi_1' \vdash F, ?\Gamma', C'$ and $\pi_2' \vdash C'^{\perp}, ?\Delta'$ such that

$$\dfrac{\dfrac{\pi_1'}{\vdash F, ?\Gamma', C'} \quad \dfrac{\pi_2'}{\vdash C'^{\perp}, ?\Delta'}}{\vdash F, ?\Gamma', ?\Delta'} \text{ (Cut)} \longrightarrow^{\star}_{\text{cut}} \pi'.$$

By setting $C = ?C'$, $\pi_1 = \dfrac{\dfrac{\pi_1'}{\vdash F, ?\Gamma', ?C'} \text{ (?d)}}{\vdash !F, ?\Gamma', ?C'} \text{ (!p)}$ and $\pi_2 = \dfrac{\pi_2'}{\vdash !C'^{\perp}, ?\Delta'}$ (!p) , one gets:

$$\dfrac{\pi_1 \quad \pi_2}{\vdash ?F, ?\Gamma', ?\Delta'} \text{ (Cut)} \longrightarrow_{\text{cut}} \dfrac{\dfrac{\dfrac{\dfrac{\pi_1'}{\vdash F, ?\Gamma', C'}}{\vdash F, ?\Gamma', C} \text{ (?d)} \quad \dfrac{\dfrac{\pi_2}{\vdash C'^{\perp}, ?\Delta'}}{\vdash C^{\perp}, ?\Delta'} \text{ (!p)}}{\vdash F, ?\Gamma', ?\Delta'} \text{ (Cut)}}{\vdash ?F, ?\Gamma', ?\Delta'} \text{ (?d)}$$

Proof

Last rule is (!p)

If $\pi = \dfrac{\dfrac{\pi'}{\vdash F, ?\Gamma', ?\Delta'}}{\vdash !F, ?\Gamma', ?\Delta'}\ (!p)$ assuming $\Gamma = !F, ?\Gamma'$ and $\Delta = ?\Delta'$.

By IH, there is an interpolant C' (with $\mathscr{L}(C') \subseteq \mathscr{L}(F, ?\Gamma') \cap \mathscr{L}(\Delta)$) and proofs $\pi_1' \vdash F, ?\Gamma', C'$ and $\pi_2' \vdash C'^{\perp}, ?\Delta'$ such that

$$\dfrac{\dfrac{\pi_1'}{\vdash F, ?\Gamma', C'} \qquad \dfrac{\pi_2'}{\vdash C'^{\perp}, ?\Delta'}}{\vdash F, ?\Gamma', ?\Delta'}\ (\mathrm{Cut}) \quad \longrightarrow^{\star}_{\mathrm{cut}} \pi'.$$

By setting $C = ?C'$, $\pi_1 = \dfrac{\dfrac{\dfrac{\pi_1'}{\vdash F, ?\Gamma', ?C'}\ (?d)}{\vdash !F, ?\Gamma', ?C'}}{}\ (!p)$ and $\pi_2 = \dfrac{\pi_2'}{\vdash !C'^{\perp}, ?\Delta'}\ (!p)$, one gets:

$$\dfrac{\pi_1 \qquad \pi_2}{\vdash ?F, ?\Gamma', ?\Delta'}\ (\mathrm{Cut}) \quad \longrightarrow_{\mathrm{cut}} \quad \dfrac{\dfrac{\pi_1' \qquad \pi_2'}{\vdash F, ?\Gamma', ?\Delta'}\ (\mathrm{Cut})}{\vdash ?F, ?\Gamma', ?\Delta'}\ (?d)$$

Proof

Last rule is (!p)

If $\pi = \dfrac{\dfrac{\pi'}{\vdash F, ?\Gamma', ?\Delta'}}{\vdash !F, ?\Gamma', ?\Delta'}$ (!p) assuming $\Gamma = !F, ?\Gamma'$ and $\Delta = ?\Delta'$.

By IH, there is an interpolant C' (with $\mathscr{L}(C') \subseteq \mathscr{L}(F, ?\Gamma') \cap \mathscr{L}(\Delta)$) and proofs $\pi'_1 \vdash F, ?\Gamma', C'$ and $\pi'_2 \vdash C'^{\perp}, ?\Delta'$ such that

$$\dfrac{\dfrac{\pi'_1}{\vdash F, ?\Gamma', C'} \quad \dfrac{\pi'_2}{\vdash C'^{\perp}, ?\Delta'}}{\vdash F, ?\Gamma', ?\Delta'} \text{ (Cut)} \longrightarrow^{\star}_{\text{cut}} \pi'.$$

By setting $C = ?C'$, $\pi_1 = \dfrac{\dfrac{\pi'_1}{\vdash F, ?\Gamma', ?C'} \text{ (?d)}}{\vdash !F, ?\Gamma', ?C'} \text{ (!p)}$ and $\pi_2 = \dfrac{\pi'_2}{\vdash !C'^{\perp}, ?\Delta'}$ (!p) , one gets:

$$\dfrac{\pi_1 \quad \pi_2}{\vdash ?F, ?\Gamma', ?\Delta'} \text{ (Cut)} \quad \longrightarrow_{\text{cut}} \quad \pi \text{ by IH.}$$

Proof

Last rule is (∀)

If $\pi = \dfrac{\dfrac{\pi'}{\vdash F, \Gamma', \Delta}}{\vdash \forall x F, \Gamma', \Delta}\ (\forall) \quad x \notin \mathrm{FV}(\Gamma', \Delta)$ assuming $\Gamma = \forall x F, \Gamma'$.

By IH, there is an interpolant C' such that $\mathscr{L}(C') \subseteq \mathscr{L}(F, \Gamma') \cap \mathscr{L}(\Delta)$ and proofs $\pi'_1 \vdash F, \Gamma', C'$ and $\pi'_2 \vdash C'^{\perp}, \Delta$ such that

$$\dfrac{\dfrac{\pi'_1}{\vdash F, \Gamma', C'} \qquad \dfrac{\pi'_2}{\vdash C'^{\perp}, \Delta}}{\vdash F, \Gamma', \Delta}\ (\mathrm{Cut}) \longrightarrow^{\star}_{\mathrm{cut}} \pi'.$$

By setting $C = \exists x. C'$, $\pi_1 = \dfrac{\dfrac{\pi'_1}{\vdash F, \Gamma', \exists x. C'}\ (\exists)}{\vdash \forall x F, \Gamma', \exists x. C'}\ (\forall)$ and $\pi_2 = \dfrac{\pi'_2}{\vdash \forall x C'^{\perp}, \Delta}\ (\forall)$ one gets:

$$\dfrac{\pi_1 \qquad \pi_2}{\vdash \forall x F, \Gamma', \Delta}\ (\mathrm{Cut}) \quad \longrightarrow^{\star}_{\mathrm{cut}} \quad \dfrac{\dfrac{\pi'_1 \qquad \pi'_2}{\vdash F, \Gamma', \Delta}\ (\mathrm{Cut})}{\vdash \forall x F, \Gamma', \Delta}\ (\forall)$$

Proof

Last rule is $(\exists)$

If $\pi = \dfrac{\dfrac{\pi'}{\vdash F\{y/x\},\Gamma',\Delta}}{\vdash \exists x F,\Gamma',\Delta}\ (\exists)$ assuming $\Gamma = \exists x F, \Gamma'$.

We assume that we have no function symbol. By IH, there is an interpolant C' with $\mathscr{L}(C') \subseteq \mathscr{L}(F\{y/x\},\Gamma') \cap \mathscr{L}(\Delta)$ and proofs $\pi_1' \vdash F\{y/x\},\Gamma',C'$ and $\pi_2' \vdash C'^{\perp},\Delta$ st.

$$\dfrac{\dfrac{\pi_1'}{\vdash F\{y/x\},\Gamma',C'} \qquad \dfrac{\pi_2'}{\vdash C'^{\perp},\Delta}}{\vdash F\{y/x\},\Gamma',\Delta}\ (\text{Cut}) \quad \longrightarrow^{\star}_{\text{cut}} \pi'.$$

In this case, we reason by case on whether y occurs in Γ',Δ:

(ii) If $y \in FV(\Gamma')$, $y \notin FV(\Delta)$, set $C = \exists y C'$,

$$\pi_1 = \dfrac{\dfrac{\pi_1'}{\exists x F,\Gamma',C'}\ (\exists)}{\exists x F,\Gamma',\exists y C'}\ (\exists), \quad \pi_2 = \dfrac{\pi_2'}{\forall y\, C'^{\perp},\Delta}\ (\forall) \quad \text{one gets:}$$

$$\dfrac{\pi_1 \qquad \pi_2}{\vdash \exists x.F,\Gamma',\Delta}\ (\text{Cut}) \quad \longrightarrow_{\text{cut}} \quad \dfrac{\dfrac{\pi_1'}{\vdash \exists x F,\Gamma',C'}\ (\exists) \qquad \pi_2'}{\vdash \exists x F,\Gamma',\Delta}\ (\text{Cut})$$

Proof

Last rule is $(\exists)$

If $\pi = \dfrac{\dfrac{\pi'}{\vdash F\{y/x\}, \Gamma', \Delta}}{\vdash \exists x F, \Gamma', \Delta}\ (\exists)$ assuming $\Gamma = \exists x F, \Gamma'$.

We assume that we have no function symbol. By IH, there is an interpolant C' with $\mathscr{L}(C') \subseteq \mathscr{L}(F\{y/x\}, \Gamma') \cap \mathscr{L}(\Delta)$ and proofs $\pi_1' \vdash F\{y/x\}, \Gamma', C'$ and $\pi_2' \vdash {C'}^{\perp}, \Delta$ st.

$$\dfrac{\dfrac{\pi_1'}{\vdash F\{y/x\}, \Gamma', C'} \quad \dfrac{\pi_2'}{\vdash {C'}^{\perp}, \Delta}}{\vdash F\{y/x\}, \Gamma', \Delta}\ (\mathrm{Cut}) \longrightarrow^{\star}_{\mathrm{cut}} \pi'.$$

In this case, we reason by case on whether y occurs in Γ', Δ:

(ii) If $y \in FV(\Gamma')$, $y \notin FV(\Delta)$, set $C = \exists y C'$,

$$\pi_1 = \dfrac{\dfrac{\pi_1'}{\exists x F, \Gamma', C'}\ (\exists)}{\exists x F, \Gamma', \exists y C'}\ (\exists), \quad \pi_2 = \dfrac{\pi_2'}{\forall y\, {C'}^{\perp}, \Delta}\ (\forall) \text{ one gets:}$$

$$\dfrac{\pi_1 \quad \pi_2}{\vdash \exists x. F, \Gamma', \Delta}\ (\mathrm{Cut}) \quad \longrightarrow^{\star}_{\mathrm{cut}} \quad \dfrac{\dfrac{\pi_1' \quad \pi_2'}{\vdash F\{y/x\}, \Gamma', \Delta}\ (\mathrm{Cut})}{\vdash \exists x F, \Gamma', C'}\ (\exists)$$

Proof

Last rule is $(\exists)$

If $\pi = \dfrac{\dfrac{\pi'}{\vdash F\{y/x\}, \Gamma', \Delta}}{\vdash \exists x F, \Gamma', \Delta} \ (\exists)$ assuming $\Gamma = \exists x F, \Gamma'$.

We assume that we have no function symbol. By IH, there is an interpolant C' with $\mathscr{L}(C') \subseteq \mathscr{L}(F\{y/x\}, \Gamma') \cap \mathscr{L}(\Delta)$ and proofs $\pi_1' \vdash F\{y/x\}, \Gamma', C'$ and $\pi_2' \vdash C'^{\perp}, \Delta$ st.

$$\dfrac{\dfrac{\pi_1'}{\vdash F\{y/x\}, \Gamma', C'} \qquad \dfrac{\pi_2'}{\vdash C'^{\perp}, \Delta}}{\vdash F\{y/x\}, \Gamma', \Delta} \ (\text{Cut}) \quad \longrightarrow^{\star}_{\text{cut}} \pi'.$$

In this case, we reason by case on whether y occurs in Γ', Δ:

(ii) If $y \in FV(\Gamma')$, $y \notin FV(\Delta)$, set $C = \exists y C'$,

$$\pi_1 = \dfrac{\dfrac{\pi_1'}{\exists x F, \Gamma', C'} \ (\exists)}{\exists x F, \Gamma', \exists y C'} \ (\exists), \quad \pi_2 = \dfrac{\pi_2'}{\forall y\, C'^{\perp}, \Delta} \ (\forall) \quad \text{one gets:}$$

$$\dfrac{\pi_1 \qquad \pi_2}{\vdash \exists x. F, \Gamma', \Delta} \ (\text{Cut}) \quad \longrightarrow^{\star}_{\text{cut}} \quad \pi.$$

Proof

Last rule is (∃)

If $\pi = \dfrac{\dfrac{\pi'}{\vdash F\{y/x\},\Gamma',\Delta}}{\vdash \exists xF,\Gamma',\Delta}\ (\exists)$ assuming $\Gamma = \exists xF,\Gamma'$.

We assume that we have no function symbol. By IH, there is an interpolant C' with $\mathscr{L}(C') \subseteq \mathscr{L}(F\{y/x\},\Gamma') \cap \mathscr{L}(\Delta)$ and proofs $\pi_1' \vdash F\{y/x\},\Gamma',C'$ and $\pi_2' \vdash C'^{\perp},\Delta$ st.

$$\dfrac{\dfrac{\pi_1'}{\vdash F\{y/x\},\Gamma',C'} \qquad \dfrac{\pi_2'}{\vdash C'^{\perp},\Delta}}{\vdash F\{y/x\},\Gamma',\Delta}\ (\mathrm{Cut}) \quad \longrightarrow^{\star}_{\mathrm{cut}} \pi'.$$

In this case, we reason by case on whether y occurs in Γ',Δ:

(iii) If $y \in FV(\Delta)$, $y \notin FV(\Gamma')$, set $C = \forall yC'$,

$$\pi_1 = \dfrac{\dfrac{\pi_1'}{\vdash \exists xF,\Gamma',C'}\ (\exists)}{\vdash \exists xF,\Gamma',\forall y.C'}\ (\forall)\ ,\quad \pi_2 = \dfrac{\pi_2'}{\vdash \exists y C'^{\perp},\Delta}\ (\exists)\ .$$ One gets:

$$\dfrac{\pi_1 \qquad \pi_2}{\vdash \exists xF,\Gamma',\Delta}\ (\mathrm{Cut}) \quad \longrightarrow_{\mathrm{cut}} \quad \dfrac{\dfrac{\pi_1'}{\vdash \exists xF,\Gamma',C'}\ (\exists) \qquad \pi_2'}{\vdash \exists xF,\Gamma',\Delta}\ (\mathrm{Cut})$$

Proof

Last rule is $(\exists)$

If $\pi = \cfrac{\cfrac{\pi'}{\vdash F\{y/x\},\Gamma',\Delta}}{\vdash \exists x F,\Gamma',\Delta}\ (\exists)$ assuming $\Gamma = \exists x F,\Gamma'$.

We assume that we have no function symbol. By IH, there is an interpolant C' with $\mathscr{L}(C') \subseteq \mathscr{L}(F\{y/x\},\Gamma') \cap \mathscr{L}(\Delta)$ and proofs $\pi_1' \vdash F\{y/x\},\Gamma',C'$ and $\pi_2' \vdash C'^{\perp},\Delta$ st.

$$\cfrac{\cfrac{\pi_1'}{\vdash F\{y/x\},\Gamma',C'} \qquad \cfrac{\pi_2'}{\vdash C'^{\perp},\Delta}}{\vdash F\{y/x\},\Gamma',\Delta}\ (\mathrm{Cut}) \quad \longrightarrow^{\star}_{\mathrm{cut}} \pi'.$$

In this case, we reason by case on whether y occurs in Γ',Δ:

(iii) If $y \in FV(\Delta)$, $y \notin FV(\Gamma')$, set $C = \forall y C'$,

$$\pi_1 = \cfrac{\cfrac{\pi_1'}{\vdash \exists x F,\Gamma',C'}\ (\exists)}{\vdash \exists x F,\Gamma',\forall y.C'}\ (\forall)\ , \quad \pi_2 = \cfrac{\pi_2'}{\vdash \exists y C'^{\perp},\Delta}\ (\exists)\ .\ \text{One gets:}$$

$$\cfrac{\pi_1 \qquad \pi_2}{\vdash \exists x F,\Gamma',\Delta}\ (\mathrm{Cut}) \quad \longrightarrow^{\star}_{\mathrm{cut}} \quad \cfrac{\cfrac{\pi_1' \qquad \pi_2'}{\vdash F\{y/x\},\Gamma',\Delta}\ (\mathrm{Cut})}{\vdash \exists x.F,\Gamma',\Delta}\ (\exists)$$

Proof

Last rule is $(\exists)$

If $\pi = \dfrac{\dfrac{\pi'}{\vdash F\{y/x\}, \Gamma', \Delta}}{\vdash \exists x F, \Gamma', \Delta}\ (\exists)$ assuming $\Gamma = \exists x F, \Gamma'$.

We assume that we have no function symbol. By IH, there is an interpolant C' with $\mathscr{L}(C') \subseteq \mathscr{L}(F\{y/x\}, \Gamma') \cap \mathscr{L}(\Delta)$ and proofs $\pi_1' \vdash F\{y/x\}, \Gamma', C'$ and $\pi_2' \vdash C'^{\perp}, \Delta$ st.

$$\dfrac{\dfrac{\pi_1'}{\vdash F\{y/x\}, \Gamma', C'} \qquad \dfrac{\pi_2'}{\vdash C'^{\perp}, \Delta}}{\vdash F\{y/x\}, \Gamma', \Delta}\ (\text{Cut}) \quad \longrightarrow^{\star}_{\text{cut}} \pi'.$$

In this case, we reason by case on whether y occurs in Γ', Δ:

(iii) If $y \in FV(\Delta)$, $y \notin FV(\Gamma')$, set $C = \forall y C'$,

$\pi_1 = \dfrac{\dfrac{\pi_1'}{\vdash \exists x F, \Gamma', C'}\ (\exists)}{\vdash \exists x F, \Gamma', \forall y. C'}\ (\forall)$, $\pi_2 = \dfrac{\pi_2'}{\vdash \exists y C'^{\perp}, \Delta}\ (\exists)$. One gets:

$$\dfrac{\pi_1 \qquad \pi_2}{\vdash \exists x F, \Gamma', \Delta}\ (\text{Cut}) \quad \longrightarrow^{\star}_{\text{cut}} \quad \pi \text{ by IH.}$$

Proof

Last rule is $(\exists)$

If $\pi = \dfrac{\dfrac{\pi'}{\vdash F\{y/x\}, \Gamma', \Delta}}{\vdash \exists x F, \Gamma', \Delta}\ (\exists)$ assuming $\Gamma = \exists x F, \Gamma'$.

We assume that we have no function symbol. By IH, there is an interpolant C' with $\mathscr{L}(C') \subseteq \mathscr{L}(F\{y/x\}, \Gamma') \cap \mathscr{L}(\Delta)$ and proofs $\pi'_1 \vdash F\{y/x\}, \Gamma', C'$ and $\pi'_2 \vdash C'^{\perp}, \Delta$ st.

$$\dfrac{\dfrac{\pi'_1}{\vdash F\{y/x\}, \Gamma', C'} \qquad \dfrac{\pi'_2}{\vdash C'^{\perp}, \Delta}}{\vdash F\{y/x\}, \Gamma', \Delta}\ (\mathrm{Cut}) \quad \longrightarrow^{\star}_{\mathrm{cut}} \pi'.$$

In this case, we reason by case on whether y occurs in Γ', Δ:

(i) If y occurs in both, take $C = C'$ as interpolant, $\pi_1 = \dfrac{\pi'_1}{\vdash \exists x F, \Gamma'}\ (\exists)$ and $\pi_2 = \pi'_2$ and we have $\mathscr{L}(C) = \mathscr{L}(C') \subseteq \mathscr{L}(F, \Gamma') \cap \mathscr{L}(\Delta) = \mathscr{L}(\exists x F, \Gamma') \cap \mathscr{L}(\Delta)$. The result follows by $(\exists)$-commutation and IH.

Proof-relevant interpolation for LK and LJ

- Either redo the proof, which goes the same way or
- using the usual linear embeddings, the result is lifted to LK and LJ, deduce the result.

Čubrić's proof-relevant interpolation

Prawitz gave a natural deduction-based proof of interpolation (for NK and NJ):

LEMMA. *Let Π be a normal deduction in C' of A depending on Γ, and let Γ_1 and Γ_2 be two disjoint sets such that $\Gamma_1 \cup \Gamma_2 = \Gamma$. Then there is a formula F, called an interpolation formula to $\langle \Gamma_1, \langle \Gamma_2, A \rangle\rangle$, such that $\Gamma_1 \vdash F$ and $\{F\} \cup \Gamma_2 \vdash A$ and such that every parameter that occurs positively [negatively] in F occurs positively [negatively] in some formula of Γ_1 and negatively [positively] in some formula of $\Gamma_2 \cup \{\sim A\}$.*

Based on Prawitz' proof of interpolation for natural deduction, Čubrić proved the following result in his PhD:

Proposition 3.3. *Let L_1 and L_2 be two languages, and let T_1 and T_2 be two $\lambda\delta$-theories on the respective languages. Let T_0 be a theory on the language $L_1 \cap L_2$ such that $T_0 \subset T_1 \cap T_2$ (we may as well assume that the theories are deductively closed). Let $(x^B \rhd t^C)$ be a term in the language $L_1 \cup L_2$ such that the type B is in L_1 and the type C is in L_2. Then, there is a type A in $L_1 \cap L_2$ and terms $(x^B \rhd r^A)$ in L_1, and $(y^A \rhd s^C)$ in L_2 such that:*

$$T_1 \cup T_2 \vdash t =_{x^B} s(r/y).$$

Outline

Interpolation as Cut-Introduction

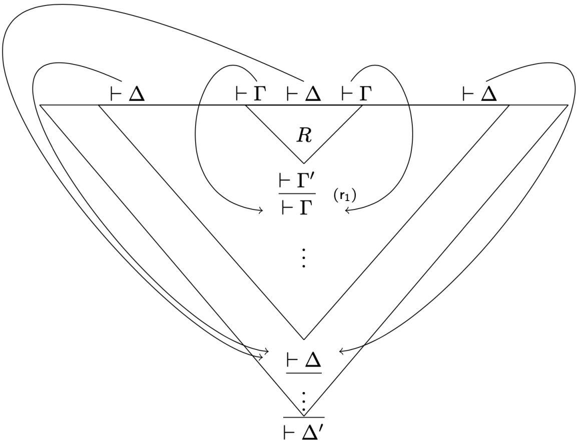
The previous synthesis of the interpolating formula / proofs can be split in two phases, an bottom-up phase and a top-down phase:

- Ascending phase: This first phase consists in traversing the initial proof π bottom-up, from root to axioms, and building, for each visited sequent Γ, a partition (Γ', Γ''). At the end of the phase, there are 4 cases for axiom rules $\vdash A^{\perp}, A$:
  - $(\{A^{\perp}, A\}, \varnothing)$;
  - $(\{A^{\perp}\}, \{A\})$;
  - $(\{A\}, \{A^{\perp}\})$;
  - $(\varnothing, \{A^{\perp}, A\})$.

  and similarly for each axiom corresponding to some unit ($\top$ or 1)
- Descending phase: Equipped with the sequents partitioning information, from root to leaves, one shall now apply cut-introduction rules to axioms, progressively moving the cuts down and merging them in such a way, ultimately, to reach the root sequent of the original proof.

Outline

Circular proofs and circular representations



Need for a correctness condition

$$\dfrac{\dfrac{\vdash F, \nu X.X}{\vdash F, \nu X.X}\,(\nu) \qquad \dfrac{\vdash \mu X.X}{\vdash \mu X.X}\,(\mu)}{\vdash F}\,(\mathsf{Cut})$$

Issues with interpolating non-wellfounded proofs (I)

Three ingredients are important to carry the proof of the previous section to circular proofs:

1. *the wellfoundedness of proof objects.* Indeed this ensures that one reaches axioms which are the base case of the induction; wellfoundedness is implicitly used to initiate the descending phase (that is, after having ended the ascending phase!)
2. *the existence of cut-free proofs.* Indeed, cut-freeness is important to reason by induction on inferences of the cut-free proofs and benefit from analyticity, which is the key for controlling the language of the interpolant, and
3. *the preservation of logical correctness during the descending phase* (that is, cut-introduction). Indeed, correctness of the interpolated proof-objects is of course necessary for the result of interpolation to simply make sense...

Issues with interpolating non-wellfounded proofs (II)

In the case of circular proofs, the first two properties are somehow lost and the third one shall be treated with great care:

1. *wellfoundedness is lost*, even in presence of circular proofs. In particular, even given a finite representation of a given circular proofs, we have leaves which are not axioms but back-edges: how can we interpolated them?
2. while it is crucial to rely on cut-freeness in the reasoning, circular proofs are not closed by cut-elimination and actually we know of sequents which are circularly provable but not cut-free circularly provable.
3. In the course of interpolation, one needs to preserve validity, that is ensure (i) that one has enough threads and (ii) they support the appropriate branches.

Splitting invariance

$$
R = \dfrac{\dfrac{\dfrac{\vdash G, H}{\vdash \underline{H}, H}\,(\mu)}{\vdash \underline{H}, \underline{\underline{H}}}\,(\mathsf{X})}{\vdash \underline{G}, H}\,(\nu)
$$

Splitting invariance

$$
R = \cfrac{\cfrac{\cfrac{\vdash G, H}{\vdash \underline{H}, H}\;(\mu)}{\vdash \underline{H}, \underline{H}}\;(\mathsf{X})}{\vdash \underline{G}, H}\;(\nu)
$$

$$
R' = \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\vdash G, H}{\vdash \underline{H}, H}\;(\mu)}{\vdash \underline{H}, \underline{H}}\;(\mathsf{X})}{\vdash \underline{G}, H}\;(\nu)}{\vdash \underline{H}, H}\;(\mu)}{\vdash \underline{H}, \underline{H}}\;(\mathsf{X})}{\vdash \underline{G}, H}\;(\nu)
$$

Splitting invariance

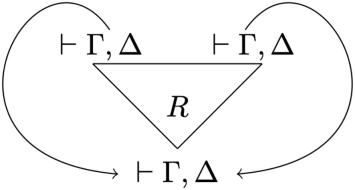
Definition (splitting invariance)

Given a finite representation R of a μLL^{∞} pre-proof $\pi \vdash \Gamma$ and a splitting s of Γ in two components Γ_l, Γ_r, R is called *s-invariant* if the result of applying the splitting-decoration phase to R, initiated with s, results into a decorated derivation R' such that for each back-edge b of R', the splitting of the source of b coincides with the splitting of the target of b.

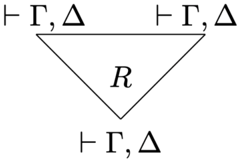
Proposition

Let π be a circular proof of a sequent $\vdash \Gamma$ and let s be a splitting of Γ in two components (Γ_L, Γ_R). To any finite representation R of π, one can associate another finite representation R' of π which is s-invariant.

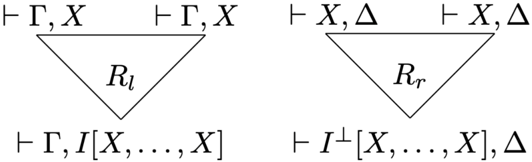
Interpolating circular pre-proofs



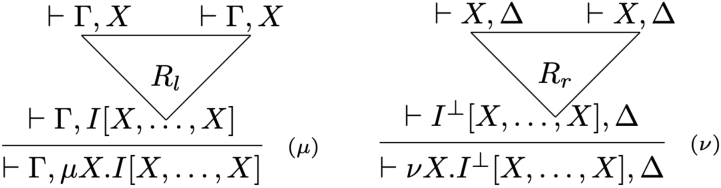
Interpolating circular pre-proofs



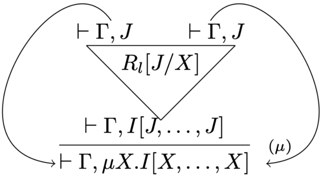
Interpolating circular pre-proofs

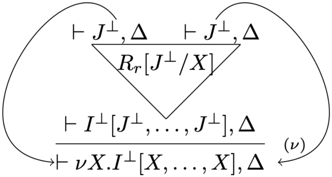


Interpolating circular pre-proofs



Interpolating circular pre-proofs





with $J = \mu X.I[X, \ldots, X]$.

How to ensure preservation of validity? (still WIP)

- Provide a distinct fixed-point variable to each source of a back-edge
- Consider a stronger notion of validity: strong validity, which ensure that the proof can be validated in such a way that each time the thread visit of sequent of the finite representation, it visits the same formula
- Reason on the strongly connected component of the finite representation to find which back-edge should be sequenced first and whether a μ or ν should be used there.

Conclusion

- Full treatment of the validity condition;
- In which other logical system can we apply the method?
- Proof nets interpolation as parsing criterion.
- Semantical counter-part of this interpolation result.
- Connection with uniform inteprolation?

Questions?