Craig-Lyndon's Interpolation as Cut-Introduction

Alexis Saurin

IRIF – CNRS, Université Paris Cité & INRIA

Logic Colloquium 2025 – June 28th, 2024

Introduction

- Various proof-techniques for Craig-Lyndon's interpolation.
  $\Rightarrow$ Maehara's method which exploits cut-admissibility.

- One often has more than cut-admissibility, but also a syntactic cut-elimination result allowing (i) to give a computational interpretation to proofs and (ii) to investigate semantics of proofs.
  $\Rightarrow$ Focus on syntactic cut-elimination relations.

- Relationship between interpolation and cut-introduction.
  $\Rightarrow$ Cut-introduction will guide the synthesis of the interpolant.

- Computational content to the interpolation theorem?
  $\Rightarrow$ Interpolation factors a computation through the interpolation type.

- Results developed for Linear Logic (LL), then extended to classical logic (LK) and intuitionistic logic (LJ) via proof translations.
  $\Rightarrow$ For simplicity, we consider mostly LK in this talk.

Outline

1. Background on Proof-theoretic Methods Craig-Lyndon Interpolation and Cut-Elimination
2. Proof-Relevant Interpolation as Cut-Introduction
3. Conclusion & Perspectives for Future Works

Outline

1. Background on Proof-theoretic Methods Craig-Lyndon Interpolation and Cut-Elimination
2. Proof-Relevant Interpolation as Cut-Introduction
3. Conclusion & Perspectives for Future Works

Statement of Craig's Interpolation Theorem

Definition (Interpolation property)

A logic L has the interpolation property if, for any formulas A, B such that $A \vdash_L B$, there is a formula C satisfying $\mathbb{V}\mathrm{oc}(C) \subseteq \mathbb{V}\mathrm{oc}(A) \cap \mathbb{V}\mathrm{oc}(B)$ and such that $A \vdash_L C$ and $C \vdash_L B$.

$\mathbb{V}\mathrm{oc}(C) \subseteq \mathbb{V}\mathrm{oc}(A) \cap \mathbb{V}\mathrm{oc}(B)$ means:

- *(in Craig's interpolation)* that all predicate symbols occurring in C occur both in A and B.
- *(in Lyndon's interpolation)* that all predicate symbols occurring positively (resp. negatively) in C occur positively (resp. negatively) in both A and B.

Remark that this amounts to a cut-rule: $$\frac{A \vdash_L C \qquad C \vdash_L B}{A \vdash_L B}\ (\mathrm{Cut})$$

Proof(-theoretic) methods for interpolation

Maehara 1960, for Classical Sequent calculus

Maehara:

- Induction on the structure of a cut-free derivation of $A \vdash B$,
- Strengthen the induction hypothesis, by showing that if $\Gamma \vdash \Delta$, then for any splitting Γ', Γ'' of Γ and Δ', Δ'' of Δ there exists an interpolant C with $\mathbb{V}\mathrm{oc}(C) \subseteq \mathbb{V}\mathrm{oc}(\Gamma', \Delta') \cap \mathbb{V}\mathrm{oc}(\Gamma'', \Delta'')$ such that $\Gamma' \vdash C, \Delta'$ and $\Gamma'', C \vdash \Delta''$.

Proof(-theoretic) methods for interpolation

Maehara 1960, for Classical Sequent calculus

Maehara:

- Induction on the structure of a cut-free derivation of $A \vdash B$,
- Strengthen the induction hypothesis, by showing that if $\Gamma \vdash \Delta$, then for any splitting Γ', Γ'' of Γ and Δ', Δ'' of Δ there exists an interpolant C with $\mathbb{V}\mathrm{oc}(C) \subseteq \mathbb{V}\mathrm{oc}(\Gamma', \Delta') \cap \mathbb{V}\mathrm{oc}(\Gamma'', \Delta'')$ such that $\Gamma' \vdash C, \Delta'$ and $\Gamma'', C \vdash \Delta''$.

Method of wide applicability for sequent calculi with cut-admissibility:

- Maehara (1960) proved it for LK;
- Schütte (1962) extended this to LJ;
- Roorda gives a proof for LL analyzing in which fragments interpolation actually holds (1994) *(actually only covers MALL...)*;
- Application to various modal logics;
- Also in natural deduction: Prawitz (1965);
- $\mathbb{V}\mathrm{oc}$ refined as $\mathbb{V}\mathrm{oc}^+$ / $\mathbb{V}\mathrm{oc}^-$ for Lyndon's inteprolation.

LK Sequent Calculus and Cut-elimination

One-sided first-order LK ***sequent calculus, with ancestor relation:***

$$\frac{}{\vdash F, F^{\perp}}\ (\mathrm{Ax}) \qquad \frac{\vdash \Gamma, F \qquad \vdash F^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (\mathrm{Cut})$$

$$\frac{\vdash \Gamma}{\vdash F, \Gamma}\ (\mathrm{W}) \qquad \frac{\vdash F, F, \Gamma}{\vdash F, \Gamma}\ (\mathrm{C}) \qquad \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ (\mathrm{X})$$

$$\frac{}{\vdash \top}\ (\top) \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{\vdash F, \Gamma \qquad \vdash G, \Delta}{\vdash F \wedge G, \Gamma, \Delta}\ (\wedge) \qquad \frac{\vdash F_1, F_2, \Gamma}{\vdash F_1 \vee F_2, \Gamma}\ (\vee)$$

$$\frac{\vdash G\{t/x\}, \Gamma}{\vdash \exists x G, \Gamma}\ (\exists) \qquad \frac{\vdash F, \Gamma}{\vdash \forall x F, \Gamma}\ (\forall) \qquad (\text{in } (\forall)\ ,\ x \notin \mathrm{FV}(\Gamma))$$

LK Sequent Calculus and Cut-elimination

One-sided first-order* LK *sequent calculus, with ancestor relation:

$$\dfrac{}{\vdash F, F^{\perp}}\;(\mathrm{Ax}) \qquad \dfrac{\vdash \Gamma, F \qquad \vdash F^{\perp}, \Delta}{\vdash \Gamma, \Delta}\;(\mathrm{Cut})$$

$$\dfrac{\vdash \Gamma}{\vdash F, \Gamma}\;(\mathrm{W}) \qquad \dfrac{\vdash F, F, \Gamma}{\vdash F, \Gamma}\;(\mathrm{C}) \qquad \dfrac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\;(\mathrm{X})$$

$$\dfrac{}{\vdash \top}\;(\top) \qquad \dfrac{\vdash \Gamma}{\vdash \bot, \Gamma}\;(\bot) \qquad \dfrac{\vdash F, \Gamma \qquad \vdash G, \Delta}{\vdash F \wedge G, \Gamma, \Delta}\;(\wedge) \qquad \dfrac{\vdash F_1, F_2, \Gamma}{\vdash F_1 \vee F_2, \Gamma}\;(\vee)$$

$$\dfrac{\vdash G\{t/x\}, \Gamma}{\vdash \exists x G, \Gamma}\;(\exists) \qquad \dfrac{\vdash F, \Gamma}{\vdash \forall x F, \Gamma}\;(\forall) \qquad (\text{in } (\forall)\text{ , } x \notin \mathrm{FV}(\Gamma))$$

(Examples of) cut-reduction relation:

Key cases & commutation cases

$$\dfrac{\dfrac{\vdash A, \Gamma \qquad \vdash B, \Gamma'}{\vdash A \wedge B, \Gamma, \Gamma'}\;(\wedge) \qquad \dfrac{\vdash A^{\perp}, B^{\perp}, \Delta}{\vdash A^{\perp} \vee B^{\perp}, \Delta}\;(\vee)}{\vdash \Gamma, \Gamma', \Delta}\;(\mathrm{Cut}) \quad \longrightarrow_{\mathrm{cut}} \quad \dfrac{\vdash A, \Gamma \qquad \dfrac{\vdash B, \Gamma' \qquad \vdash A^{\perp}, B^{\perp}, \Delta}{\vdash A^{\perp}, \Gamma', \Delta}\;(\mathrm{Cut})}{\vdash \Gamma, \Gamma', \Delta}\;(\mathrm{Cut})$$

$$\dfrac{\dfrac{\vdash A, \Gamma \qquad \vdash B, \Gamma', C}{\vdash A \wedge B, \Gamma, \Gamma', C}\;(\wedge) \qquad \vdash C^{\perp}, \Delta}{\vdash A \wedge B, \Gamma, \Gamma', \Delta}\;(\mathrm{Cut}) \quad \longrightarrow_{\mathrm{cut}} \quad \dfrac{\vdash A, \Gamma \qquad \dfrac{\vdash B, \Gamma', C \qquad \vdash C^{\perp}, \Delta}{\vdash B, \Gamma', \Delta}\;(\mathrm{Cut})}{\vdash A \wedge B, \Gamma, \Gamma', \Delta}\;(\wedge)$$

System L : Term calculi for the Sequent calculus

Back to 2-sided sequent calculus!

$$\dfrac{\dfrac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\,(\Rightarrow_{\mathrm{r}}) \qquad \dfrac{\Gamma' \vdash A, \Delta' \qquad \Gamma'', B \vdash \Delta''}{\Gamma', \Gamma'', A \Rightarrow B \vdash \Delta', \Delta''}\,(\Rightarrow_{\mathrm{l}})}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta''}\,(\mathrm{Cut})$$

$$\longrightarrow_{\mathrm{cut}} \quad \dfrac{\Gamma' \vdash A, \Delta' \qquad \dfrac{\Gamma, A \vdash B, \Delta \qquad \Gamma'', B \vdash \Delta''}{\Gamma, \Gamma'', A \vdash \Delta, \Delta''}\,(\mathrm{Cut})}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta'}\,(\mathrm{Cut})$$

System L : Term calculi for the Sequent calculus

Back to 2-sided sequent calculus!

$$\dfrac{\dfrac{\Gamma, A \overset{t}{\vdash} B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ (\Rightarrow_r) \qquad \dfrac{\Gamma' \vdash \overset{u}{A}, \Delta' \qquad \Gamma'', \overset{e}{B} \vdash \Delta''}{\Gamma', \Gamma'', A \Rightarrow B \vdash \Delta', \Delta''}\ (\Rightarrow_l)}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta''}\ (\mathrm{Cut})$$

$$\longrightarrow_{\mathrm{cut}} \quad \dfrac{\Gamma' \vdash \overset{u}{A}, \Delta' \qquad \dfrac{\Gamma, A \overset{t}{\vdash} B, \Delta \qquad \Gamma'', \overset{e}{B} \vdash \Delta''}{\Gamma, \Gamma'', A \vdash \Delta, \Delta''}\ (\mathrm{Cut})}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta'}\ (\mathrm{Cut})$$

Corresponds (roughly) to a variant of Krivine's Asbtract Machine (after Curien-Herbelin's *Duality of computation*):

$$\langle \lambda x.t \mid u \cdot e \rangle \longrightarrow_{\mathrm{cut}} \langle u \mid \tilde{\mu} x.\langle t \mid e \rangle \rangle$$

($u \cdot e$ corresponds to evaluation context $e[\Box u]$ while $\tilde{\mu} x.\langle t \mid e \rangle$ corresponds to let $x = \Box$ in $e[t]$.*)*

$\Rightarrow$ System L calculi are structured around:

terms t (right rules),
contexts e (left rules),
commands $\langle t \mid e \rangle$ (cut rules).

Outline

1. Background on Proof-theoretic Methods Craig-Lyndon Interpolation and Cut-Elimination
2. Proof-Relevant Interpolation as Cut-Introduction
3. Conclusion & Perspectives for Future Works

Proof-Relevant Interpolation as Cut-Introduction

Refining Maehara's method

Prove that for any splitting of the conclusion sequent, one can find (i) an interpolant formula and (ii) two interpolating proofs such that cutting together the interpolating proofs cut-reduces to the original proof.

Proof-Relevant Interpolation as Cut-Introduction

Refining Maehara's method

Prove that for any splitting of the conclusion sequent, one can find (i) an interpolant formula and (ii) two interpolating proofs such that cutting together the interpolating proofs cut-reduces to the original proof.

Interpolation as Cut-Introduction will proceed in two phases:

- Bottom-up phase: Starting with a splitting of the root sequent, traverse the initial proof π bottom-up, from root to axioms, and split each visited sequent Γ, as (Γ', Γ'') according to the ancestor relation. In the end, there are 4 cases for an axiom rule $\vdash A^{\perp}, A$:

  $$\vdash A^{\perp}, A \qquad \vdash A^{\perp}, A \qquad \vdash A^{\perp}, A \qquad \vdash A^{\perp}, A$$

  and similarly for each axiom corresponding to $\top$.

- Top-down phase: Equipped with the sequents splitting information, one shall now apply cut-introduction rules to axioms, progressively moving the cuts down and merging them in such a way, ultimately, to reach the root sequent of the original proof.

Proof-Relevant Interpolation as Cut-Introduction

Refining Maehara's method

Prove that for any splitting of the conclusion sequent, one can find (i) an interpolant formula and (ii) two interpolating proofs such that cutting together the interpolating proofs cut-reduces to the original proof.

Interpolation as Cut-Introduction will proceed in two phases:

- Bottom-up phase: Starting with a splitting of the root sequent, traverse the initial proof π bottom-up, from root to axioms, and split each visited sequent Γ, as (Γ', Γ'') according to the ancestor relation. In the end, there are 4 cases for an axiom rule $\vdash A^{\perp}, A$:

  $$\vdash A^{\perp}, A \qquad \vdash A^{\perp}, A \qquad \vdash A^{\perp}, A \qquad \vdash A^{\perp}, A$$

  and similarly for each axiom corresponding to $\top$.

- Top-down phase: Equipped with the sequents splitting information, one shall now apply cut-introduction rules to axioms, progressively moving the cuts down and merging them in such a way, ultimately, to reach the root sequent of the original proof.

$$\overset{\pi}{\vdash \Gamma, \Delta} \qquad \text{interpolated as} \qquad \dfrac{\overset{\pi^L}{\vdash \Gamma, I} \qquad \overset{\pi^R}{\vdash I^{\perp}, \Delta}}{\vdash \Gamma, \Delta}\ (\mathrm{Cut})$$

Proof-Relevant Interpolation Situation

Definition (Proof-relevant Interpolation Situation – PRIS)

A PRIS for (Γ, Δ) is given by:

- a cut-free LK proof π of conclusion $\vdash \Gamma, \Delta$ and with $n \geq 0$ open premises $(\vdash \Gamma_i, \Delta_i)_{1 \leq i \leq n}$ such that for each $1 \leq i \leq n$ the formulas in Γ_i (resp. Δ_i) are ancestors of formulas in Γ (resp. of Δ); – the goal
- for each $1 \leq i \leq n$, a formula I_i st. $\mathbb{V}\mathrm{oc}^+(I_i) \subseteq \mathbb{V}\mathrm{oc}^-(\Gamma_i) \cap \mathbb{V}\mathrm{oc}^+(\Delta_i)$ and $\mathbb{V}\mathrm{oc}^-(I_i) \subseteq \mathbb{V}\mathrm{oc}^+(\Gamma_i) \cap \mathbb{V}\mathrm{oc}^-(\Delta_i)$; – the partial interpolants
- for each $1 \leq i \leq n$, derivations π_i^L (resp. π_i^R) of conclusion $\vdash \Gamma_i, I_i$ (resp. $\vdash I_i^\perp, \Delta_i$). – the partial solutions

Represented as:

$$
\dfrac{\dfrac{\pi_1^L}{\vdash \Gamma_1, I_1} \quad \dfrac{\pi_1^R}{\vdash I_1^\perp, \Delta_1}}{\vdash \Gamma_1, \Delta_1}\ (\mathrm{Cut}) \qquad \dots \qquad \dfrac{\dfrac{\pi_n^L}{\vdash \Gamma_n, I_n} \quad \dfrac{\pi_n^R}{\vdash I_n^\perp, \Delta_n}}{\vdash \Gamma_n, \Delta_n}\ (\mathrm{Cut})
$$

$$
\pi
$$

$$
\vdash \Gamma, \Delta
$$

Special cases of PRIS

$$\begin{array}{c}
\dfrac{\begin{array}{cc}\pi_1^L & \pi_1^R \\ \vdash \Gamma_1, I_1 & \vdash I_1^\perp, \Delta_1\end{array}}{\vdash \Gamma_1, \Delta_1}\ (\mathrm{Cut}) \quad \dots \quad \dfrac{\begin{array}{cc}\pi_n^L & \pi_n^R \\ \vdash \Gamma_n, I_n & \vdash I_n^\perp, \Delta_n\end{array}}{\vdash \Gamma_n, \Delta_n}\ (\mathrm{Cut}) \\
\pi \\
\vdash \Gamma, \Delta
\end{array}$$

- *Initial PRIS*: when $n = 0$, $\begin{array}{c}\pi \\ \vdash \Gamma, \Delta\end{array}$
- *Solved PRIS*: when $n = 1$ and π is reduced to one open premise node $\vdash \Gamma, \Delta$:

$$\pi = \dfrac{\begin{array}{cc}\pi_1^L & \pi_1^R \\ \vdash \Gamma, I_1 & \vdash I_1^\perp, \Delta\end{array}}{\vdash \Gamma, \Delta}\ (\mathrm{Cut}) .$$

  I_1 is the interpolant and π_1^L, π_1^R are the interpolating proofs.
- *Elementary PRIS*: for (r) an n-ary inference rule,

$$\pi = \dfrac{\dfrac{\begin{array}{cc}\pi_1^L & \pi_1^R \\ \vdash \Gamma_1, I_1 & \vdash I_1^\perp, \Delta_1\end{array}}{\vdash \Gamma_1, \Delta_1}\ (\mathrm{Cut}) \quad \dots \quad \dfrac{\begin{array}{cc}\pi_n^L & \pi_n^R \\ \vdash \Gamma_n, I_n & \vdash I_n^\perp, \Delta_n\end{array}}{\vdash \Gamma_n, \Delta_n}\ (\mathrm{Cut})}{\vdash \Gamma, \Delta}\ (\mathrm{r})$$

How to relate initial and solved PRIS via cut-introduction?

Solving PRIS

$$\dfrac{\dfrac{\begin{matrix}\pi_1^L \\ \vdash \Gamma_1, I_1\end{matrix} \quad \begin{matrix}\pi_1^R \\ \vdash I_1^\perp, \Delta_1\end{matrix}}{\vdash \Gamma_1, \Delta_1}\,(\mathrm{Cut}) \quad \dots \quad \dfrac{\begin{matrix}\pi_n^L \\ \vdash \Gamma_n, I_n\end{matrix} \quad \begin{matrix}\pi_n^R \\ \vdash I_n^\perp, \Delta_n\end{matrix}}{\vdash \Gamma_n, \Delta_n}\,(\mathrm{Cut})}{\vdash \Gamma, \Delta}\,(\mathrm{r})$$

Lemma

For any n-ary inference rule (r) *and any elementary PRIS π there exist I, π^L, π^R such that* $\pi' = \dfrac{\begin{matrix}\pi^L \\ \vdash \Gamma, I\end{matrix} \quad \begin{matrix}\pi^R \\ \vdash I^\perp, \Delta\end{matrix}}{\vdash \Gamma, \Delta}\,(\mathrm{Cut})$ *is a solved PRIS and* $\pi \longleftarrow^{\star}_{\mathrm{cut}} \pi'$.

Solving PRIS

$$\dfrac{\dfrac{\dfrac{\pi_1^L}{\vdash \Gamma_1, I_1} \quad \dfrac{\pi_1^R}{\vdash I_1^\perp, \Delta_1}}{\vdash \Gamma_1, \Delta_1}\,(\mathrm{Cut}) \quad \dots \quad \dfrac{\dfrac{\pi_n^L}{\vdash \Gamma_n, I_n} \quad \dfrac{\pi_n^R}{\vdash I_n^\perp, \Delta_n}}{\vdash \Gamma_n, \Delta_n}\,(\mathrm{Cut})}{\vdash \Gamma, \Delta}\,(\mathrm{r})$$

Lemma

For any n-ary inference rule (r) *and any elementary PRIS* π *there exist* I, π^L, π^R *such that* $\pi' = \dfrac{\dfrac{\pi^L}{\vdash \Gamma, I} \quad \dfrac{\pi^R}{\vdash I^\perp, \Delta}}{\vdash \Gamma, \Delta}\,(\mathrm{Cut})$ *is a solved PRIS and* $\pi \longleftarrow^{\star}_{\mathrm{cut}} \pi'$.

Corollary

Any initial PRIS can be reduced, by cut-expansions, to a solved PRIS.

(Indeed, each application of the above lemma decreases by 1 the size of interpolation goal: the sequence of cut-introductions ends in a solved PRIS.)

Solving PRIS

$$\dfrac{\dfrac{\overset{\pi_1^L}{\vdash \Gamma_1, I_1} \quad \overset{\pi_1^R}{\vdash I_1^\perp, \Delta_1}}{\vdash \Gamma_1, \Delta_1}\,(\mathrm{Cut}) \quad \dots \quad \dfrac{\overset{\pi_n^L}{\vdash \Gamma_n, I_n} \quad \overset{\pi_n^R}{\vdash I_n^\perp, \Delta_n}}{\vdash \Gamma_n, \Delta_n}\,(\mathrm{Cut})}{\vdash \Gamma, \Delta}\,(\mathrm{r})$$

Lemma

For any n-ary inference rule (r) *and any elementary PRIS* π *there exist* I, π^L, π^R *such that* $\pi' = \dfrac{\overset{\pi^L}{\vdash \Gamma, I} \quad \overset{\pi^R}{\vdash I^\perp, \Delta}}{\vdash \Gamma, \Delta}\,(\mathrm{Cut})$ *is a solved PRIS and* $\pi \longleftarrow^\star_{\mathrm{cut}} \pi'$.

Corollary

Any initial PRIS can be reduced, by cut-expansions, to a solved PRIS.

(Indeed, each application of the above lemma decreases by 1 the size of interpolation goal: the sequence of cut-introductions ends in a solved PRIS.)

Theorem

Let A, B *be* LL *formulas and* π *be a cut-free* LL *proof of* $A \vdash B$.
There exists a LL *formula* C *such that* $\mathbb{V}\mathrm{oc}^+(C) \subseteq \mathbb{V}\mathrm{oc}^+(A) \cap \mathbb{V}\mathrm{oc}^+(B)$ *and* $\mathbb{V}\mathrm{oc}^-(C) \subseteq \mathbb{V}\mathrm{oc}^-(A) \cap \mathbb{V}\mathrm{oc}^-(B)$ *and two cut-free* LL *proofs* π_1, π_2 *of* $A \vdash C$ *and* $C \vdash B$ *respectively such that* $\dfrac{\overset{\pi_1}{A \vdash C} \quad \overset{\pi_2}{C \vdash B}}{A \vdash B}\,(\mathrm{Cut}) \longrightarrow^\star_{\mathrm{cut}} \pi$.

(Similar results for LK *and* LJ*.)*

Proof-Relevant Interpolation in System L

terms t (right rules),
contexts e (left rules),
commands $\langle t \mid e \rangle$ (cut rules).

For classical, intuitionistic or linear versions of System L:

Proposition

Assume t (resp. e, resp. c) is a normal L-term (resp. normal L-context, resp. normal L-command). The following interpolating results hold:

1. *If* $c : (\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2)$*, there exist an* $I \in \mathbb{V}\mathrm{oc}(\Gamma_1, \Delta_1) \cap \mathbb{V}\mathrm{oc}(\Gamma_2, \Delta_2)$ *and* t, e *such that* $\Gamma_1 \vdash t : I \mid \Delta_1$ *and* $\Gamma_2 \mid e : I \vdash \Delta_2$*, and* $\langle t \mid e \rangle \longrightarrow^\star c$*.*

Proof-Relevant Interpolation in System L

Computational Content of Interpolation

terms t (right rules),
contexts e (left rules),
commands $\langle t \mid e\rangle$ (cut rules).

For classical, intuitionistic or linear versions of System L:

Proposition

Assume t (resp. e, resp. c) is a normal L-term (resp. normal L-context, resp. normal L-command). The following interpolating results hold:

1. *If* $c : (\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2)$, *there exist an* $I \in \mathbb{V}\mathrm{oc}(\Gamma_1, \Delta_1) \cap \mathbb{V}\mathrm{oc}(\Gamma_2, \Delta_2)$ *and* t, e *such that* $\Gamma_1 \vdash t : I \mid \Delta_1$ *and* $\Gamma_2 \mid e : I \vdash \Delta_2$, *and* $\langle t \mid e\rangle \longrightarrow^\star c$.
2. *If* $\Gamma_1, \Gamma_2 \vdash t : A \mid \Delta_1, \Delta_2$, *there exist an* $I \in \mathbb{V}\mathrm{oc}(\Gamma_1, \Delta_1, A) \cap \mathbb{V}\mathrm{oc}(\Gamma_2, \Delta_2)$ *and* α, t', e' *st* $\Gamma_1 \vdash t' : A \mid \alpha : I, \Delta_1$, $\Gamma_2 \mid e' : I \vdash \Delta_2$, *and* $t'\{e'/\alpha\} \longrightarrow^\star t$.
3. *If* $\Gamma_1, \Gamma_2 \mid e : A \vdash \Delta_1, \Delta_2$, *there exist an* $I \in \mathbb{V}\mathrm{oc}(\Gamma_1, \Delta_1, A) \cap \mathbb{V}\mathrm{oc}(\Gamma_2, \Delta_2)$ *and* α, e', e'' *st* $\Gamma_1 \mid e' : A \mid \alpha : I, \Delta_1$, $\Gamma_2 \mid e'' : I \vdash \Delta_2$, *and* $e'\{e''/\alpha\} \longrightarrow^\star e$.
4. *If* $\Gamma_1, \Gamma_2 \vdash t : A \mid \Delta_1, \Delta_2$, *there exist an* $I \in \mathbb{V}\mathrm{oc}(\Gamma_1, \Delta_1) \cap \mathbb{V}\mathrm{oc}(\Gamma_2, \Delta_2, A)$ *and* α, t', t'' *st.* $\Gamma_1 \vdash t'' : I \mid \Delta_1$, $\Gamma_2, x : I \vdash t' : A \mid \Delta_2$, *and* $t'\{t''/\alpha\} \longrightarrow^\star t$.
5. *If* $\Gamma_1, \Gamma_2 \mid e : A \vdash \Delta_1, \Delta_2$, *there exist an* $I \in \mathbb{V}\mathrm{oc}(\Gamma_1, \Delta_1) \cap \mathbb{V}\mathrm{oc}(\Gamma_2, \Delta_2, A)$ *and* x, t', e' *st.* $\Gamma_1 \vdash t' : I \mid \Delta_1$, $\Gamma_2, x : I \mid e' : A \vdash \Delta_2$, *and* $e'\{t'/x\} \longrightarrow^\star e$.

Related proof-relevant Interpolation results

Quite surprisingly, Maehara's usual proof technique essentially provides this result, even though this is not noticed in any published reference.

Related proof-relevant Interpolation results

Quite surprisingly, Maehara's usual proof technique essentially provides this result, even though this is not noticed in any published reference.

Prawitz:

LEMMA. *Let* Π *be a normal deduction in* C' *of* A *depending on* Γ, *and let* Γ_1 *and* Γ_2 *be two disjoint sets such that* $\Gamma_1 \cup \Gamma_2 = \Gamma$. *Then there is a formula* F, *called an interpolation formula to* $\langle \Gamma_1, \langle \Gamma_2, A \rangle\rangle$, *such that* $\Gamma_1 \vdash F$ *and* $\{F\} \cup \Gamma_2 \vdash A$ *and such that every parameter that occurs positively* [*negatively*] *in* F *occurs positively* [*negatively*] *in some formula of* Γ_1 *and negatively* [*positively*] *in some formula of* $\Gamma_2 \cup \{\sim A\}$.

(for NK and NJ)

Related proof-relevant Interpolation results

Quite surprisingly, Maehara's usual proof technique essentially provides this result, even though this is not noticed in any published reference.

Prawitz:

LEMMA. *Let Π be a normal deduction in C' of A depending on Γ, and let Γ_1 and Γ_2 be two disjoint sets such that $\Gamma_1 \cup \Gamma_2 = \Gamma$. Then there is a formula F, called an interpolation formula to $\langle \Gamma_1, \langle \Gamma_2, A \rangle\rangle$, such that $\Gamma_1 \vdash F$ and $\{F\} \cup \Gamma_2 \vdash A$ and such that every parameter that occurs positively [negatively] in F occurs positively [negatively] in some formula of Γ_1 and negatively [positively] in some formula of $\Gamma_2 \cup \{\sim A\}$.*

(for NK and NJ)

Čubrić: Similar proof-relevant result established in 1993 for simply typed λ-calculus by refining Prawitz's result, in a paper that was almost never cited:

Proposition 3.3. *Let L_1 and L_2 be two languages, and let T_1 and T_2 be two $\lambda\delta$-theories on the respective languages. Let T_0 be a theory on the language $L_1 \cap L_2$ such that $T_0 \subset T_1 \cap T_2$ (we may as well assume that the theories are deductively closed). Let $(x^B \rhd t^C)$ be a term in the language $L_1 \cup L_2$ such that the type B is in L_1 and the type C is in L_2. Then, there is a type A in $L_1 \cap L_2$ and terms $(x^B \rhd r^A)$ in L_1, and $(y^A \rhd s^C)$ in L_2 such that:*

$$T_1 \cup T_2 \vdash t =_{x^B} s(r/y).$$

Outline

1. Background on Proof-theoretic Methods Craig-Lyndon Interpolation and Cut-Elimination
2. Proof-Relevant Interpolation as Cut-Introduction
3. Conclusion & Perspectives for Future Works

Conclusion & Perspectives for future works

	classical	intuitionistic	linear
Sequent calculus	✓[Maehara]	✓[Schütte]	✓[Roorda]
PR int	✓[S]	✓[S]	✓[S]
System L	✓[S]	✓[S]	✓[S]
Natural deduction	✓[Prawitz]	✓[Prawitz]	✓[Fiorillo, Osorio, S]
PR int	ongoing	✓[Čubrić]	✓[S]
λ-calculi	✓[Fiorillo]	✓[Čubrić]	?
Denotational semantics	?	✓[Čubrić]	ongoing

Conclusion & Perspectives for future works

	classical	intuitionistic	linear
Sequent calculus	✓[Maehara]	✓[Schütte]	✓[Roorda]
PR int	✓[S]	✓[S]	✓[S]
System L	✓[S]	✓[S]	✓[S]
Natural deduction	✓[Prawitz]	✓[Prawitz]	✓[Fiorillo, Osorio, S]
PR int	ongoing	✓[Čubrić]	✓[S]
λ-calculi	✓[Fiorillo]	✓[Čubrić]	?
Denotational semantics	?	✓[Čubrić]	ongoing

- Cut-introduction as a mean to synthetise PR interpolants.
- Computational interpretation: interpolant as an interface datatype through which a given computation can be factored.
- Computational interpretation of uniform interpolation?
- Other directions of ongoing and future works:
  - PR interpolation in the μ-calculus (ongoing with Osorio).
  - To which other proof systems can we apply the method?
  - Semantical interpolation result (ongoing with Fiorillo).
  - Impact of the proof-relevant approach on other applications of interpolation (in database theory or model-checking)?

Proof of the Main Lemma

Axiom case

- If $\pi = \dfrac{}{\vdash F, F^\perp}\ (\mathrm{Ax})$, one simply takes $I = F^\perp$, $\pi_1^l = \dfrac{}{\vdash F, F^\perp}\ (\mathrm{Ax})$ and $\pi_1^r = \dfrac{}{\vdash F, F^\perp}\ (\mathrm{Ax})$.
  The cut between π_1^l and π_1^r reduces to π by one cut-axiom reduction step.

- If $\pi = \dfrac{}{\vdash F, F^\perp}\ (\mathrm{Ax})$, the case is symmetrical taking $I = F$.

- If $\pi = \dfrac{}{\vdash F, F^\perp}\ (\mathrm{Ax})$, one takes $I = \perp$, $\pi_1^l = \dfrac{\pi}{\vdash F, F^\perp, \perp}\ (\perp)$ and $\pi_1^r = \dfrac{}{\vdash \top}\ (\top)$.
  The cut of π_1^l and π_1^r reduces to π by a key $\top/\perp$ case.

- If $\pi = \dfrac{}{\vdash F, F^\perp}\ (\mathrm{Ax})$, the case is symmetrical, taking $I = \top$.

Proof of the Main Lemma

- If $\pi = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash \Gamma, I} \qquad \overset{\pi_1^R}{\vdash I^\perp, \Delta, A, B}}{\vdash \Gamma, \Delta, A, B}\ (\mathrm{Cut})}{\vdash \Gamma, \Delta, A \vee B}\ (\vee)$ then taking $I' = I$, $\pi^L = \pi_1^L$ and

  $\pi^R = \dfrac{\overset{\pi_1^R}{\vdash I^\perp, \Delta, A, B}}{\vdash I^\perp, \Delta, A \vee B}\ (\vee)$ we obtain a solved PRIS π' such that $\pi \longleftarrow_{\mathrm{cut}} \pi'$

  by a commutative reduction of (Cut).

Proof of the Main Lemma

- If $\pi = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash \Gamma, I} \qquad \overset{\pi_1^R}{\vdash I^\perp, \Delta, A, B}}{\vdash \Gamma, \Delta, A, B}\ (\text{Cut})}{\vdash \Gamma, \Delta, A \vee B}\ (\vee)$ then taking $I' = I$, $\pi^L = \pi_1^L$ and

  $\pi^R = \dfrac{\overset{\pi_1^R}{\vdash I^\perp, \Delta, A, B}}{\vdash I^\perp, \Delta, A \vee B}\ (\vee)$ we obtain a solved PRIS π' such that $\pi \longleftarrow_{\text{cut}} \pi'$ by a commutative reduction of (Cut).

- If $\pi = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash \Gamma_1, I_1} \qquad \overset{\pi_1^R}{\vdash I_1^\perp, \Delta_1, A}}{\vdash \Gamma_1, \Delta_1, A}\ (\text{Cut}) \qquad \dfrac{\overset{\pi_2^L}{\vdash \Gamma_2, I_2} \qquad \overset{\pi_2^R}{\vdash I_2^\perp, \Delta_2, B}}{\vdash \Gamma_2, \Delta_2, B}\ (\text{Cut})}{\vdash \Gamma_1, \Gamma_2, \Delta_1, \Delta_2, A \wedge B}\ (\wedge)$, then setting

  $I = I_1 \wedge I_2$, $\pi^L = \dfrac{\overset{\pi_1^L}{\vdash \Gamma_1, I_1} \qquad \overset{\pi_2^L}{\vdash \Gamma_2, I_2}}{\vdash \Gamma_1, \Gamma_2, I_1 \wedge I_2}\ (\wedge)$ and $\pi^R = \dfrac{\dfrac{\overset{\pi_1^R}{\vdash I_1^\perp, \Delta_1, A} \qquad \overset{\pi_2^R}{\vdash I_2^\perp, \Delta_2, B}}{\vdash I_1^\perp, I_2^\perp, \Delta_1, \Delta_2, A \wedge B}\ (\wedge)}{\vdash (I_1 \wedge I_2)^\perp, \Delta_1, \Delta_2, A \wedge B}\ (\vee)$

  one gets a solved PRIS π' such that $\pi \longleftarrow^{\star}_{\text{cut}} \pi'$ by a commutative reduction of (Cut) and a key $(\wedge)/(\vee)$ case.

- Other cases are treated similarly.