Virtuous circles in proofs

Virtual proof theory seminar

Alexis Saurin
IRIF, CNRS, INRIA & Université de Paris

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Introduction and Background

Logics with least and greatest fixed points

Logics with least and greatest fixed points, modelling *inductive and coinductive* reasoning:

- Very useful to encode and reason about inductive and coinductive data structures.
- Their proof theory is not very well studied and understood.
- Not only to express statements, but also a proof system in *sequent calculus*: LL *with fixed points*
  - μLL: proofs are finite trees. Includes rules for induction, local correctness, cut-elimination and focalization but not subformula property
  - μLL^{∞}: proofs are infinite trees. Simple inference rules for fixed points, global correctness criterion, cut-elimination with subformula property. Of particular interest is the fragment of *circular proofs*, which are presentable as finite graphs.
- Extends the *proof–program correspondence* to recursive and co-recursive programming, with coinductive datatypes.

Outline

1. Introduction
2. μLL$^\infty$: circular and non-wellfounded proofs
3. Cut-elimination for μMALL$^\infty$ *(joined work with Baelde & Doumane)*
4. Cut-elimination for μLL$^\infty$
5. Relaxing the thread validity condition *(joined work with Baelde, Doumane & Kuperberg)*
6. On sequentiality and parallelism in non-wellfounded proofs *(joined work with De & Pellissier)*
7. Conclusion

Knaster-Tarski fixed-point theorem

Let C be a complete lattice and F a monotonic operator on C.

Theorem

F has a **least** fixed-point μF.

μF is the **least prefixed**-point:
– $F(\mu F) \sqsubseteq \mu F$ and
– $\forall S, F(S) \sqsubseteq S \;\Rightarrow\; \mu F \sqsubseteq S$.

Theorem

F has a **greatest** fixed-point νF.

νF is the greatest postfixed-point:
– $\nu F \sqsubseteq F(\nu F)$ and
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Proof by induction:

To prove that $\mu F \subseteq P$, it is sufficient to find some $S \subseteq P$ and to prove that $\forall x \in F(S)$, $x \in S$.

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Proof by coinduction:

To prove that $P \subseteq \nu F$, it is sufficient to find some $S \supseteq P$ and to prove that $\forall x \in S$, $x \in F(S)$.

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$$\frac{H \vdash F[\mu X.F/X]}{H \vdash \mu X.F}\,[\mu_r] \qquad \frac{F[S/X] \vdash S}{\mu X.F \vdash S}\,[\mu_l]$$

Proof by coinduction:
To prove that $P \subseteq \nu F$, it is sufficient to find some $S \supseteq P$ and to prove that $\forall x \in S$, $x \in F(S)$.

$$\frac{F[\nu X.F/X] \vdash H}{\nu X.F \vdash H}\,[\nu_l] \qquad \frac{S \vdash F[S/X]}{S \vdash \nu X.F}\,[\nu_r]$$

Martin-Löf's induction definitions, LKID

A sequent calculus parameterized by a set of inductive definitions.

Idea: inductive predicates described by production rules.

$$\frac{Q_1(u_1) \;\ldots\; Q_k(u_k) \quad P_1(t_1) \;\ldots\; P_l(t_l)}{P(t)}$$

Example:

$$\frac{}{N(0)} \qquad \frac{N(x)}{N(s(x))}$$

LK + ***inferences for the inductively defined predicates:***

$$\frac{}{\Gamma \vdash N(0), \Delta}\ (N_R^1) \qquad \frac{\Gamma \vdash N(u), \Delta}{\Gamma \vdash N(s(u)), \Delta}\ (N_R^2)$$

$$\frac{\Gamma \vdash F(0), \Delta \qquad \Gamma, F(x) \vdash F(s(x)), \Delta \qquad \Gamma, F(t) \vdash \Delta}{\Gamma, N(t) \vdash \Delta}\ (Ind\ N)$$

Mutually dependency:

$$\frac{}{E(0)} \qquad \frac{O(x)}{E(sx)} \qquad \frac{E(x)}{O(sx)}$$

Fixed-point logics and (co)induction

Some examples from (co)inductive predicates to μ-calculus

- $Nat(x) \triangleq_{ind} (x = 0) \vee \exists y. x = s(y) \wedge Nat(y)$
- $ListNat(l) \triangleq_{ind} (l = nil) \vee \exists h, t. l = h :: t \wedge (Nat(h) \wedge ListNat(t))$
- $StreamNat(l) \triangleq_{coind} \exists h, t. l = h :: t \wedge (Nat(h) \wedge StreamNat(t))$

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- $Nat \triangleq \mu N. \top \vee N$
- $ListNat \triangleq \mu L. \top \vee (Nat \wedge L)$
- $StreamNat \triangleq \nu S. Nat \wedge S$

$\Rightarrow$ in the following, the propositional μ-calculus only.

Fixed-point logics and (co)induction

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Interleavings of inductive/coinductives behaviours; *eg.* allowing to express fairness properties:

$$\nu X. \mu Y. (P \wedge \bigcirc X) \vee \bigcirc Y.$$

$\mu MALL$: MALL with least and greatest fixed points

$\mu MALL$ formulas and sequent calculus

(Baelde & Miller 2007, Baelde 2012)

$\mu MALL$ formulas

$F \quad ::= \quad a \mid \top \mid \bot \mid F \parr F \mid F \& F$	**negative *MALL* formulas**
$\mid a^\perp \mid 0 \mid 1 \mid F \otimes F \mid F \oplus F$	**positive *MALL* formulas**
$\mid X \mid \mu X.F \mid \nu X.F$	**least and greatest fixed points**

- Negation $(\ )^\perp$: involutive operator on formula, not a connective.
- μ and ν are binders, consider closed formulas only.
- μ and ν are dual. Ex: $(\nu X.X \otimes X)^\perp = \mu X.X \parr X$.
- One-sided sequents: $\vdash A_1, \ldots, A_n$. *($\Gamma \vdash \Delta$ is a short for $\vdash \Gamma^\perp, \Delta$)*
- Data types encodings:

$$\begin{aligned} \text{Nat} &\triangleq \mu X.1 \oplus X \\ \text{List}(A) &\triangleq \mu X.1 \oplus (A \otimes X) \\ \text{Stream}(A) &\triangleq \nu X.1 \& (A \otimes X) \end{aligned}$$

μMALL sequent Calculus

μMALL Inference Rules

$$\frac{}{\vdash F, F^\perp}\ [\mathrm{Ax}] \qquad \frac{\vdash \Gamma, F \quad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta}\ [\mathrm{Cut}] \qquad \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ [\mathrm{X}]$$

$$\frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ [\parr] \qquad \frac{\vdash F, \Gamma \quad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ [\otimes] \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ [\bot] \qquad \frac{}{\vdash 1}\ [1]$$

$$\frac{\vdash F, \Gamma \quad \vdash G, \Gamma}{\vdash F \& G, \Gamma}\ [\&] \qquad \frac{\vdash A_i, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ [\oplus_i] \qquad \frac{}{\vdash \top, \Gamma}\ [\top] \qquad \text{(no rule for 0)}$$

$$\frac{\vdash \Gamma, S \quad \vdash S^\perp, G[S/X]}{\vdash \nu X.G, \Gamma}\ [\nu] \qquad \frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\ [\mu]$$

μMALL sequent Calculus

μMALL Inference Rules (with explicit ancestor relation)

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Theorem

Cut elimination holds in μMALL.

Proof theory of least and greatest fixed points

	μMALL
Proof objects	Finite trees
Inferences	Induction rules
MALL rules +	$\dfrac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F}\ [\mu]$ $\dfrac{\vdash \Gamma, S \quad \vdash S^{\perp}, F[S/X]}{\vdash \Gamma, \nu X.F}\ [\nu]$
Log. correctness	local
Cut-elimination	sort of: $[\nu]$ hides a cut
Subformula prop.	NO
Focalization	✓, but μ/ν have arbitrary polarities

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Proof theory of least and greatest fixed points

	μMALL	μMALL$^\infty$
Proof objects	Finite trees	Non well-founded trees
Inferences	Induction rules	Fixed points unfoldings (+ validity conditions)
MALL rules +	$\dfrac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F}\ [\mu]$ $\dfrac{\vdash \Gamma, S \quad \vdash S^\perp, F[S/X]}{\vdash \Gamma, \nu X.F}\ [\nu]$	$\dfrac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F}\ [\mu]$ $\dfrac{\vdash \Gamma, F[\nu X.F/X]}{\vdash \Gamma, \nu X.F}\ [\nu]$
Log. correctness	local	global
Cut-elimination	sort of: $[\nu]$ hides a cut	✓
Subformula prop.	NO	✓
Focalization	✓, but μ/ν have arbitrary polarities	✓ μ pos. and ν neg.

μLL$^{\infty}$: circular and non-wellfounded proofs for linear logic with least and greatest fixed-points

Circular proofs: an old mathematical story

Back to Euclid's *Elements* (Book VII)

another example

PROPOSITION 31

Any composite number is measured by some prime number.

Let A be a composite number;

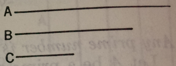
I say that A is measured by some prime number.

For, since A is composite,

some number will measure it.

Let a number measure it, and let it be B.

Now, if B is prime, what was enjoined will have been done.



But if it is composite, some number will measure it.

Let a number measure it, and let it be C.

Then, since C measures B,

and B measures A,

therefore C also measures A.

And, if C is prime, what was enjoined will have been done.

But if it is composite, some number will measure it.

Thus, if the investigation be continued in this way, some prime number will be found which will measure the number before it, which will also measure A.

For, if it is not found, an infinite series of numbers will measure the number A, each of which is less than the other:

which is impossible in numbers.

Therefore some prime number will be found which will measure the one before it, which will also measure A.

Therefore any composite number is measured by some prime number.

Q. E. D.

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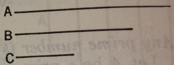

Now, if B is prime, what was enjoined will have been done.

But if it is composite, some number will measure it.

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therefore C also measures A.

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Root of Fermat's **infinite descent** proof method.

Non-wellfounded proofs: inductive and coinductive cases

Inductive case:

$$\dfrac{\dfrac{}{\vdash nat\ 0} \quad \dfrac{\dfrac{\dfrac{\vdots}{even\ y \vdash nat\ y}}{even\ y \vdash nat\ (s\ y)}}{even\ y \vdash nat\ (s\ (s\ y))}}{even\ x \vdash nat\ x}$$

The infinite branch unfolds the inductive predicate *even* infinitely often on the left: valid!

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Coinductive case:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{step\ p\ \alpha\ q \vdash step\ p\ \alpha\ q} \quad \dfrac{\vdots}{\vdash sim\ q\ q}}{step\ p\ \alpha\ q \vdash step\ p\ \alpha\ q \wedge sim\ q\ q}}{\vdash \forall\alpha\forall q.\ step\ p\ \alpha\ q \supset \exists q'.\ step\ p\ \alpha\ q' \wedge sim\ q\ q'}}{\vdash sim\ p\ p}$$

The infinite branch unfolds the coinductive predicate *sim* infinitely often on the right: valid!

Circular & non-wellfounded proofs in the litterature

- **As verification device or for completeness arguments**: Complete deduction sytem giving algorithms for checking validity (Tableaux, sequent calculi), intermediate objects between syntax and semantics for modal μ-calculus (Kozen, Kaivola, Walukiewicz)

μ-calulus formula $\rightarrow$ Circular proof $\rightarrow$ Finite axiomatization

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- **But rarely as proof–program objects in themselves**:
  - develop such a proof-theoretical study, from a Curry-Howard perspective;
  - establish focalization and cut-elimination (prior works: cut-admissibility by Brotherston, additive fragment by Fortier & Santocanale)

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  - develop such a proof-theoretical study, from a Curry-Howard perspective;
  - establish focalization and cut-elimination (prior works: cut-admissibility by Brotherston, additive fragment by Fortier & Santocanale)
- Recently, development of numerous circular/cyclic proof systems (Afshari & Leigh, Das, Doumane & Pous, Cohen & Rowe, Tatsuta et al. etc.)

Non-Wellfounded Sequent Calculus

Consider your favourite logic $\mathscr{L}$ & add fixed points as in μMALL:

Pre-proofs are the trees **coinductively** generated by:

- $\mathscr{L}$ inference rules
- inference for μ, ν:

$$\frac{\Gamma, F[\mu X.F/X] \vdash \Delta}{\Gamma, \mu X.F \vdash \Delta}\,[\mu_l] \qquad \frac{\Gamma, F[\nu X.F/X] \vdash \Delta}{\Gamma, \nu X.F \vdash \Delta}\,[\nu_l]$$

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Circular (pre-)proofs: the regular fragment of infinite (pre-)proofs, ie finitely many sub-(pre)proofs.

Pre-proofs are unsound!! ***Need for a validity condition***

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X}\,[\mu]}{\vdash \mu X.X}\,[\mu] \qquad \dfrac{\dfrac{\vdots}{\vdash \nu X.X, F}\,[\nu]}{\vdash \nu X.X, F}\,[\nu]}{\vdash F}\,[\mathrm{Cut}]$$

μLL$^\infty$ Non-Wellfounded Sequent Calculus

Consider your favourite logic LL & add fixed points as in μMALL:

μLL$^\infty$ Pre-proofs are the trees **coinductively** generated by:

- LL inference rules
- inference for μ, ν:

$$\frac{\vdash F[\mu X.F/X], \Delta}{\vdash \mu X.F, \Delta}\ [\mu_r] \qquad \frac{\vdash F[\nu X.F/X], \Delta}{\vdash \nu X.F, \Delta}\ [\nu_r]$$

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μLL$^\infty$ Inferences

μLL$^\infty$ Inference Rules

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$$\frac{\vdash F, \Gamma}{\vdash ?F, \Gamma}\,[?\mathrm{d}] \qquad \frac{\vdash F, ?\Gamma}{\vdash !F, ?\Gamma}\,[!\mathrm{p}] \qquad \frac{\vdash \Gamma}{\vdash ?F, \Gamma}\,[?\mathrm{w}] \qquad \frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma}\,[?\mathrm{c}]$$

$$\frac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma}\,[\nu] \qquad \frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\,[\mu]$$

μLL$^\infty$ Inferences

μLL$^\infty$ Inference Rules (with ancestor relation)

$$\frac{}{\vdash F, F^\perp}\ [\mathrm{Ax}] \qquad \frac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta}\ [\mathrm{Cut}] \qquad \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ [\mathrm{X}]$$

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Fischer-Ladner subformulas

$FL(F)$ is the least set of formula occurrences such that:

- $F \in FL(F)$;
- $G_1 \star G_2 \in FL(F) \Rightarrow G_1, G_2 \in FL(F)$ for $\star \in \{\oplus, \&, \invamp, \otimes\}$;
- $\sigma X.B \in FL(F) \Rightarrow B[\sigma X.B/X] \in FL(F)$ for $\sigma \in \{\mu, \nu\}$;
- $mG \in FL(F) \Rightarrow G \in FL(F)$ for $m \in \{!, ?\}$.

Fact

$FL(F)$ is a finite set for any formula F.

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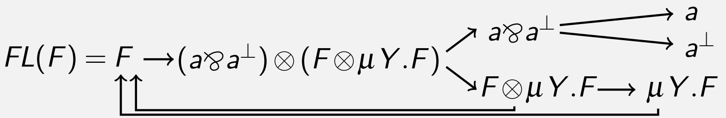
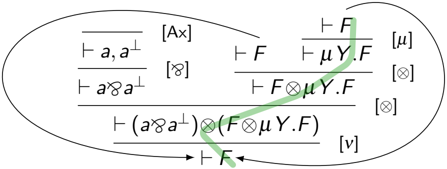


Infinite threads, validity

$F = \nu X.((a \parr a^\perp) \otimes (X \otimes \mu Y.X))$.



A ***thread*** on an infinite branch $(\Gamma_i)_{i\in\omega}$ is an infinite sequence of formula occurrences $(F_i)_{i\geq k}$ such that for any $i \geq k$, $F_i \in \Gamma_i$ and F_{i+1} is an immediate ancestor of F_i.

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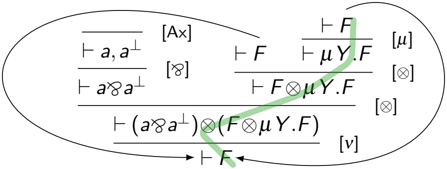

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A thread is valid if it unfolds infinitely many ν. More precisely, if the minimal ***recurring*** principal formula of the thread is a ν-formula.

A proof is valid if every infinite branch contains a valid thread.

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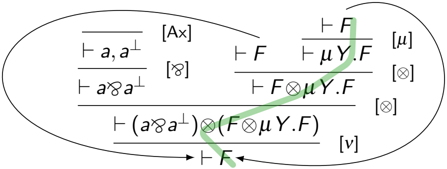

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Theorem (Nollet, Tasson & S, 2019)

Validity of $\mu\mathrm{LL}^{\omega}$ (circular) pre-proofs is PSPACE-complete. Details

Theorem (Baelde, Doumane & S, 2016)

$\mu\mathrm{MALL}^{\infty}$ is sound, and admits cut-elimination.

Examples of circular proofs

- **Inductive and coinductive definitions**

$$\mathsf{N} = \mu X.1 \oplus X \qquad\qquad \mathsf{S} = \nu X.(1 \& (N \otimes X))$$

- **Proofs-programs over these data types**

$$\begin{array}{lcll} double & : & N \to N & \\ double(n) & = & 0 & \text{if } n = 0 \\ & = & succ(succ(double(m))) & \text{if } n = succ(m) \end{array}$$

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$$\pi_0 = \cfrac{\cfrac{\overline{\vdash 1}\;[1]}{\vdash 1 \oplus N}\;[\oplus_1]}{\vdash N}\;[\mu]$$

$$\pi_{k+1} = \cfrac{\cfrac{\cfrac{\pi_k}{\vdash N}}{\vdash 1 \oplus N}\;[\oplus_2]}{\vdash N}\;[\mu]$$

$$\Pi_{double} \quad = \quad \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overline{\vdash 1}\;[1]}{\vdash 1 \oplus N}\;[\oplus_1]}{\vdash N}\;[\mu]}{1 \vdash N}\;[\bot] \qquad \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Pi_{double}}{N \vdash N}}{N \vdash 1 \oplus N}\;[\oplus_2]}{N \vdash N}\;[\mu]}{N \vdash 1 \oplus N}\;[\oplus_2]}{N \vdash N}\;[\mu]}{1 \oplus N \vdash N}\;[\&]}{N \vdash N}\;[\nu]$$

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$$\begin{array}{lcl} enum & : & N \to S \\ enum(n) & = & n :: enum(succ(n)) \end{array}$$

$$\pi_{\text{succ}} = \dfrac{\dfrac{\dfrac{}{N \vdash N}\,[\text{Ax}]}{N \vdash 1 \oplus N}\,[\oplus_2]}{N \vdash N}\,[\mu]$$

$$\Pi_{enum} = \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,[1]}{!N \vdash 1}\,[?\text{w}] \quad \dfrac{\dfrac{\dfrac{}{!N \vdash !N}\,[\text{Ax}] \quad \dfrac{\dfrac{\dfrac{\dfrac{\pi_{\text{succ}}}{N \vdash N}}{!N \vdash N}\,[?\text{d}]}{!N \vdash !N}\,[!\text{p}] \quad \dfrac{\Pi_{enum}}{!N \vdash S}}{!N \vdash S}\,[\text{Cut}]}{!N, !N \vdash N \otimes S}\,[\otimes]}{!N \vdash N \otimes S}\,[?\text{c}]}{!N \vdash 1 \& (N \otimes S)}\,[\&]}{!N \vdash S}\,[\nu]$$

Circular & finitary proofs

From finitary to circular proofs

Theorem

Finitary proofs can be transformed to (valid) circular proofs.

The key translation step is the following:

$$\dfrac{\dfrac{\pi_1}{\vdash \Gamma, S} \qquad \dfrac{\pi_2}{\vdash S^\perp, F[S]}}{\vdash \Gamma, \nu X.F}\,[\nu] \quad \longmapsto \quad \dfrac{\dfrac{[\pi_1]}{\vdash \Gamma, S} \qquad \dfrac{\dfrac{\dfrac{[\pi_2]}{\vdash S^\perp, F[S]} \qquad \dfrac{\vdash S^\perp, \nu X.F}{\vdash F[S]^\perp, F[\nu X.F]}\,[\mathsf{r}_F]}{\vdash S^\perp, F[\nu X.F]}\,[\mathsf{Cut}]}{\vdash S^\perp, \nu X.F}\,[\nu]}{\vdash \Gamma, \nu X.F}\,[\mathsf{Cut}]$$

From circular to finitary proofs

Open problem for $\mu\mathsf{LL}^\omega$.

$\mu MALL^{\infty}$ Cut elimination

μMALL$^\infty$ Cut Elimination Theorem

Theorem (Baelde, Doumane & S, 2016)

Fair μMALL$^\infty$ cut-reduction sequences converge to cut-free μMALL$^\infty$ proofs.

Previous result by Santocanale and Fortier
for the purely additive fragment of μLL$^\infty$.
Proof uses a locative treatment of occurrences.

- **Strategy:** "push" the cuts away from the root.
- **Cut-Cut:**

$$\dfrac{\dfrac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta, G}{\vdash \Gamma, \Delta, G}\,[\mathrm{Cut}] \qquad \vdash G^\perp, \Sigma}{\vdash \Gamma, \Delta, \Sigma}\,[\mathrm{Cut}] \quad \longleftrightarrow \quad \dfrac{\vdash \Gamma, F \qquad \dfrac{\vdash F^\perp, \Delta, G \qquad \vdash G^\perp, \Sigma}{\vdash F^\perp, \Delta, \Sigma}\,[\mathrm{Cut}]}{\vdash \Gamma, \Delta, \Sigma}\,[\mathrm{Cut}]$$

μMALL$^\infty$ Cut Elimination Theorem

Theorem (Baelde, Doumane & S, 2016)

Fair μMALL$^\infty$ mcut-reduction sequences converge to cut-free μMALL$^\infty$ proofs.

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Cut elimination procedure

External phase: Cut-commutation cases

$$\dfrac{\dfrac{\vdash \Delta, F, G}{\vdash \Delta, F \parr G}[\parr] \qquad \dots}{\vdash \Sigma, F \parr G}[\text{mcut}] \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F, G \qquad \dots}{\vdash \Sigma, F, G}[\text{mcut}]}{\vdash \Sigma, F \parr G}[\parr]$$

$$\dfrac{\dfrac{\vdash \Delta, F \qquad \vdash \Delta, G}{\vdash \Delta, F \& G}[\&] \qquad \dots}{\vdash \Sigma, F \& G}[\text{mcut}] \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F \qquad \dots}{\vdash \Sigma, F}[\text{mcut}] \qquad \dfrac{\vdash \Delta, G \qquad \dots}{\vdash \Sigma, G}[\text{mcut}]}{\vdash \Sigma, F \& G}[\&]$$

$$\dfrac{\dfrac{\vdash \Delta, F[\mu X.F/X]}{\vdash \Delta, \mu X.F}[\mu] \qquad \dots}{\vdash \Sigma, \mu X.F}[\text{mcut}] \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F[\mu X.F/X] \qquad \dots}{\vdash \Sigma, F[\mu X.F/X]}[\text{mcut}]}{\vdash \Sigma, \mu X.F}[\mu]$$

+ additional cases

Cut-commutation steps are productive

Cut elimination procedure

Internal Phase: Key cases

$$\dfrac{\dots \quad \dfrac{\vdash \Delta, F_2 \quad \vdash \Delta, F_1}{\vdash \Delta, F_2 \& F_1}[\&] \quad \dfrac{\vdash \Gamma, F_i^\perp}{\vdash \Gamma, F_1^\perp \oplus F_2^\perp}[\oplus_i]}{\vdash \Sigma}[\text{mcut}] \Rightarrow \dfrac{\dots \quad \vdash \Delta, F_i \quad \vdash \Gamma, F_i^\perp}{\vdash \Sigma}[\text{mcut}]$$

$$\dfrac{\dots \quad \dfrac{\vdash \Delta, F[\mu X.F/X]}{\vdash \Delta, \mu X.F}[\mu] \quad \dfrac{\vdash \Gamma, F^\perp[\nu X.F^\perp/X]}{\vdash \Gamma, \nu X.F^\perp}[\nu]}{\vdash \Sigma}[\text{mcut}]$$

$$\Rightarrow \dfrac{\dots \quad \vdash \Delta, F[\mu X.F/X] \quad \vdash \Gamma, F^\perp[\nu X.F^\perp/X]}{\vdash \Sigma}[\text{mcut}]$$

+ additional cases

Key cases are not productive

Cut elimination algorithm

- Internal phase: Perform key case reductions while you cannot do anything else.
- External phase: Build a part of the output tree by applying cut-commutation steps as soon possible.
- Repeat.

Cut elimination algorithm

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- Repeat.

Remark: We consider a **fair** strategy ie. every reduction which is available at some point will be performed eventually.

Theorem

more details

Internal phases always halt. Cut-elimination produces a pre-proof.

Theorem

more details

The pre-proof obtained by the cut elimination algorithm is valid.

μLL^ω is not stable by cut-elimination

Eliminating cuts from a μLL^ω proof (circular) may result in a μLL^∞, non circular, proof.

Cut-elimination for $\mu\mathsf{LL}^{\infty}$

Cut-elimination for μLL$^\infty$

Theorem

Fair μLL$^\infty$ mcut-reduction sequences converge to cut-free μLL$^\infty$ proofs.

Idea

The proof goes by:

- considering the following encoding of LL exponential modalities:

$$\begin{array}{rcl} ?^\bullet F & = & \mu X.F \oplus (\bot \oplus (X \parr X)) \\ !^\bullet F & = & \nu X.F \& (1 \& (X \otimes X)) \end{array}$$

- translating μLL$^\infty$ sequents and proofs in μMALL$^\infty$,
- simulating μLL$^\infty$ cut-reduction sequences in μMALL$^\infty$ and
- applying μMALL$^\infty$ cut-elimination theorem.

Encoding μLL^∞ in $\mu MALL^\infty$

$?^\bullet F = \mu X.F \oplus (\bot \oplus (X \parr X)) \qquad !^\bullet F = \nu X.F \& (1 \& (X \otimes X))$

$\mu MALL^\infty$ derivability of the exponential rules ($?d^\bullet, ?c^\bullet, ?w^\bullet, !p^\bullet$):

Dereliction :

$$\dfrac{\dfrac{\vdash F, \Delta}{\vdash F \oplus (\bot \oplus (?^\bullet F \parr ?^\bullet F)), \Delta}[\oplus_1]}{\vdash ?^\bullet F, \Delta}[\mu]$$

Contraction :

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash ?^\bullet F, ?^\bullet F \Delta}{\vdash ?^\bullet F \parr ?^\bullet F, \Delta}[\parr]}{\vdash \bot \oplus (?^\bullet F \parr ?^\bullet F), \Delta}[\oplus_2]}{\vdash F \oplus (\bot \oplus (?^\bullet F \parr ?^\bullet F)), \Delta}[\oplus_2]}{\vdash ?^\bullet F, \Delta}[\mu]$$

Weakening :

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Delta}{\vdash \bot, \Delta}[\bot]}{\vdash \bot \oplus (?^\bullet F \parr ?^\bullet F), \Delta}[\oplus_1]}{\vdash F \oplus (\bot \oplus (?^\bullet F \parr ?^\bullet F)), \Delta}[\oplus_2]}{\vdash ?^\bullet F, \Delta}[\mu]$$

Promotion:

$$\dfrac{\vdash F, ?^\bullet \Delta \qquad \dfrac{\dfrac{}{\vdash 1}[1]}{\vdash 1, ?^\bullet \Delta}[?w^\bullet] \qquad \dfrac{\dfrac{\dfrac{(\star)}{\vdash !^\bullet F, ?^\bullet \Delta} \quad \dfrac{(\star)}{\vdash !^\bullet F, ?^\bullet \Delta}}{\vdash !^\bullet F \otimes !^\bullet F, ?^\bullet \Delta, ?^\bullet \Delta}[\otimes]}{\vdash !^\bullet F \otimes !^\bullet F, ?^\bullet \Delta}[?c^\bullet]}{(\star) \vdash !^\bullet F, ?^\bullet \Delta}[\nu], [\&], [\&]$$

Preservation of validity

π is a valid μMLL^∞ pre-proof of $\vdash \Gamma$ iff
$\pi^\bullet$ is a valid $\mu MALL^\infty$ pre-proof of $\vdash \Gamma^\bullet$.

Simulation of $\mu\mathsf{LL}^\infty$ cut-elimination steps

$\mu\mathsf{LL}^\infty$ cut-elimination steps can be simulated by the previous encoding.

For instance, the following reduction can be simulated by applying the external reduction rule $[\mu]/[\mathsf{Cut}]$ followed by the external reduction rule $[\oplus]/[\mathsf{Cut}]$.

$$\dfrac{\dfrac{\vdash F, G, \Gamma}{\vdash ?^\bullet F, G, \Gamma}\,[?\mathsf{d}^\bullet] \qquad \vdash G^\perp, \Delta}{\vdash ?^\bullet F, \Gamma, \Delta}\,[\mathsf{Cut}] \longrightarrow^2 \dfrac{\dfrac{\vdash F, G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash F, \Gamma, \Delta}\,[\mathsf{Cut}]}{\vdash ?^\bullet F, \Gamma, \Delta}\,[?\mathsf{d}^\bullet]$$

Challenge: to show that the simulation of derivation also holds

(i) for the reductions involving $[!p]$ as well as
(ii) for reductions occurring ***above*** a promotion rule (aka. in a box) since the encoding of $[!p]$ uses an infinite, circular derivation.

Simulation of μLL$^\infty$ cut-elimination steps

External phase: Cut-commutation rules

$$\dfrac{\dfrac{\vdash F, G, \Gamma}{\vdash ?^\bullet F, G, \Gamma}\,[?\mathrm{d}^\bullet] \qquad \vdash G^\perp, \Delta}{\vdash ?^\bullet F, \Gamma, \Delta}\,[\mathrm{Cut}] \quad \longrightarrow^2 \quad \dfrac{\dfrac{\vdash F, G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash F, \Gamma, \Delta}\,[\mathrm{Cut}]}{\vdash ?^\bullet F, \Gamma, \Delta}\,[?\mathrm{d}^\bullet]$$

$$\dfrac{\dfrac{\vdash ?^\bullet F, ?^\bullet F, G, \Gamma}{\vdash ?^\bullet F, G, \Gamma}\,[?\mathrm{c}^\bullet] \qquad \vdash G^\perp, \Delta}{\vdash ?^\bullet F, \Gamma, \Delta}\,[\mathrm{Cut}] \quad \longrightarrow^3 \quad \dfrac{\dfrac{\vdash ?^\bullet F, ?^\bullet F, G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash ?^\bullet F, ?^\bullet F, \Gamma, \Delta}\,[\mathrm{Cut}]}{\vdash ?^\bullet F, \Gamma, \Delta}\,[?\mathrm{c}^\bullet]$$

$$\dfrac{\dfrac{\vdash G, \Gamma}{\vdash ?^\bullet F, G, \Gamma}\,[?\mathrm{w}^\bullet] \qquad \vdash G^\perp, \Delta}{\vdash ?^\bullet F, \Gamma, \Delta}\,[\mathrm{Cut}] \quad \longrightarrow^3 \quad \dfrac{\dfrac{\vdash G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash \Gamma, \Delta}\,[\mathrm{Cut}]}{\vdash ?^\bullet F, \Gamma, \Delta}\,[?\mathrm{w}^\bullet]$$

$$\dfrac{\dfrac{\vdash F, ?^\bullet G, ?^\bullet \Gamma}{\vdash !^\bullet F, ?^\bullet G, ?^\bullet \Gamma}\,[!\mathrm{p}^\bullet] \qquad \dfrac{\vdash G, ?^\bullet \Delta}{\vdash !^\bullet G^\perp, ?^\bullet \Delta}\,[!\mathrm{p}^\bullet]}{\vdash !^\bullet F, ?^\bullet \Gamma, ?^\bullet \Delta}\,[\mathrm{Cut}] \quad \longrightarrow^\omega \quad \dfrac{\dfrac{\vdash F, ?^\bullet G, ?^\bullet \Gamma \qquad \dfrac{\vdash G, ?^\bullet \Delta}{\vdash !^\bullet G^\perp, ?^\bullet \Delta}\,[!\mathrm{p}^\bullet]}{\vdash F, ?^\bullet \Gamma, ?^\bullet \Delta}\,[\mathrm{Cut}]}{\vdash !^\bullet F, ?^\bullet \Gamma, ?^\bullet \Delta}\,[!\mathrm{p}^\bullet]$$

Simulation of μLL$^\infty$ cut-elimination steps

Internal phase: Key-cut rules

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash F,\Gamma}}{\vdash ?^\bullet F,\Gamma}\,[?\mathrm{d}^\bullet] \qquad \dfrac{\dfrac{\pi'}{\vdash F^\perp, ?^\bullet\Delta}}{\vdash !^\bullet F^\perp, ?^\bullet \Delta}\,[!\mathrm{p}^\bullet]}{\vdash \Gamma, ?^\bullet\Delta}\,[\mathrm{Cut}] \longrightarrow^2 \dfrac{\dfrac{\pi}{\vdash F,\Gamma}\qquad\dfrac{\pi'}{\vdash F^\perp,?^\bullet\Delta}}{\vdash\Gamma,?^\bullet\Delta}\,[\mathrm{Cut}]$$

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash ?^\bullet F, ?^\bullet F,\Gamma}}{\vdash ?^\bullet F,\Gamma}\,[?\mathrm{c}^\bullet] \qquad \dfrac{\dfrac{\pi'}{\vdash F^\perp, ?^\bullet\Delta}}{\vdash !^\bullet F^\perp, ?^\bullet \Delta}\,[!\mathrm{p}^\bullet]}{\vdash \Gamma, ?^\bullet\Delta}\,[\mathrm{Cut}] \longrightarrow^{4int, 4\times\#\Delta ext}$$

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash ?^\bullet F, ?^\bullet F,\Gamma} \qquad \dfrac{\pi'}{\vdash !^\bullet F^\perp, ?^\bullet\Delta} \qquad \dfrac{\pi'}{\vdash !^\bullet F^\perp, ?^\bullet\Delta}}{\vdash \Gamma, ?^\bullet\Delta, ?^\bullet\Delta}\,[\mathrm{mcut}]}{\vdash \Gamma, ?^\bullet\Delta}\,[?\mathrm{c}^\bullet]^\star$$

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash \Gamma}}{\vdash ?^\bullet F,\Gamma}\,[?\mathrm{w}^\bullet] \qquad \dfrac{\dfrac{\pi'}{\vdash F^\perp, ?^\bullet\Delta}}{\vdash !^\bullet F^\perp, ?^\bullet \Delta}\,[!\mathrm{p}^\bullet]}{\vdash \Gamma, ?^\bullet\Delta}\,[\mathrm{Cut}] \longrightarrow^{3int, 3\times\#\Delta ext} \dfrac{\dfrac{\pi}{\vdash \Gamma}}{\vdash \Gamma, ?^\bullet\Delta}\,[?\mathrm{w}^\bullet]^\star$$

Cut-elimination for μLL$^\infty$

- Consider a fair cut-reduction sequence $\sigma = (\pi_i)_{i \in \omega}$ in μLL$^\infty$ from π.
- σ converges to a cut-free μLL$^\infty$ pre-proof. Otherwise, a suffix τ of σ would contain only key-cut steps. The encoding of τ in μMALL$^\infty$, $\tau^\bullet$, would be unproductive (contradicting productivity of cut-elimination).
- As σ is productive, it strongly converges to some μLL$^\infty$pre-proof π'.
- $\sigma^\bullet$ is therefore a transfinite reduction sequence from $\pi^\bullet$ strongly converging to $\pi'^\bullet$, cut-free (as it is the encoding of π').
- The compression lemma applies: there exists ρ an ω-indexed μMALL$^\infty$ cut-reduction sequence converging to $\pi'^\bullet$.
- By compression, fairness of $\sigma^\bullet$ transfers to ρ which is fair.
- Therefore, ρ has a limit, $\pi'^\bullet$ which is a valid cut-free μMALL$^\infty$ proof. $\pi'^\bullet$ is cut-free and valid and so is π', by the validity preservation property. □

Cut-elimination for μLK$^\infty$, μLJ$^\infty$

The usual call-by-value embedding of LJ in ILL (intuitionnistic LL) can be lifted to μLJ$^\infty$: indeed, the translation of proofs does not introduce cuts. For μLK$^\infty$, it is slightly trickier as the well-known T/Q-translations introduce cuts breaking validity. An alternative translation which does not introduce cuts can be used.

Moreover, one gets the skeleton of a μLL$^\infty$ (resp. μILL$^\infty$) proof which is a μLK$^\infty$ (resp. μLJ$^\infty$) proof, simply by erasing the exponentials (connectives and inferences), preserving validity.
The skeleton of a μLL$^\infty$ (resp. μILL$^\infty$) cut-reduction sequence is a μLK$^\infty$ (resp. μLJ$^\infty$) cut-reduction sequence. As a result, one has:

Theorem

If π is an μLK$^\infty$ (resp. μLJ$^\infty$) proof of $\vdash \Gamma$ (resp. $\Gamma \vdash F$), there exists a μLL$^\infty$ (resp. μILL$^\infty$) proof of the translated sequents.

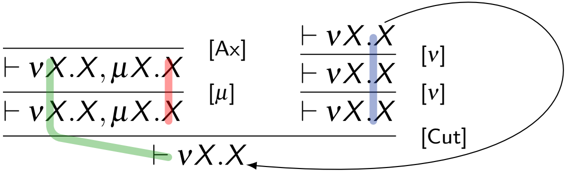
Theorem

There are productive cut-reduction strategies producing cut-free μLK$^\infty$ (resp. μLJ$^\infty$) proofs.

Bouncing validity

A invalid, though productive, proof with cut

Problem: Cuts are not well-managed by the validity condition.



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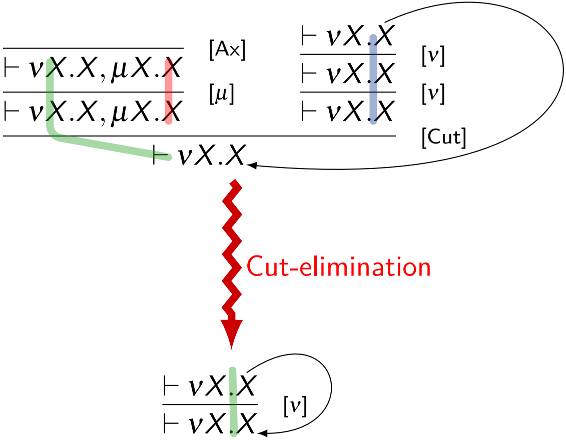
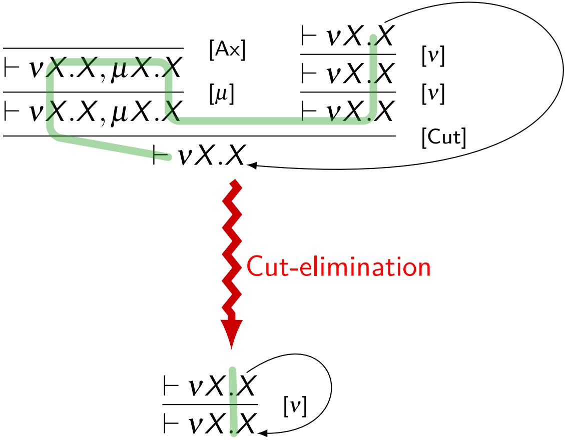


From now, we will refer to s-valid pre-proof for the previous validity condition and will consider alternative validity conditions.

A invalid, though productive, proof with cut

Problem: Cuts are not well-managed by the validity condition.



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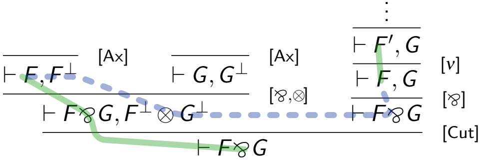
Bouncing threads: visible part

Visible part: survives the cut-elimination.

Hidden part: Must satisfy matching constaints.



Bouncing thread valid: ∞ ν-unfoldings in visible part.

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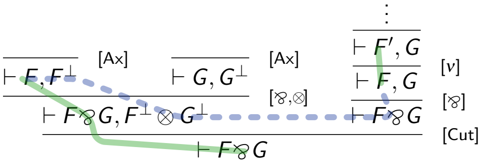

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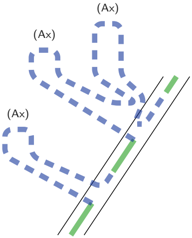
Valid branch B: exists a valid bouncing thread with visible part included in B.

B-valid proof: all infinite branches are valid.



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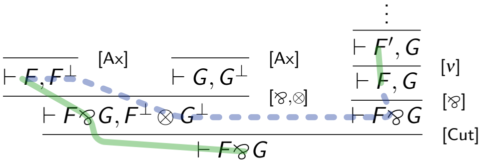

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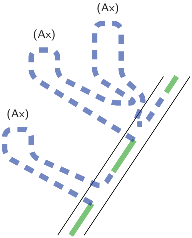

B-valid proof: all infinite branches are valid.

Theorem (Baelde, Doumane, Kuperberg & S)

Soundness and cut-elimination hold for μMALL$^\infty$ b-valid proofs.

Decidability of the bouncing validity condition ?

Given a **circular** proof, can we decide b-validity ?

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$\Longrightarrow$ Reduce termination of Minsky machines to bouncing validity.

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$\Longrightarrow$ Reduce termination of Minsky machines to bouncing validity.

A hierarchy of decidable conditions: Height of a b-thread: parameter binding the height of bounces.

$b(k)$-valid proof: b-valid proof using only threads of height $\leq k$.

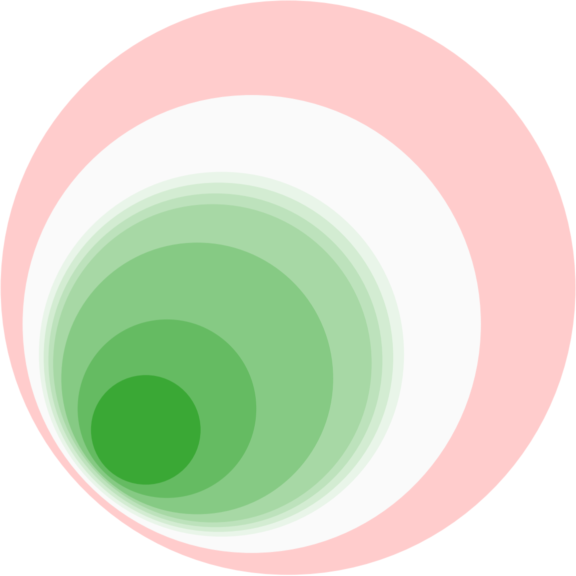
Theorem

Every b-valid circular proof is a $b(k)$-valid for some $k \in \mathbb{N}$.

Theorem

For all $k \in \mathbb{N}$, it is decidable whether a circular proof is a k-proof.

Hierarchy of validity criteria



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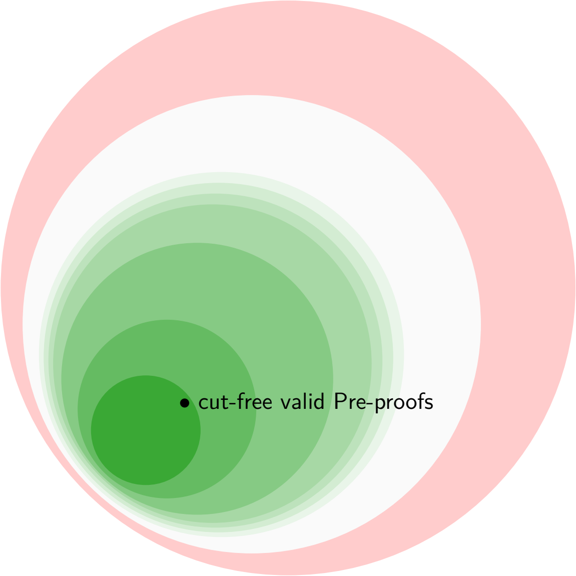

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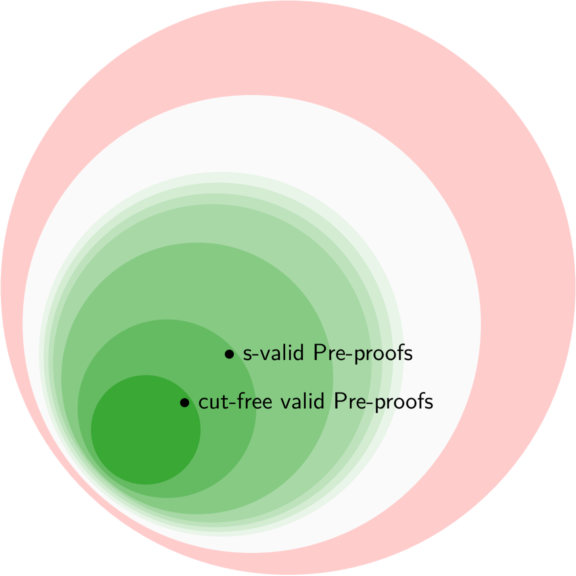

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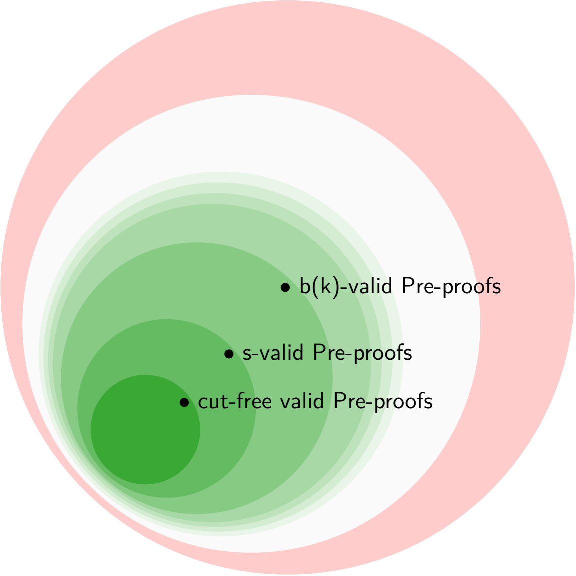

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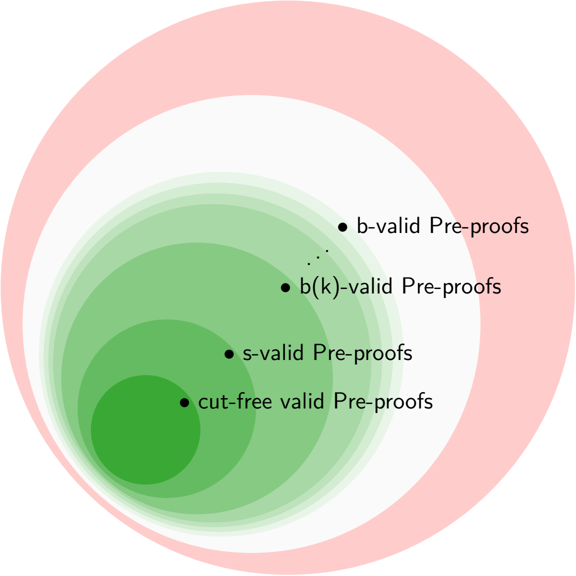

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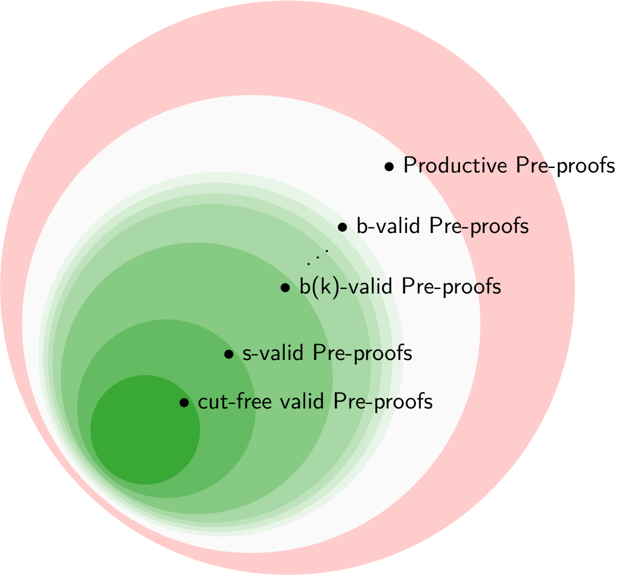

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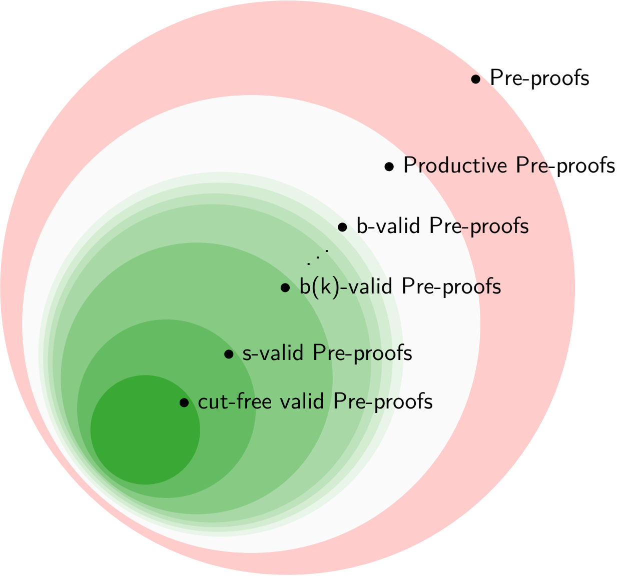
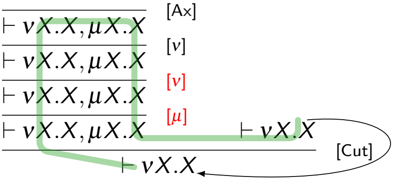
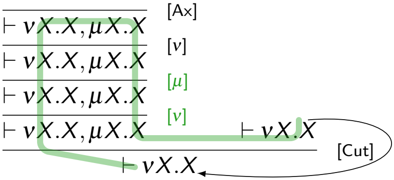


Sequentiality & parallelism in non-wellfounded proofs: proof-nets for μMLL$^{\infty}$

Mismatch between the parallel nature of threads and the sequential nature of sequent proofs.



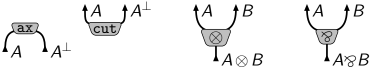
Non-productive cut-elimination

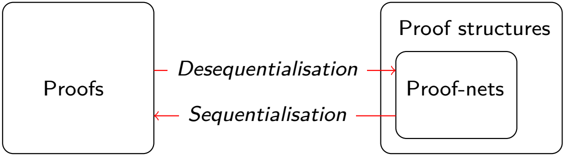


Productive cut-elimination

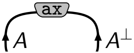
MLL proof-nets

An MLL proof structure is a directed finite graph composed of:





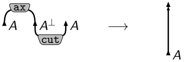
A proof structure that represents no sequent proof:

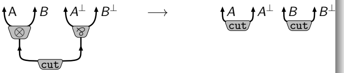


Canonicity

Two proofs are equivalent up to permutation of rules iff they have the same proof-net.

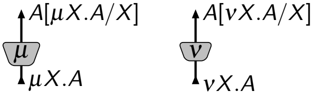
Confluent and terminating cut-elimination



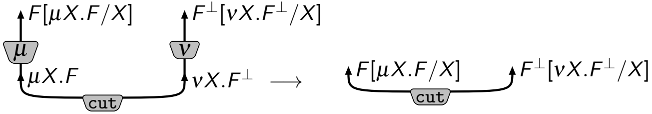


μMLL$^\infty$ proof structures

An MLL proof structure + the following decorated nodes:



New cut-elimination rules for new operators:



Is that enough?

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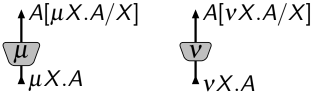

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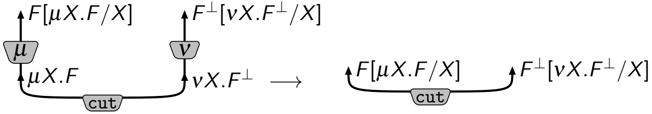


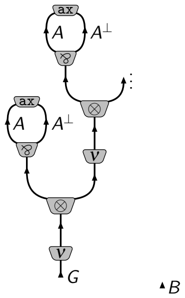
Is that enough? No! Need more structure & more reductions:

- Need to consider "infinite axioms" as invariants of infinite branches;
- Need to add visitable paths to infinite axioms, to prevent disconnectedness of the proof structure;
- cut-elimination shall be adapted to those infinite axioms.

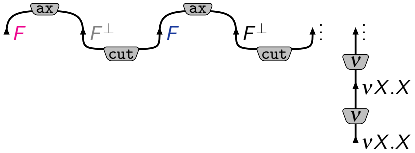
Infinite axioms and visitable paths

Let $G = \nu X.(A \parr A^\perp) \otimes X$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,[\text{Ax}]}{\vdash A \parr A^\perp}\,[\parr] \qquad \star \vdash G, B}{\vdash (A \parr A^\perp) \otimes G, B}\,[\otimes]}{\star \vdash G, B}\,[\nu]$$



$$\dfrac{\dfrac{}{\vdash F, F^\perp}\,[\text{Ax}] \qquad \dfrac{\dfrac{\star}{\vdash F, \nu X.X}}{\vdash F, \nu X.X}\,[\nu]}{\star \vdash F, \nu X.X}\,[\text{Cut}]$$



Infinite axioms are invariants of infinite branches in proofs. They may contain “visitable" sequences of axioms and cuts/tensors.

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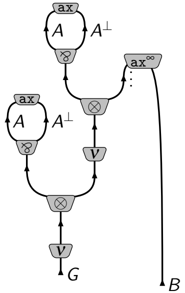
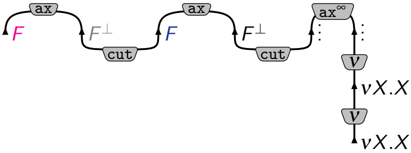


$$\dfrac{\dfrac{}{\vdash {\color{magenta}F}, {\color{gray}F^{\perp}}}\,[\mathrm{Ax}] \qquad \dfrac{\dfrac{\star}{\vdash {\color{blue}F}, \nu X.X}}{\vdash {\color{blue}F}, \nu X.X}\,[\nu]}{\star \vdash {\color{magenta}F}, \nu X.X}\,[\mathrm{Cut}]$$



Infinite axioms are invariants of infinite branches in proofs. They may contain “visitable" sequences of axioms and cuts/tensors.

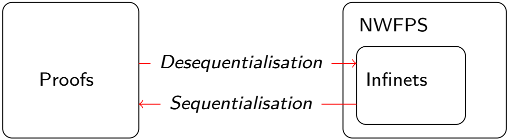
Non-wellfounded proof-structures

An NWFPS has the following components:

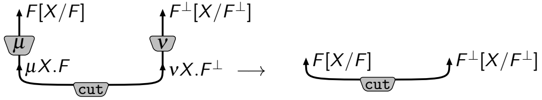
- Formulas $\{F_1, F_2, \ldots\}$ and their corresponding syntax trees $\{T_1, T_2, \ldots\}$
- Cuts of the form $(C, C^\perp)$ where $C = F_i$ and $C^\perp = F_j$.
- Axioms $(L, L^\perp)$ of leaves of some trees T_i, T_j.
- Visitable paths: infinite sequences of the form $APAPAP\ldots$ where A is an axiom and P is either a cut or a $\otimes$.
- Infinite axioms that contain leaves and visitable paths.

Correctness criterion

A correctness criterion ensures sequentialisation and cut-elimination.

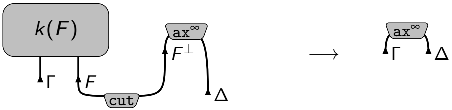


Infinet cut-elimination



But what about the cut/inf-ax case?

Consider $k(F)$ the smallest subnet with F as the conclusion (corresponding to the kingdom of F).



Theorem (De, Pellissier & S, 2021)

The limit of any sequence of (fair) reductions is a (cut-free) infinet.

Conclusion

Conclusion

- Fixed-point logics extending LL with finite circular or non-wellfounded proofs;
- Syntactic cut elimination for various nwf sequent calculi: $\mu\mathsf{MALL}^{\infty}$, $\mu\mathsf{LL}^{\infty}$, $\mu\mathsf{LJ}^{\infty}$, $\mu\mathsf{LK}^{\infty}$;
- More expressive validity condition;
- Proof-nets in the non-wellfounded multiplicative case.
- Ongoing and future work:
  - Equivalence of circular fragment of $\mu\mathsf{MALL}^{\infty}$ and $\mu\mathsf{MALL}$: Translate infinitary proofs to finitary ones. Same question as above by preserving the computational content.
  - Relax the conditions on bouncing threads retaining cut-elimination in infinets.
  - Design a good notion of ***circularity*** for infinets.
  - Extend to circular natural deduction and circular λ-calculus.
  - Provability and denotational semantics of circular proofs (jww De, Ehrhard and Jafarrahmani).

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Thank you for your attention!

Appendix

For any integer m, $\sqrt{m}$ is either an integer, or irrational.

Another example of infinite descent

another example

Proof

Let $m \in \mathbb{N}$ and for the sake of contradiction, assume $\sqrt{m} \in \mathbb{Q} \setminus \mathbb{N}$.

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another example

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1. Choose $q, a_0, b_0 \in \mathbb{N}$ st. $0 < \sqrt{m} - q < 1$ and $\sqrt{m} = a_0/b_0$.
   One has $b_0\sqrt{m} = a_0 \in \mathbb{N}$ and $a_0\sqrt{m} = mb_0 \in \mathbb{N}$.

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2. Therefore by setting $a_1 \triangleq m b_0 - a_0 q = a_0(\sqrt{m} - q)$ and $b_1 \triangleq a_0 - b_0 q = b_0(\sqrt{m} - q)$, we have
   - a_0, a_1 are integers,
   - $0 < a_1 < a_0, \quad 0 < b_1 < b_0$ and
   - $\sqrt{m} = a_1 / b_1$.

For any integer m, $\sqrt{m}$ is either an integer, or irrational.

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3. In a similar way, one can build $(a_i)_{i \in \mathbb{N}}$ and $(b_i)_{i \in \mathbb{N}}$ infinite sequences of integers, *which are strictly decreasing*.

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3. In a similar way, one can build $(a_i)_{i \in \mathbb{N}}$ and $(b_i)_{i \in \mathbb{N}}$ infinite sequences of integers, *which are strictly decreasing*.
4. This is impossible. Therefore $\sqrt{m}$ is either integer or irrational. □

Decidability of the validity condition

Parity automata

back to main slide

Definition

A *parity automaton* is a finite state word automaton,
whose states are ordered and given a parity bit ν/μ,
which accepts runs $(q_i)_{i\in\omega}$ such that $\min(\inf((q_i)_i))$ has parity ν.

Parity automata

back to main slide

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Remarks

- States are usually given a color in $\mathbb{N}$, equivalently.
- Only co-accessible states need to be ordered.

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Remarks

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- Only co-accessible states need to be ordered.

Properties

- PA can be determinized,
- PA are closed by complementation and intersection,
- The emptiness problem is decidable,
- (Thus) inclusion of parity automata is decidable.

Theorem: The validity of circular pre-proofs is decidable.

Proof.

Consider a pre-proof Π i.e. a graph with nodes $s_i = (F_i^j)_{j \in [1;n_i]}$.

The proof goes as follows:

- One builds a parity automaton recognizing the language $\mathscr{L}_B$ of infinite branches of Π;
- One builds a parity automaton recognizing the language $\mathscr{L}_T$ the valid branches of Π.
- Validity amounts to the inclusion of $\mathscr{L}_B$ in $\mathscr{L}_T$, that is showing that $\mathscr{L}_B \setminus \mathscr{L}_T = \emptyset$ which is decidable.

Branch automaton: Let $\mathscr{A}_B$ be the **branch automaton** with states s_i, transitions $s_i \to^k s_j$ when s_j is the k-th premise of s_i, and which accepts all runs.

(...)

Theorem: The validity of circular pre-proofs is decidable.

Proof.

Consider a pre-proof Π i.e. a graph with nodes $s_i = (F_i^j)_{j\in[1;n_i]}$.
(...)

Thread automaton: Let $\mathscr{A}_T$ be the **thread automaton** with states F_i^{j+}, F_i^{j-} or s_i, with transitions:

- $s_i \to^k s_p$ and $s_i \to^k F_p^{q-}$ when s_p is the k-th premise of s_i
- $F_i^{j+} \to^k F_p^{q\varepsilon} \quad (\varepsilon \in \{+,-\})$ when $s_i \to^k s_p$ and F_i^j is active in the rule of conclusion s_i and has ancestor F_p^q
- $F_i^{j-} \to^k F_p^{q\varepsilon} \quad (\varepsilon \in \{+,-\})$ when $s_i \to^k s_p$ and F_i^j is passive in the rule of conclusion s_i and has ancestor F_p^q

acceptance based on subformula ordering with the active/passive distinction: only active ν-formulas have coinductive parity.

Validity of Π equivalent to $\mathscr{L}(\mathscr{A}_B) \setminus \mathscr{L}(\mathscr{A}_T) = \emptyset$, thus decidable.

□

Cut elimination is productive

back to the statement

Theorem

Internal phase always halts.

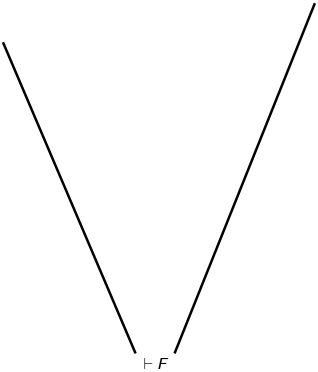
Cut elimination is productive

back to the statement

Theorem

Internal phase always halts.

Proof by contradiction: Suppose that there is a proof of F for which the internal phase does not halt.



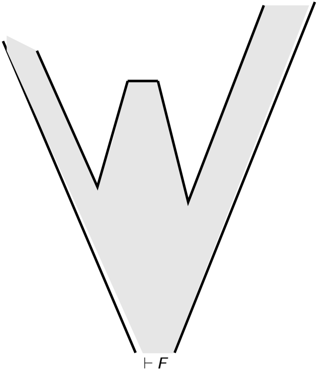
Cut elimination is productive

back to the statement

Theorem

Internal phase always halts.

Proof by contradiction: Consider the trace of this divergent reduction.



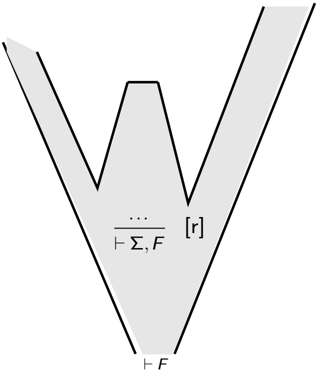
Cut elimination is productive

back to the statement

Theorem

Internal phase always halts.

Proof by contradiction: No rule on F is applied in the trace, otherwise the internal phase would halt.



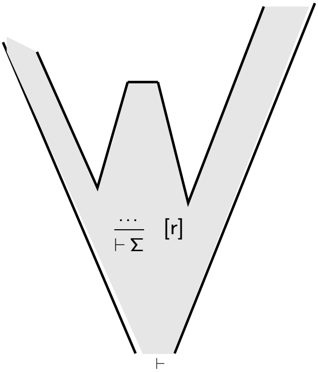
Cut elimination is productive

back to the statement

Theorem

Internal phase always halts.

Proof by contradiction: We can eliminate the occurrences of F from the trace. This yields a "proof" of $\vdash$.



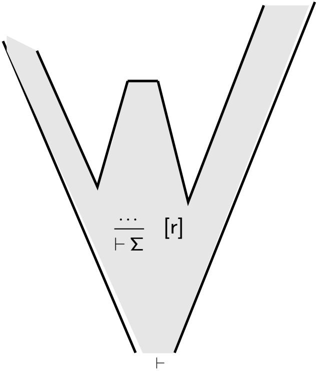
Cut elimination is productive

back to the statement

Theorem

Internal phase always halts.

Proof by contradiction: We show that the proof system is sound.
Contradiction.



Cut elimination is productive (Details)

back to the statement

Theorem

Internal phase always halts.

Cut elimination is productive (Details)

back to the statement

Theorem

Internal phase always halts.

Proof: Suppose that the internal phase diverges for a proof $\pi \vdash \Delta$.

- Let θ be the sub-derivation of π explored by the reduction.
- No rule is applied to a formula of Δ in θ, as this would contradict the divergence of internal phase.
- Let $\overline{\theta}$ be the proof obtained from θ by dropping all the formulas from Δ.
- $\overline{\theta}$ is then a proof for $\vdash$ in a proof system with "truncation".
- We define a truth semantics for $\mu MALL^{\infty}$ formulas and show soundness of the proof system with truncation *wrt.* it.
- Contradiction. □

Cut elimination produces a proof

back to the statement

Theorem

The pre-proof obtained by the cut elimination algorithm is valid.

Cut elimination produces a proof

back to the statement

Theorem

The pre-proof obtained by the cut elimination algorithm is valid.

Proof: Let $\pi^\star$ be the pre-proof obtained from $\pi \vdash \Delta$ by cut elimination. Suppose that a branch b of $\pi^\star$ is not valid.

- Let θ be the sub-derivation of π explored by the reduction that produces b.
- **Fact:** Threads of θ are the threads of b, together with threads starting from cut formulas.
- The validity of θ cannot rely on the threads of b.
- Define θ^μ to be θ where we replace in Δ any ν by a μ and any $1, \top$ by $\bot, 0$.
- Show that formulas containing only $\mu, \bot, 0$ and *MALL* binary connectives are false.
- θ^μ proves a false sequent which contradicts soundness.