# Interpolation as Cut-introduction 

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#### Abstract

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Analyzing Maehara's method for proving Craig's interpolation theorem, we extract a "proof relevant" interpolation theorem for first-order LL in the sense that if $\pi$ is a cut-free sequent proof of $A \vdash B$, we can find a formula $C$ in the common vocabulary of $A$ and $B$ and proofs $\pi_{1}, \pi_{2}$ of $A \vdash C$ and $C \vdash B$ respectively such that $\pi_{1}$ composed with $\pi_{2}$ cut-reduces to $\pi$. As a direct corollary, we get similar proof relevant interpolation results for LJ and LK using linear translations. This refined interpolation is then rephrased in terms of a cut-introduction process synthetizing the interpolant.

Finally, we analyze how to extend our methodology beyond the wellfounded setting, showing as a preliminary step, how to proof-relevantly interpolate $\mu \mathrm{LL}^{\infty}$ circular pre-proofs.

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## 1 "- Why Not a Proof-Relevant Interpolation Theorem? - Introduce Cuts, Of Course!"

In the words of Solomon Feferman, "though deceptively simple and plausible on the face of it, Craig's interpolation theorem (...) has proved to be a central logical property that has been used to reveal the deep harmony between the syntax and semantics of first order logic" [20]. Indeed, Craig's interpolation (which states that in the predicate calculus, if $A \vdash B$, there exists a formula $C$ built from the relation symbols occurring both in $A$ and $B$ such that $A \vdash C$ and $C \vdash B$ ) and its developments suggest far deeper connections between models and proofs that the simple correspondence between probability and validity given by Gödel completeness theorem. This could be argued to be in line with and pursue structural proof-theoretic proofs of Gödel completeness theorem such as Schütte proof [37] or the more recent analysis by Basaldella and Terui of completeness in Ludics [5, 4].

First of all, one should recall that while the original proof of interpolation by Craig [9, 10] was proof-theoretic as well as Maehara's method [27] its most striking applications were model-theoretic results from the early results that could be reproved from interpolation, such as Beth definability theorem [6] or Robinson's consistency theorem [33] to modern use of interpolation in model-checking [24, 29]. The success of Maehara's method is probably due to its applicability to a wide range of logics and proof-systems, from intuitionistic logic [30, 37] to modal logics $[21,26,1,38]$ or in infinitary logics and abstract model theory $[20,17]$.
Contributions and organization of the paper. While in most proof theory textbooks [23, 37, 40, 41] Craig's interpolation theorem is presented as an application of cut-elimination, one shall see here that it also has in fact much to do with cut-introduction, giving a proofrelevant content to Interpolation theorem. More precisely, we shall establish in Section 3 the following result for first-order LL: For any first-order LL formulas $A, B$, if $\pi$ proves $A \vdash B$, there exists a formula $C$ in the common vocabulary (that refers to the subset of the first-order language occurring in a formula, in terms of relation symbols) of $A$ and $B$ and proofs $\pi_{1}, \pi_{2}$ of $A \vdash C$ and $C \vdash B$ respectively such that $\pi_{1}$ composed with $\pi_{2}$ is equivalent to $\pi$ : (Cut) $\left(\pi_{1}, \pi_{2}\right)={ }_{\text {cut }} \pi$. Interpolation can therefore be achieved while preserving the computational / denotational content of proofs, while factoring the computation through an interfacing, interpolant type made only of the base types used in both the input and output

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types. This result is (easily) extended to classical and intuitionistic logics (thanks to linear translations) and sheds an interesting light on the relationship between Lyndon and Craig's interpolation.

We then consider two extensions. First, by a further analysis of Maehara's method, we show in Section 4 that the interpolation process for a cut-free proof deriving $A \vdash B$ can be in fact decomposed in two phases: (i) an ascending phase which equips each sequent of $\pi$ with a splitting is followed by (ii) a descending phase which solves the interpolation problem. This latter descending phase happens to be a cut-introduction phase providing an alternative proof of our proof-relevant interpolation result. The resulting proof is, by construction, denotationally equal to $\pi$. Finally, we consider in Section 5 the question of extending our approach to the $\mu$-calculus and non-wellfounded proofs, which is an interesting stress-test for our approach since Maehara's method strongly relies on wellfoundedness of the cut-free proof to interpolate. In that setting, we show that the method extends smoothly to circular pre-proofs and that, maybe surprisingly, the construction can even be achieved mostly neglecting the validity condition for non-wellfounded proofs.

Related works. Surprisingly we could not find any occurrence in the literature analyzing Maehara's method in terms of cut-elimination (or rather, cut-introduction) even though all ingredients were there since Maehara's seminal work.

On the other hand, another early proof-theoretic proof of interpolation theorem was proposed by Prawitz for natural deduction [31]. Just like for Maehara's method the strengthened, proof-relevant interpolation result was at hand in this work as well and Čubrić actually showed this in the setting of the simply typed $\lambda$-calculus as well as a corresponding factorization result for bicartesian closed categories in the early 90s [11, 12]. Sadly, Čubrić's paper as well as his PhD thesis supervised by Makkai, received too little attention and very few following works refer to his results: we could only find less than 10 references to these works among them only three truly consider the interpolation aspect $[18,28,25]^{1}$. We hope that the present work can contribute to foster interest in Čubrić results.

Another related work is that of Carbone [8] where she establishes a strengthened form of Maehara's interpolation paying a great attention to the ancestor relation (formulated in terms of flow graphs in that work) which allows her to get bounds on the complexity of the interpolant but did not led her to a study of proof-relevant interpolation, invariance by cut-elimination nor interpolation as cut-introduction. Only few works consider interpolation in (fragments of) linear logic, starting with Roorda [34]. In the framework of the calculus of structure, Strassburger proves a decomposition theorem for MELL [39] that is advocated to correspond to an interpolation theorem and may have a more fine-grained proof-theoretical content. More recently, several papers investigated and formalized interpolation theorems in substructural logics, including exponential-free linear logic [7, 16, 32].

## 2 Background on LL and $\mu \mathrm{LL}^{\infty}$ proof theory

In the following, we provide the necessary background on first-order LL as well as for its circular and non-wellfounded extensions with least and greatest fixed-points, $\mu \mathrm{LL}{ }^{\infty}$.

[^0]As usual, we assume a first-order language $\mathcal{L}$ (without equality) as well as a countable set of fixed-point variables $\mathcal{V}$ (ranged over by $X, Y, Z, \ldots$ ). We introduce a language of first-order $\operatorname{LL}$ formulas with least and greatest fixed points, called $\mu \mathrm{LL}$ pre-formulas:

- Definition 1. The grammar of first-order $\mu \mathrm{LL}$ pre-formulas is defined inductively as:

| $F$ | $::=$ | $a$ | $\top$ | $\mid \perp$ | $\mid F \diamond F$ | $\mid F \& F$ | $\mid \forall x . F$ | negative $M A L L$ formulas |
| ---: | :---: | :---: | :---: | :---: | :---: | :--- | :--- | ---: |
|  | $\mid$ | $a^{\perp}$ | 0 | 1 | $\mid F \otimes F$ | $\mid F \oplus F$ | $\mid \exists x . F$ | positive $M A L L$ formulas |

An LL formula is a pre-formula built using only the first two line of the above grammar. A $\mu \mathrm{LL}$ formula is a pre-formula containing no free fixed-point variable.

Negation is defined as usual as an involution on LL (resp. $\mu \mathrm{LL}$ ) formulas. Notice that negation does not change fixed-points variables which are really just used for the binding structure of the fixed-point definition: $(\mu X . F)^{\perp}=\nu X . F^{\perp}$ and $X^{\perp}=X$.

- Definition 2 (LL \& $\mu \mathrm{LL}^{\infty}$ sequent calculi). The inference of LL and $\mu \mathrm{LL}{ }^{\infty}$ sequent calculi are given in Figure 1 considering inferences in (a-c) for first-order LL sequent calculus, inferences in (a-d) for first-order $\mu \mathrm{LL}^{\infty}$ sequent calculus and inferences in (a-b) $+(d)$ for propositional $\mu \mathrm{LL}{ }^{\infty}$ sequent calculus. The inferences of Figure 1 show a relation between conclusion and premises formulas, the ancestor relation (or sub-occurrence relation) that is extended by transitivity to formulas of non consecutive sequents of a derivation.
- Remark 3. The ancestor relation defined above is (implicitly) used in designing a cutreduction system and plays a crucial role in expressing the validity condition for nonwellfounded and circular proofs as well as in the extension of Maehara's method we will show next, in order to propagate sequent splittings from conclusions to premises.

Note that some $\mu \mathrm{LL}^{\infty}$ sequent calculi [3, 2] adopt a locative approach: instead of ordered lists of formulas, sequents are lists of occurrences of formula, which are pairs of a formula and an address; the sub-occurrence relation then coincides with the sub-address relation.

- Definition 4 ( $\mu \mathrm{LL}^{\infty}$ (circular) pre-proof). $A \mu \mathrm{LL}^{\infty}$ pre-proof is a possibly infinite tree built from $\mu \mathrm{LL}{ }^{\infty}$ sequent calculus inferences, that is a finitely branching, possibly non-wellfounded derivation tree. A $\mu \mathrm{LL}{ }^{\infty}$ pre-proof having finitely many distinct subtrees is call circular. (They are the pre-proof having regular infinite trees.)
- Remark 5. Circular pre-proofs admit finite representations in the form of finite trees with back-edges, for more details on such finite representations we refer to [19]. To such a finite tree with back-edges, one associates its infinite unfolding, uniquely defined by guardedness of the back-edges. They typically have the structure depicted in Figure 2.

As is well known, such non-wellfounded or circular derivations shall be tamed to ensure logical soundness. Indeed, one can trivially derive circularly any formula $F$ as in Figure 3.

The validity condition is expressed $[3,2,19,36]$ as a condition requiring that every infinite branch contains we can choose formulas in consecutive sequents and form a sequence of ancestor-related formulas such that, (i) those formulas are infinitely often principal in their sequent (ie. their main connective is introduced) and (ii) among the formulas that occur infinitely often, there exists a minimal one which is a $\nu$ formula (in two-sided sequent calculi, we ha the possibility that the minimal formula is a $\mu$-formula on the left of the sequent. Such validity conditions can be refined when considering finite representations, such as with the strong validity, or invariance, condition [19]. We do not detail more as most of the paper will neglect validity when dealing with non-wellfounded and circular derivations.


Figure 1 (a) Propositional MALL Inferences; (b) LL Exponential Inferences; (c) First-order Inferences; (d) Fixed-point Inferences - explicitly depicting the ancestor relation.

## 3 Proof-relevant interpolation theorem

- Theorem 6. Let $\Gamma, \Delta$ be lists of LL formulas and $\pi \vdash \Gamma, \Delta$. There exists a LL formula $C$ such that $\mathcal{L}(C) \subseteq \mathcal{L}(\Gamma) \cap \mathcal{L}(\Delta)$ and two cut-free proofs $\pi_{1}, \pi_{2}$ of $\vdash \Gamma, C$ and $\vdash C^{\perp}, \Delta$ respectively such that $\frac{\frac{\pi_{1}}{\vdash \Gamma, C} \frac{\pi_{2}}{\vdash C^{\perp}, \Delta}}{\vdash \Gamma, \Delta}$ (Cut) $^{\vdash \text { cut }} \pi$.

In fact, we shall prove a refined version for cut-free proofs from which Theorem 6 follows directly by LL cut-elimination theorem:

- Theorem 7. Let $\Gamma, \Delta$ be lists of LL formulas and $\pi \vdash \Gamma, \Delta$ be cut-free. There exists a LL formula $C$ such that $\mathcal{L}(C) \subseteq \mathcal{L}(\Gamma) \cap \mathcal{L}(\Delta)$ and two cut-free proofs $\pi_{1}, \pi_{2}$ of $\vdash \Gamma, C$ and $\vdash C^{\perp}, \Delta$ respectively such that $\frac{\frac{\pi_{1}}{\vdash \Gamma, C} \frac{\pi_{2}}{\vdash C^{\perp}, \Delta}}{\vdash \Gamma, \Delta}(\mathrm{Cut}) \longrightarrow_{\text {cut }}^{\star} \pi$.

We prove the theorem by induction on the structure of $\pi$ and by case on the last inference. We only detail few significant cases below, the remaining cases can be found in Appendix A.

The proof can precisely be viewed as Maehara's method for which we are precise and explicit about the proof built and the relation ensured by cut-elimination, so that we preserve the denotational equivalence of the interpolated proof with the proof being constructed.

Proof sketch. Let $\pi$ be a cut-free proof as in the theorem statement and let us prove the theorem by induction on the structure of $\pi$ and by case on the last inference.
If $\pi=\overline{\vdash \boldsymbol{F}, \boldsymbol{F}^{\perp}}{ }^{(\mathrm{Ax})}, \Gamma=F$, one simply takes $C=F^{\perp}, \pi_{1}=\pi_{2}=\overline{\vdash F, F^{\perp}} \quad$ (Ax) . (the case when $\Gamma=F^{\perp}$ is symmetrical, taking $C=F$.) A cut between $\pi_{1}$ and $\pi_{2}$ simply reduces to $\pi$ by a cut-axiom reduction case.
If $\pi=\overline{\vdash \boldsymbol{F}, \boldsymbol{F}^{\perp}}{ }^{(\mathbf{A x})}, \Gamma=F, F^{\perp}$, one simply takes $C=\perp, \pi_{1}=\frac{\pi}{\vdash \Gamma, \perp}(\perp)$ and $\pi_{2}=$
$\overline{\vdash 1}{ }^{(1)}$. (the case when $\Gamma$ is empty is symmetrical, taking $C=1$.) Again, the cut of $\pi_{1}$ and $\pi_{2}$ reduces to $\pi$ by a key $1 / \perp$ case.
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Figure 2 General structure of a finite representation.

$\square$ Figure 3 Example of an unsound pre-proof

If the last rule is $(\otimes)$, that is $\pi=\frac{\frac{\pi^{\prime}}{\vdash F, \Gamma^{\prime}, \Delta^{\prime}} \frac{\pi^{\prime \prime}}{\vdash F, G, \Gamma^{\prime \prime}, \Delta^{\prime \prime}}}{\vdash\left(\otimes, \Gamma^{\prime}, \Gamma^{\prime \prime}, \Delta^{\prime}, \Delta^{\prime \prime}\right.} \quad$, assuming $\Gamma=F \otimes G, \Gamma^{\prime}, \Gamma^{\prime \prime}$. By induction hypothesis, there are interpolants $C^{\prime}, C^{\prime \prime}$, as well as interpolating proofs (i) $\pi_{1}^{\prime} \vdash F, \Gamma^{\prime}, C^{\prime}$, (ii) $\pi_{2}^{\prime} \vdash C^{\prime}, \Delta^{\prime}$, (iii) $\pi_{1}^{\prime \prime} \vdash G, \Gamma^{\prime \prime}, C^{\prime \prime}$ and (iv) $\pi_{2}^{\prime \prime} \vdash C^{\prime \prime \perp}, \Delta^{\prime \prime}$ such that $\frac{\pi_{1}^{\prime} \pi_{2}^{\prime}}{\vdash F, \Gamma^{\prime}, \Delta^{\prime}}($ Cut $) \longrightarrow_{\text {cut }}^{\star} \pi^{\prime}$ and $\frac{\pi_{1}^{\prime \prime} \pi_{2}^{\prime \prime}}{\vdash G, \Gamma^{\prime \prime}, \Delta^{\prime \prime}}($ Cut $) \longrightarrow_{\text {cut }}^{\star} \pi^{\prime}$. Let $C=C^{\prime} 8 C^{\prime \prime}$, $\pi_{1}=\frac{\pi_{1}^{\prime} \pi_{1}^{\prime \prime}}{\vdash F \otimes G, \Gamma^{\prime}, \Gamma^{\prime \prime}, C^{\prime}, C^{\prime \prime}} \stackrel{(\otimes)}{\vdash F \otimes G, \Gamma^{\prime}, \Gamma^{\prime \prime}, C^{\prime} 8 C^{\prime \prime}} \quad$ (8) $\quad$ and $\pi_{2}=\frac{\pi_{2}^{\prime} \pi_{2}^{\prime \prime}}{\vdash C^{\prime \perp} \otimes C^{\prime \prime \perp}, \Delta^{\prime}, \Delta^{\prime \prime}} \quad(\otimes)$. We have the reduction of the box labelled $(\otimes)$ in Figure 4 which cut reduces to $\pi$ by IH.
If the last rule is (४), that is $\pi=\frac{\frac{\pi^{\prime}}{\vdash F, G, \Gamma^{\prime}, \Delta}}{\vdash F \gamma G, \Gamma^{\prime}, \Delta}$ (४) , assuming $\Gamma=F \ngtr G, \Gamma^{\prime}$. By IH, there is an interpolant $C^{\prime}$ such that $\mathcal{L}\left(C^{\prime}\right) \subseteq \mathcal{L}\left(F, G, \Gamma^{\prime}\right) \cap \mathcal{L}(\Delta)$ as well as proofs $\pi_{1}^{\prime} \vdash F, G, \Gamma^{\prime}, C^{\prime}$ and $\pi_{2}^{\prime} \vdash C^{\prime \perp}, \Delta$ such that $\frac{\pi_{1}^{\prime}}{\frac{\vdash F, G, \Gamma^{\prime}, C^{\prime}}{\vdash F, G, \Gamma^{\prime}, \Delta^{\prime}} \frac{\pi_{2}^{\prime}}{\vdash C^{\prime}, \Delta^{\prime}}}(\mathrm{Cut}) \longrightarrow_{\mathrm{cut}}^{\star} \pi^{\prime}$.
Setting $C=C^{\prime}, \pi_{1}=\frac{\pi_{1}^{\prime}}{\vdash F 8 G, \Gamma^{\prime}, C}$ (8) and $\pi_{2}=\pi_{2}^{\prime}$ we get the reduction of the box labelled (४) in Figure 4 which cut reduces to $\pi$ by induction hypothesis.
If the last rule is $\left(\oplus^{\mathrm{i}}\right)(i \in\{1,2\})$, that is $\pi=\frac{\frac{\pi^{\prime}}{\vdash F_{i}, \Gamma^{\prime}, \Delta}}{\vdash F_{1} \oplus F_{2}, \Gamma^{\prime}, \Delta} \quad$ ( $\oplus^{\mathrm{i})}$, assuming $\Gamma=F_{1} \oplus$ $F_{2}, \Gamma^{\prime}$. By IH, there is an interpolant $C^{\prime}$ such that $\mathcal{L}\left(C^{\prime}\right) \subseteq \mathcal{L}\left(F_{i}, \Gamma^{\prime}\right) \cap \mathcal{L}(\Delta)$ as well as proofs $\pi_{1}^{\prime} \vdash F_{i}, \Gamma^{\prime}, C^{\prime}$ and $\pi_{2}^{\prime} \vdash C^{\prime \perp}, \Delta$ such that $\frac{\pi_{1}^{\prime}}{\frac{\vdash F_{i}, \Gamma^{\prime}, C^{\prime}}{\vdash F_{i}, \Gamma^{\prime}, \Delta^{\prime}} \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta^{\prime}}}$ (Cut) $\longrightarrow_{\mathrm{cut}}^{\star} \pi^{\prime}$. Setting $C=C^{\prime}, \pi_{1}=\frac{\pi_{1}^{\prime}}{\vdash F_{1} \oplus F_{2}, \Gamma^{\prime}, C} \quad\left(\oplus^{\prime}\right)$ and $\pi_{2}=\pi_{2}^{\prime}$ we get the following cut-

| $(\otimes) \frac{\pi_{1}}{\frac{\vdash F \otimes G, \Gamma^{\prime}, C}{\vdash F \otimes G, \Gamma^{\prime}, \Delta^{\prime}, \Delta^{\prime \prime}} \quad \Delta^{\prime}, \Delta^{\prime \prime}}($ (Cut) | $\longrightarrow_{\text {cut }}$ |  | $\underline{\pi}_{2}^{\prime}$ |
| :---: | :---: | :---: | :---: |
| (४) $\frac{\pi_{1}}{\frac{\vdash F \gamma G, \Gamma^{\prime}, C}{\vdash F \ngtr G, \Gamma^{\prime}, \Delta^{\prime}} \frac{\pi_{2}}{\vdash C^{\perp}, \Delta^{\prime}}}$ (Cut) | $\longrightarrow_{\mathrm{cut}}$ | $\begin{array}{cc} \frac{\pi_{1}^{\prime}}{\vdash F, G, \Gamma^{\prime}, C^{\prime}} & \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta^{\prime}} \\ \frac{\vdash F, G, \Gamma^{\prime}, \Delta^{\prime}}{\vdash F \diamond G, \Gamma^{\prime}, \Delta^{\prime}} \end{array}$ |  |
| $\left(\oplus^{\mathrm{i}}\right) \frac{\pi_{1}}{\frac{\vdash F_{1} \oplus F_{2}, \Gamma^{\prime}, C}{\vdash F_{1} \oplus F_{2}, \Gamma^{\prime}, \Delta^{\prime}} \frac{\pi_{2}}{\vdash C^{\perp}, \Delta^{\prime}}} \text { (Cut) }$ | $\longrightarrow_{\text {cut }}$ | $\frac{\frac{\pi_{1}^{\prime}}{\vdash F_{i}, \Gamma^{\prime}, C^{\prime}} \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta^{\prime}}}{\frac{\vdash F, G, \Gamma^{\prime}, \Delta^{\prime}}{\vdash F_{1} \oplus F_{2}, \Gamma^{\prime}, \Delta^{\prime}}}\left(\oplus^{\mathrm{I}}\right)$ |  |
| (?d) $\frac{\frac{\pi_{1}}{\vdash ? F, \Gamma^{\prime}, C} \frac{\pi_{2}}{\vdash C^{\perp}, \Delta^{\prime}}}{\vdash ? F, \Gamma^{\prime}, \Delta^{\prime}}$ (Cut) | $\longrightarrow_{\text {cut }}$ | $\frac{\frac{\pi_{1}^{\prime}}{\vdash F, \Gamma^{\prime}, C^{\prime}} \quad \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta^{\prime}}}{\frac{\vdash F, \Gamma^{\prime}, \Delta^{\prime}}{\vdash ? F, \Gamma^{\prime}, \Delta^{\prime}}} \text { (?(Cut)}$ |  |
| (!p) $\frac{\pi_{1}}{\frac{\pi_{2}}{\vdash!F, ? \Gamma^{\prime}, C} \quad \frac{\pi^{\perp}}{\vdash!F, ? \Gamma^{\prime}, ? \Delta^{\prime}}}$ (Cut) | $\longrightarrow_{\mathrm{cut}}$ |  |  |
| ( $\forall$ ) $\frac{\pi_{1} \pi_{2}}{\vdash \forall x F, \Gamma^{\prime}, \Delta^{\prime}}$ (Cut) | $\longrightarrow_{\mathrm{cut}}$ $\longrightarrow_{\text {cut }}$ | $\begin{aligned} & \left.\frac{\pi_{1}^{\prime}}{\frac{\vdash F, \Gamma^{\prime}, \exists x \cdot C^{\prime}}{}{ }^{\frac{\vdash F, \Gamma^{\prime}, \Delta^{\prime}}{\vdash \forall x C^{\prime \perp}, \Delta^{\prime}}}}{ }^{\text {(Cut) }} \text { ( } \text { ( }\right) \\ & \frac{\pi_{1}^{\prime} \quad \pi_{2}^{\prime}}{\vdash \forall x F, \Gamma^{\prime}, \Delta^{\prime}} \\ & \frac{\text { (Cut) }}{\vdash F, \Gamma^{\prime}, \Delta^{\prime}} \end{aligned}$ |  |
| $(\sigma) \in\{(\mu),(\nu)\} \frac{\pi_{1}}{\frac{\vdash \sigma X \cdot F, \Gamma^{\prime}, C}{\vdash \sigma X . F, \Gamma^{\prime}, \Delta^{\prime}} \frac{\pi_{2}}{\vdash C^{\perp}, \Delta^{\prime}}} \text { (Cut) }$ | $\longrightarrow_{\text {cut }}$ | $\frac{\pi_{1}^{\prime}}{\frac{\pi_{2}^{\prime}}{\vdash F[\sigma X . F / X], \Gamma^{\prime}, C^{\prime}}} \frac{\stackrel{\pi^{\prime}}{\vdash C^{\prime \perp}, \Delta^{\prime}}}{\frac{\vdash F[\sigma X . F / X], \Gamma^{\prime}, \Delta^{\prime}}{\vdash \sigma X . F, \Gamma^{\prime}, \Delta^{\prime}}} \text { (८) }$ |  |

Figure 4 Cases of cut-reduction for the proof-relevant interpolation proof (Theorem 7). (The last line is used in Section 5 only.)
reduction starting with a cut-commutation case in the box labelled ( $\oplus^{i}$ ) in Figure 4 which cut reduces to $\pi$ by induction hypothesis.
If the last rule is (?d), that is $\pi=\frac{\frac{\pi^{\prime}}{\vdash F, \Gamma^{\prime}, \Delta}}{\vdash ? F, \Gamma^{\prime}, \Delta} \quad$ (?d) $\quad$ assuming $\Gamma=? F, \Gamma^{\prime}$. By induction hypothesis, there is an interpolant $C^{\prime}$ such that $\mathcal{L}\left(C^{\prime}\right) \subseteq \mathcal{L}\left(F, \Gamma^{\prime}\right) \cap \mathcal{L}(\Delta)$ as well as proofs $\pi_{1}^{\prime} \vdash F, \Gamma^{\prime}, C^{\prime}$ and $\pi_{2}^{\prime} \vdash C^{\prime \perp}, \Delta$ such that $\frac{\pi_{1}^{\prime}}{\frac{\vdash F, \Gamma^{\prime}, C^{\prime}}{\vdash F, \Gamma^{\prime}, \Delta} \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta}}\left(\right.$ (Cut) $\longrightarrow_{\text {cut }}^{\star} \pi^{\prime}$. By setting $C=C^{\prime}, \pi_{1}=\frac{\pi_{1}^{\prime}}{\vdash ? F, \Gamma^{\prime}, C^{\prime}} \quad$ (?d), one gets the reduction of the box labelled (?d) in Figure 4 which cut reduces to $\pi$ by induction hypothesis.
If the last rule is (!p), that is $\pi=\frac{\frac{\pi^{\prime}}{\vdash F, ? \Gamma^{\prime}, ? \Delta^{\prime}}}{\vdash!F, ? \Gamma^{\prime}, ? \Delta^{\prime}} \quad$ (!p) $\quad$ assuming $\Gamma=!F, ? \Gamma^{\prime}$ and $\Delta=? \Delta^{\prime}$. By IH, there is an interpolant $C^{\prime}$ such that $\mathcal{L}\left(C^{\prime}\right) \subseteq \mathcal{L}\left(F, ? \Gamma^{\prime}\right) \cap \mathcal{L}(\Delta)$ as well as proofs $\pi_{1}^{\prime} \vdash F, ? \Gamma^{\prime}, C^{\prime}$ and $\pi_{2}^{\prime} \vdash C^{\prime \perp}, ? \Delta^{\prime}$ such that $\frac{\pi_{1}^{\prime}}{\frac{\vdash F, ? \Gamma^{\prime}, C^{\prime}}{\vdash F, ? \Gamma^{\prime}, ? \Delta^{\prime}} \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, ? \Delta^{\prime}}}\left(\right.$ (Cut) $\longrightarrow_{\text {cut }}^{\star} \pi^{\prime}$.
By setting $C=? C^{\prime}, \quad \pi_{1}=\frac{\pi_{1}^{\prime}}{\frac{\vdash F, ? \Gamma^{\prime}, ? C^{\prime}}{\vdash!F, ? \Gamma^{\prime}, ? C^{\prime}}}{ }^{(? \mathrm{~d})} \quad$ (!p) $\quad$ and $\pi_{2}=\frac{\pi_{2}^{\prime}}{\vdash!C^{\prime \perp}, ? \Delta^{\prime}} \quad(\mathrm{pp})$, one gets the reduction of the box labelled (!p) in Figure 4 which cut reduces to $\pi$ by IH.
If the last rule is (?w) or (?c), the interpolation is built as before by adding the structural rule at the root of one of the interpolated proofs obtained by the IH.
If the last rule is $(\forall)$, that is $\pi=\frac{\frac{\pi^{\prime}}{\vdash F, \Gamma^{\prime}, \Delta}}{\vdash \forall x F, \Gamma^{\prime}, \Delta}(\forall) \quad x \notin \mathrm{FV}\left(\Gamma^{\prime}, \Delta\right) \quad$ assuming $\Gamma=\forall x F, \Gamma^{\prime}$. By IH, there is an interpolant $C^{\prime}$ such that $\mathcal{L}\left(C^{\prime}\right) \subseteq \mathcal{L}\left(F, \Gamma^{\prime}\right) \cap \mathcal{L}(\Delta)$ as well as proofs $\pi_{1}^{\prime} \vdash F, \Gamma^{\prime}, C^{\prime}$ and $\pi_{2}^{\prime} \vdash C^{\prime \perp}, \Delta$ such that $\frac{\pi_{1}^{\prime}}{\frac{\vdash F, \Gamma^{\prime}, C^{\prime}}{\vdash F, \Gamma^{\prime}, \Delta} \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta}}$ (Cut) $\longrightarrow_{\text {cut }}^{\star} \pi^{\prime}$. By setting $C=\exists x \cdot C^{\prime}, \pi_{1}=\frac{\pi_{1}^{\prime}}{\vdash F, \Gamma^{\prime}, \exists x \cdot C^{\prime}} \stackrel{(\exists)}{\vdash \forall x F, \Gamma^{\prime}, \exists x \cdot C^{\prime}} \quad$ ( $\forall$ ) $\quad$ and $\pi_{2}=\frac{\pi_{2}^{\prime}}{\vdash \forall x C^{\prime \perp}, \Delta}$ ( ) one gets the reduction of the box labelled $(\forall)$ in Figure 4 which cut reduces to $\pi$ by IH.
If the last rule is $(\exists)$, that is $\pi=\frac{\frac{\pi^{\prime}}{\vdash F\{y / x\}, \Gamma^{\prime}, \Delta}}{\vdash \exists x F, \Gamma^{\prime}, \Delta} \quad$ (ヨ) $\quad$ assuming $\Gamma=\exists x F, \Gamma^{\prime}$. Note that we treat only the case of a language containing no function symbols for simplicity. In that case, we do a case analysis of the possible occurrences of $y$ in $\Gamma^{\prime}$ and $\Delta$ and depending on the case, either reuse the interpolant given by the induction hypothesis of use an interpolant obtained by quantifying $y$ universally of existentially in the interpolant.

### 3.1 Proof-relevant interpolation for LK and LJ thanks to linear embeddings

A similar proof-relevant interpolation can be proved for LK and LJ by expliciting Maehara's method as we did. On the other hand, one can directly deduce this result by using linear
embeddings of LK and $\mathrm{LJ}[22,15]$ together with the result of the previous section. We sketch this here for LK and provide details in appendices:

- First, we notice that proof-relevant interpolation presented in the previous section holds also for the two-sided section calculus for LL.
- Second, a cut-free LK proof $\pi$ (it would be similar for LJ) can be decorated with exponential modalities and inferences in order to turn it into a cut-free LL proof $\pi^{\prime}$.
- After interpolating this proof (obtaining $I^{\prime}, \pi_{1}^{\prime}$ and $\pi_{2}^{\prime}$ ), one can erase the linear information of the interpolants and the two interpolating proofs (that is, taking the classical skeleton of the proofs) and get back a pair of LK (resp. LJ) proofs $\pi_{1}, \pi_{2}$ together with a formula $I$ in LK (resp. LJ).
- The properties of the linear embeddings ensure that the skeleton of a cut-free proof obtained from $\operatorname{Cut}\left(\pi_{1}^{\prime}, \pi_{2}^{\prime}\right)$ can be obtained by eliminating the classical cuts from $\operatorname{Cut}\left(\pi_{1}, \pi_{2}\right)$.


## 4 Interpolation as cut-introduction

Now that we have obtained the proof-relevant interpolation theorem, we will explain, in the present section, how the synthesis of the interpolant is in fact a cut-introduction process.

The usual proof method for interpolation, made more informative in the previous section, actually obfuscates the fact that the interpolating formula and proofs are indeed built using a cut-introduction mechanism: this is due to the structure of the inductive reasoning used to establish the interpolation theorem under Maehara's method. On the other hand, one can analyze what happens in constructing the interpolating formula and proofs by structuring the process in two phases, an bottom-up phase and a top-down phase:

Ascending phase. This first phase consists in traversing the initial proof $\pi$ bottom-up, from root (conclusion) to leaves (axioms), and building, for each visited sequent $\Gamma$, a splitting ( $\Gamma^{\prime}, \Gamma^{\prime \prime}$ ) inherited from the splitting of the conclusion of the proof by the ancestor relation. In this way, each node of the proof is ultimately decorated with some additional information on how to splitting the sequent labelling the node.
Ultimately, for each logical axiom rule $\vdash A^{\perp}$, $A$, we are in one of the following situations:
(i) $\left(\left\{A^{\perp}, A\right\}, \emptyset\right)$;
(ii) $\left(\left\{A^{\perp}\right\},\{A\}\right)$;
(iii) $\left(\{A\},\left\{A^{\perp}\right\}\right)$;
(iv) $\left(\emptyset,\left\{A^{\perp}, A\right\}\right)$.
(and similarly for each axiom corresponding to some unit, $\top$ or 1.) This corresponds to the various base cases of the inductive proof of the previous section.
Once every axiom has been reached, we switch to the descending phase, traversing again the proof, top to bottom, in an asynchronous manner.
Descending phase. Equipped with the sequents splitting information one shall now apply cut-introduction rules to axioms, progressively moving the cuts down and merging them in such a way, ultimately, to reach the root sequent of the original proof. We call active a sequent such that all its premises are concluded with cut inferences. (Initially, since $\pi$ is cut-free, only the conclusions of logical axioms or 0 -ary unit rules are active sequents.)
We apply cut-introductions to active sequents, maintaining the following two invariants:
= when a sequent is active with splitting $\left(\Gamma^{\prime}, \Gamma^{\prime \prime}\right)$, the cut formulas of its premises are interpolants for the premise sequents wrt. their splitting (Note that this condition trivially satisfied initially since the active axioms have no premise).

- when an inference ( $r$ ) has conclusion $\mathcal{S}$ which is active, we apply a (sequence of) cutintroduction step(s) on this inference, in such a way that (i) $\mathcal{S}$ becomes the conclusion of the introduced cut and (ii) the premises of this cut correspond to the splitting associated with sequent $\mathcal{S}$.

It is easy to check that applying cut-introductions as specified in the boxed reductions of Figure 4 (in reverse order) preserves the above invariants. The descending phase therefore terminates when the cut reaches the root. Ultimately, one builds a cut formula $C$ and two cut-free proofs $\pi_{1}$ and $\pi_{2}$ such that $C$ is an interpolant of the conclusion sequent with respect to the original splitting and such that (Cut) $\left(\pi_{1}, \pi_{2}\right) \longrightarrow_{\text {cut }}^{\star} \pi$, a condition which is satisfied by construction. To state precisely the properties of the two phases, we introduce the following notions:

- A decorated proof is an LL proof st. each sequent is equipped with a splitting.
- A coherent decorated proof is a decorated proof such that for each node, the splitting of the conclusion and of its premises is coherent wrt. the ancestor relation: a formula belonging to the left (resp. right) component of the splitting has all its ancestors belonging to the left (resp. right) component of the splitting.
- Remark 8. The notion of coherent decorations can be refined to two-sided calculi (see Appendix C): checking that the ascending phase builds such a refined notion of coherent decorated proof directly entails Lyndon's interpolation, in addition to Craig's.

With the above notions, the following lemma is clear by induction on the proof:

- Lemma 9. For any LL proof and any splitting of its conclusion sequent, the ascending phase terminates with a coherent decorated proof.

Coherent decorated proofs satisfy a progress condition wrt. the descending phase, which follows from a simple inspection of the cut-reduction sequences in Figure 4:

- Lemma 10. The descending phase applied to a coherent decorated proof is never blocked, unless the active sequent is a roof of the proof itself.

Combining the ascending and descending phases, the two previous lemmas provide us with the following theorem stating interpolation as cut-introduction:

- Theorem 11. Given a cut-free proof $\pi \vdash \Gamma, \Delta$ and a coherent decorated proof $\pi^{\prime}$ with respect to splitting $(\Gamma ; \Delta)$, the descending phase determines a cut-introduction sequence from $\pi$ ending in a proof $(\mathrm{Cut})\left(\pi_{1}, \pi_{2}\right)$, the cut formula $C$ of which is an interpolant of $\Gamma$ and $\Delta$.
- Remark 12. The above result does not rely on the cut-freeness of $\pi$, only the notion of interpolant does. This suggests an immediate generalization of proof-relevant Craig and Lyndon's interpolation for proofs with cuts in which each (pair of) cut-formula is assigned to the left/right component of a splitting (in a consistent way for a given cut inference): the language of the interpolant is then constructed by taking into account the language of the cut-formulas depending on the choice of splitting. In that case, an additional degree of freedom appears in the choice of the interpolant when assigning each cut to a component of the splitting: strategies for optimizing the language of the interpolant could therefore been investigated. Of course analytic cuts are of particular interest here.


## 5 Towards proof-relevant interpolation for $\mu \mathrm{LL}^{\infty}$ circular proofs

In this section, we demonstrate the applicability of our method by considering the question of extending the proof-relevant interpolation theorem of the previous section to a fragment of circular derivations for linear logic with fixed points, ie. $\mu \mathrm{LL}{ }^{\infty}$. We first explain how when considering all circular cut-free pre-proofs and not necessarily valid proofs, one can perform proof-relevant interpolation. Then, we provide a discussion on how to integrate a validity condition in this picture and suggest several directions as well as treating some examples.

### 5.1 Interpolating finite $\mu \mathrm{LL}{ }^{\infty}$ proofs

The first and most trivial case that we have to address is the case of wellfounded derivations of $\mu \mathrm{LL}{ }^{\infty}$. In that case, the proof of Section 3 can be trivially extended: since the fixed-point unfolding rule are non-branching, the interpolant does not need to be updated and the interpolating proofs are simply obtained by expanding one them with the adequate fixedpoint rule, as determined by the splitting. The equality by cut-reduction is given in Figure 4. The approach from Section 4 thus extends trivially as in the previous section.

### 5.2 A non-avoidable restriction for the non-wellfounded case?

Three ingredients are important to carry the proof of the previous section to circular proofs:

1. the wellfoundedness of proof objects. Indeed this ensures that one reaches axioms which are the base case of the induction; wellfoundedness is implicitly used to initiate the descending phase (that is, after having ended the ascending phase!)
2. the existence of cut-free proofs. Indeed, cut-freeness is important to reason by induction on inferences of the cut-free proofs and benefit from analyticity, which is the key for controlling the language of the interpolant, and
3. the preservation of logical correctness during the descending phase (that is, cut-introduction). Indeed, correctness of the interpolated proof-objects is of course necessary for the result of interpolation to simply make sense...

In the case of circular proofs, the first two properties are somehow lost and the third one shall be treated with great care:

1. wellfoundedness is lost, even in presence of circular proofs. In particular, even given a finite representation of a given circular proofs, we have leaves which are not axioms but back-edges: how can we interpolated them?
2. while it is crucial to rely on cut-freeness in the reasoning, circular proofs are not closed by cut-elimination are actually we know of sequents which are circularly provable but not cut-free circularly provable.
3. As we will see below, while cut-free circular pre-proofs can be interpolated easily (that is, finding an interpolant formula $C$ and two interpolating cut-free pre-proofs $\pi_{1}, \pi_{2}$ such that there cut can be eliminated to produce $\pi$ ), ensuring the preservation of validity (ie requiring that a valid cut-free circular $\mu \mathrm{LL}{ }^{\infty}$ proof can be interpolated by two valid cut-free circular $\mu \mathrm{LL}{ }^{\infty}$ proofs) is much more complex. This is due to the fact that it is difficult to control where threads will end up after the cut-introduction process.

The above analysis justifies that we first consider the case of cut-free circular pre-proof, neglecting the issue of validity in first approximation and only then considering whether this can be made compatible with validity requirements.

- Remark 13. In fact, were we trying to interpolated cut-free non-wellfounded proofs, we would produce infinite interpolants for which is is unclear (and actually quite doubtful) whether one can find a finite structure capturing this interpolant.


### 5.3 Interpolating cut-free circular pre-proofs

In this section, we consider a cut-free pre-proof $\pi \vdash \Gamma$, together with a finite representation $R$ of $\pi$. We show that, for any splitting $\Gamma_{l}, \Gamma_{r}$ we can interpolate $\pi$ with $\pi_{l} \vdash \Gamma_{l}, I$ and
(i) $R=$
$\frac{\vdash G, H}{\vdash \underline{H}, H}$
$\frac{(\mu)}{\vdash \underline{H}, \underline{H}}$
$\vdash \underline{G}, H$

Figure 5 Examples of splitting-invariance and non-splitting invariance.
$\pi_{r} \vdash I^{\perp}, \Gamma_{r}$ with $I$ in the common language of $\Gamma_{l}$ and of $\Gamma_{r}$ and such that $C u t\left(\pi_{l}, \pi_{r}\right)$ (infinitarily) reduces to $\pi$.

The main difficulty to address is the existence of back-edges in the finite representation (and the associated non-wellfoundedness in $\pi$ ). Indeed, Maehara's method as well as our approaches rely on the wellfoundedness of the proof objects to initiate the synthesis of the interpolant. In the following, we will treat sources of back-edges as generalized axioms (they are indeed leaves of the tree with back-edge that in a finite representation) and initiate the interpolation from them as well. At some point, we will have to reconstruct back-edges so that we will have to satisfy some constraints on the interpolant and use the ability to use inductive and coinductive statement for that, that is to build fixed point of formulas.

But first, we need to address an issue with the splitting of the conclusion sequent. Indeed, while the finite representation $R$ has back-edges such that sources and targets of back-edges coincide, once a splitting has been selected, nothing ensures that the splitting at the level of the sources of back-edges matches the splitting we had at the target of the back-edge.

- Definition 14 (splitting invariance). Given a finite representation $R$ of a $\mu \mathrm{LL}^{\infty}$ pre-proof $\pi \vdash \Gamma$ and a splitting sof $\Gamma$ in two components $\Gamma_{l}, \Gamma_{r}, R$ is called s-invariant if the result of applying the splitting-decoration phase to $R$ (ie. the ascending phase of Section 4), initiated with s , results into a decorated derivation $R^{\prime}$ such that for each back-edge $b$ of $R^{\prime}$, the splitting of the source of $b$ coincides with the splitting of the target of $b$.

After exemplifying the issue discussed above, we show how this condition can be enforced by unfolding the finite representation and then how splitting-invariance enables interpolation.

### 5.3.1 A mismatch between splittings and back-edges

The following very simple example should make clear the issue with splittings that prevents us to directly interpolate finite representation.

Let $G$ be $\nu X . \mu Y . X$ and $H$ be $\mu Y . G$. We can consider the circular proof $\pi$ given by the finite representation in Figure 5.(i) together with the splitting sof the conclusion sequent $\{G\},\{H\}$ and propagate it through upwards to decorate each sequent of $R$, we end up with the target of the back-edge being decorated with the splitting $\{H\},\{G\}$. This would prevent us from pursuing the interpolation construction. On the other hand, by unfolding once the back-edge of $R$, we can consider an alternative finite representation for $\pi, R^{\prime}$, such that the decoration process initiated with splitting $s$ (but also initiated with any other splitting of the conclusion sequent as the reader can easily check) result in splittings which coincide on the source and target of the back-edge: $R^{\prime}$ is splitting-invariant.

This is in fact a general fact that to any finite representation $R$ of a pre-proof $\pi$, one can find another finite representation of $\pi$ which is splitting-invariant and is obtained by simply unfolding the back-edges of $R$ a certain number of times.

### 5.3.2 Enforcing splitting-invariance

Interpolating circular pre-proofs will rely on the following proposition:

- Proposition 15. Let $\pi$ be a circular proof of a sequent $\vdash \Gamma$ and let sbe a splitting of $\Gamma$ in two components $\left(\Gamma_{L}, \Gamma_{R}\right)$. To any finite representation $R$ of $\pi$, one can associate another finite representation $R^{\prime}$ of $\pi$ which is s-invariant.

To understand the intuitive idea behind the previous result, let us consider the simpler case of a finite representation $R$ with several back-edges, all pointing to the root of the derivation. Given a splitting $\mathrm{s}_{0}$, one then propagates it upwards in $R$. When the source of a back-edge $b$ is encountered, one check whether the splitting at the source of $b$ coincides with that of the target of $b$. If so, s-invariance is locally satisfied. Otherwise, we have an alternative splitting $\mathrm{s}_{1}$ : let us unfold $R$ and keep propagating $\mathrm{s}^{\prime}$. When reaching sources of back-edges of $R$, we check the splitting at the source: if it is $\mathrm{s}_{0}$, we place a back-edge to the root of the derivation, if it is $s_{1}$, we place a back-edge to the root of the unfolding, otherwise, we unfold again and propagate the new splitting $s_{2}$ : this process will surely terminates since there are only finitely many splittings of $\Gamma$. See Figure 5 for an example.

In fact, one proves a stronger result about splitting-invariance which entails Proposition 15:

- Lemma 16. Let $\pi$ be a circular proof of a sequent $\vdash \Gamma$ and $R$ be a finite representation of $\pi$. One can unfold $R$ into another finite representation $R^{\prime}$ of $\pi$ which is splitting-invariant.

Proof. The proof follows the same idea as before but, instead of propagating upwards a single splitting, we propagate a list containing all possible splittings of the root sequent of $R$. The finiteness condition that ensured termination of the unfolding is preserved ${ }^{2}$ and when we encounter another target of back-edges of $R$, one can safely follow the same unfolding process since, when a back-edge is introduce, invariance is globally ensured on all possible splittings. The resulting finite representation $R^{\prime}$ is therefore splitting-invariant as expected.

- Remark 17. Choosing between Proposition 15 and Lemma 16 is a matter of trade-off between uniformity and size-efficiency. By applying the reasoning on a single splitting, we obtain in general smaller invariant representations but a distinct representation has to be chosen for distinct splittings, contrarily to the global invariance ensured by Lemma 16.
- Remark 18. The above lemmas can be viewed as a strengthening of a result by Shamkanov [38] for Gödel-Löb circular proofs where he shows the existence of a s-invariant circular proof without control on the finite representation. In their work on Lyndon interpolation for the modal $\mu$-calculus, Afshari and Leigh [1] do not show such a result as they benefit from the annotation system to directly prove there exists a proof that is invariant. The disadvantage is that at that step their approach is inherently not proof-relevant.


### 5.3.3 Interpolation-as-cut-introduction for invariant finite representations of circular pre-proofs

We shall now explain how to adapt our cut-introduction process of Section 4 to circular pre-proof. For this, we shall temporarily consider pre-formulas (ie. $\mu$ formulas which may have free fixed-point variable) and shall recall that the linear negation of a fixed-point variable $X$ is $X$ itself. As a consequence, the following is a correct instance of a cut-inference:

[^1]$$
\frac{\vdash \Gamma, X \quad \vdash X, \Delta}{\vdash \Gamma, \Delta}
$$

- Theorem 19. Let $\pi$ be a cut-free circular $\mu \mathrm{LL}^{\infty}$ pre-proof of $\vdash \Gamma$, sa splitting of $\Gamma$ (into $\Gamma_{l}$ and $\Gamma_{r}$ ) and $R$ be as-invariant finite representation of $\pi$. There exists a $\mu \mathrm{LL}^{\infty}$ formula $I$ built on the common language of $\Gamma_{l}$ and $\Gamma_{r}$ (that is $\mathcal{L}(I) \subseteq \mathcal{L}\left(\Gamma_{l}\right) \cap \mathcal{L}\left(\Gamma_{r}\right)$ ) and two cut-free circular $\mu \mathrm{LL}^{\infty}$ pre-proofs $\pi_{1} \vdash \Gamma_{l}, I$ and $\pi_{2} \vdash I^{\perp}, \Gamma_{r}$ such that $\frac{\frac{\pi_{1}}{\vdash \Gamma_{l}, I} \frac{\pi_{2}}{\vdash I^{\perp}, \Gamma_{r}}}{\vdash \Gamma}$ (Cut) $\longrightarrow_{\text {cut }}^{\omega} \pi$.

Proof. The ascending phase does not need to be adapted. As for the descending phase, we shall simply consider two additional cases, in order to treat sources and targets of back-edges.

First, let us assign to each (occurrence of a) sequent $s$ which is the target of a back-edge in $R$, a fixed-point variable $X_{s}$. We can now express how to adapt the descending phase

Sources of back-edges are a novel way to initiate the descending phase. For this, to each source $\vdash \Gamma, \Delta$ of a back-edge $b$ with target $s$, for which the splitting is $(\Gamma ; \Delta)$, we associate the cut $\frac{\vdash \Gamma, X_{s} \vdash X_{s}, \Delta}{\vdash \Gamma, \Delta}$ (Cut)
Targets of back-edges are reached during the descending phase and induce a specific construction, in order to reconstruct the back-edges (to each back-edge of $R$, one will introduce a back-edge in each of the interpolating derivations). Assume that when reaching the target $s$ of some back-edges, we have: $\frac{\frac{R_{l}}{\vdash \Gamma, I} \frac{R_{r}}{\vdash I^{\perp}, \Delta}}{\vdash \Gamma, \Delta}$ (Cut)
Remember that we are currently working with pre-formulas, which may have free fixedpoint variables and that $R_{l}$ (resp. $R_{r}$ ) has some leaves $\vdash \Gamma, X_{s}\left(\right.$ resp. $\left.\vdash X_{s}, \Delta\right)$ and that $I$ (resp. $I^{\perp}$ ) has $X_{s}$ as free fixed-point variable (as well as all the variables corresponding to sources of back-edges, but no more fixed-point variables that those coming from back-edges. In such a situation, we first apply to $R_{l}$ substitution $\left[\mu X_{s} . I\left(X_{s}, \ldots, X_{s}\right) / X_{s}\right]$ and to $R_{r}$ substitution $\left[\nu X_{s} . I^{\perp}\left(X_{s}, \ldots, X_{s}\right) / X_{s}\right]$ reaching: $\frac{\frac{R_{l}\left[\mu X_{s} . I / X_{s}\right]}{\vdash \Gamma, I\left[\mu X_{s} . I / X_{s}\right]} \frac{R_{r}\left[\nu X_{s} \cdot I^{\perp} / X_{s}\right]}{\vdash I^{\perp}\left[\nu X_{s} \cdot I^{\perp} / X_{s}\right], \Delta}}{\vdash \Gamma, \Delta}$
and apply a $\mu / \nu$-introduction reaching: $\frac{\frac{R_{l}\left[\mu X_{s} \cdot I / X_{s}\right]}{\vdash \Gamma, I\left[\mu X_{s} \cdot I / X_{s}\right]}}{\frac{\vdash \Gamma, \mu X_{s} \cdot I}{}{ }_{(\mu)} \frac{\frac{R_{r}\left[\nu X_{s} \cdot I^{\perp} / X_{s}\right]}{\vdash I^{\perp}\left[\nu X_{s} \cdot I^{\perp} / X_{s}\right], \Delta}}{\vdash \nu X_{s} \cdot I^{\perp}, \Delta}}{ }_{(\nu)}^{\vdash \Gamma, \Delta}$
The previous two steps updated the leaves $\vdash \Gamma, X_{s}\left(\right.$ resp. $\left.\vdash X_{s}, \Delta\right)$ of $R_{l}$ (resp. $R_{r}$ ) to $\vdash \Gamma, \mu X_{s} . I$ (resp. $\vdash \nu X_{s} . I^{\perp}, \Delta$ ) and one can thus place a back-edge from each of those leaves to the corresponding premise of the cut inference. The process is summed up in Appendix D. (Here, the choice of introducing a $(\mu)$ in $R_{l}$ and a $(\nu)$ in $R_{r}$ is arbitrary as we only consider pre-proof. This choice is analyzed in the next section about validity.)

As in Section 4, when the cut reached the root of $R$, we get a triple $\left(I, R_{l}, R_{r}\right)$ such that:

- $I$ is a $\mu \mathrm{LL}^{\infty}$ formula: it contains no free fixed-point variable since we crossed all back-edges.
- $R_{l}$ and $R_{r}$ are circular representations of respective conclusions $\vdash \Gamma, I$ and $\vdash I^{\perp}, \Delta$
- $I$ is in the common language of $\Gamma$ and $\Delta$ : indeed, this is ensured by the result of Sections 3 and 4 for LL inferences, encountering $(\mu) /(\mu)$ does not modify the interpolating formula and both the initialization of a interpolant at a source of a back-edge or closure or introduction of fixed-points in the interpolant when crossing the target of a back-edge does not modify the language of the interpolant.

Cutting $R_{l}$ and $R_{r}$ results in a finite representation of a proof of which the cuts can be eliminated reaching $\pi$ as a limit: indeed, the interpolant is synthesized by cut-introduction, their cut-elimination progressively reconstructs the infinite unfolding of $R$.

- Remark 20. Notice that for this reasoning to hold, it is useful (needed) to have the form of reasoning that separate the ascending and descending phase as we did in Section 4.

Note also that in the locative sequent calculi for $\mu \mathrm{LL}^{\infty}[3,2]$, the ascending phase can be skipped since if $F$ is sub-occurrence of $G$, the address of $F$ is a sub-address of that of $G$.

### 5.4 Towards a proof-relevant interpolation for valid circular proofs

Proof-relevant circular interpolation presented in the previous section totally neglected the validity condition: this is of course unsatisfactory. While falling beyond the scope of this paper, we discuss in the present section the make questions that have to be addressed and outline a solution that will be the subject of a future work.

First, let us make a general remark on the impact of the interpolation process of Section 5.3.3: assuming the circular pre-proof is indeed valid, that means it is equipped with a set of valid threads covering every infinite branch. While building the two interpolating pre-proofs $\pi_{l}, \pi_{r}$, this set of threads will be split into two sets, one set of threads being sent to $\pi_{l}$ while the other goes to $\pi_{r}$. In particular, we can notice that in the (very restricted) case that $\pi$ contains a single infinite branch, we have a solution to ensure validity of both interpolating proofs by setting the interpolating formula to contain a $\mu$ is the branch that goes with the valid thread and its dual to contain a $\nu$. For instance with the finite representation of Figure 5 which is splitting-invariant and with the splitting $(\{G, H\}, \emptyset)$, one can pick the interpolant to be $\mu X . X$. But in more complex cases where there are multiple back-edges which are interleaved, the method of the previous section cannot be expected to work and produce valid interpolating proofs as it treats all back-edges pointing to a given sequent uniformly while there is no reason that validity should be justified in the same way for all infinite branches generated by iterating those back-edges.

There is a natural adaptation for this: instead of choosing one fresh fixed-point formula for each target of a back-edge as we did above, we can choose a fresh fixed-point variable for each source of a back-edge, say $X_{1}, \ldots X_{n}$. When reaching the target of back-edges the interpolant under synthesis will now depend on all those fixed-point variables, $I\left(X_{1}, \ldots X_{n}\right)$ and we have the freedom: (i) to choose a distinct fixed-point, $\tau_{i} \in\{\mu, \nu\}$ for $X_{i}$ and (ii) to choose any sequentialization in the formation of the fixed-point formula to be used as interpolator, that is choosing some permutation $\sigma \in \mathfrak{S}_{n}$ and set the interpolating formula be $\tau_{\sigma(1)} X_{\sigma(1)} \ldots \tau_{\sigma(n)} X_{\sigma(n)} . I\left(X_{1}, \ldots X_{n}\right)$ based on the validating mode of the back-edges. For proving the existence of such a permutation $\tau$ ensuring validity of the interpolating proofs (and actually constructing it), one can take inspiration of the work by Afshari \& Leigh on Lyndon interpolation for the modal $\mu$-calculus [1] but at the moment we only know how to handle the fragment of strongly valid $\mu \mathrm{LL}{ }^{\infty}$-proofs as considered by Doumane [19]. Doing this precisely will be the focus of a future paper.

## 6 Conclusion

In this paper, we established a refined, proof-relevant, version of Craig-Lyndon interpolation theorems for first-order linear logic and then deduced it, using completely standard tools of LL proof theory, to LK and LJ. A most striking fact, in our opinion, is that the result was almost there for decades, since the early proofs by Maehara (and its broad dissemination in proof
theory textbooks, not to speak of applications to broader logical frameworks) and Prawitz. Borrowing Feferman's words, "though deceptively simple and plausible on the face of it", we think that this approach to proof-relevant interpolation in sequent calculus emphasizes a deep duality between interpolation and cut elimination : more specifically, the process of synthesis of the interpolant and the two interpolating proofs is reformulated as a cut-introduction process. Finally, we showed that our results can be applied even in non-wellfounded settings: we proved that it can be extended to $\mu \mathrm{LL}^{\infty}$ circular pre-proofs and discuss some directions to be taken to take validity into account, leaving this for a future work.

While we think that interpolation as cut-introduction is both a new conceptual and technical contribution of this work, a proof-relevant interpolation theorem has already been established by Čubrić [11, 12] in the early 90 's for propositional intuitionistic natural deduction in the form of an interpolation for the typed $\lambda$-calculus and for bicartesian closed categories. Our approach is similarly subject to a computational interpretation that we plan to develop in a future work about interpolation in system $\mathrm{L}[13,14]$. In fact we also hope that our interpolation-as-cut-introduction can pave the way for a broader analysis of the computational content of interpolation as a manner to factor computation through interfacing (that is, interpolating) types. Indeed, while the computational interpretation of Čubrić's result, stated in the $\lambda$-calculus, is certainly more transparent than the sequent calculus that we presented here, it has not been extended in more than 30 years, except once by Matthes [28]. A reason for this is might be that while both allows for a proof-relevant phrasing, Maehara's method is more modular and easily extensible than Prawitz as it rests on a logical framework, the sequent calculus, that is inherently more modular that natural deduction. For instance, we conjecture that it is possible to state a computational version of interpolation in System L, that is in a classical framework featuring continuations, while it is very much unclear how Čubrić's results can be extended to Parigot's $\lambda \mu$-calculus.

Among other future works that we plan to tackle, one can list the following directions.

- We will develop a full treatment of the validity condition along the line of Section 5.4;
- In LL, proof nets are a proof system that satisfies canonicity properties akin to natural deduction for intuitionistic logic. It seems that interpolation as cut-introduction in (multiplicative) proof-nets can be reformulated in terms of the parsing correctness criterion.
- We hope to establish factorization properties for models of LL, similar to Čubrić's results.
- An intrinsic advantage of sequent calculus over natural deduction, and therefore of our approach over Čubric's, is that many more logics can be formulated as sequent calculi than in natural deduction. Can we extend our results to other logics having cut-elimination?
- An important question and clearly non-trivial question that we would like to explore is whether such a proof-relevant approach to interpolation can be extended to uniform interpolation. That would mean that all computations that can be performed from a piece of data $u$ in a type A to data sharing with A only a fixed set $\mathcal{L}$ of primitive datatypes can be factored through a program that computes a value $v$ in the uniform interpolant datatype build from $\mathcal{L}$ such that everything that can be computed from $u$ can be computed from $v$ as well.
- Finally, we plan to reconsider Čubrić's results from the cut-introduction perspective.

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## A Full proof of proof-relevant interpolation for LL

Proof. By LL cut-elimination theorem, one can assume that $\pi$ is cut-free and reason by induction on the structure of $\pi$ and by case on the last inference. We will proceed by "introducing cuts" and build new interpolants in such a way as to preserve the denotational equivalence of the interpolated proof with the proof being constructed.

In fact, we shall prove a slightly stronger result, that is the cut of the interpolating proofs reduces, by cut-elimination, to the interpolated proof (up to exchange rules which are neglected in the following).

If $\pi=\overline{\vdash \boldsymbol{F}, \boldsymbol{F}^{\perp}}{ }^{(\mathbf{A x})}, \Gamma=F$, one simply takes $C=F^{\perp} \pi_{1}=\pi_{2}=\overline{\vdash F, F^{\perp}}{ }^{(\mathbf{A x})}$. (The case when $\Gamma=F^{\perp}$ is symmetrical, taking $C=F$.)
If $\pi=\overline{\vdash \boldsymbol{F}, \boldsymbol{F}^{\perp}}{ }^{(\mathrm{Ax})}, \Gamma=F, F^{\perp}$, one simply takes $C=\perp, \pi_{1}=\frac{\pi}{\vdash \Gamma, \perp}$ ( $\perp$ ) and $\pi_{2}=$
$\overline{\vdash 1}{ }^{(1)}$. (The case when $\Gamma$ is empty is symmetrical, taking $C=1$.)
If the last rule is $(\otimes)$, that is $\pi=\frac{\frac{\pi^{\prime}}{\vdash F, \Gamma^{\prime}, \Delta^{\prime}} \frac{\pi^{\prime \prime}}{\vdash F G, \Gamma^{\prime \prime}, \Delta^{\prime \prime}}}{\vdash F \theta, \Gamma^{\prime}, \Gamma^{\prime \prime}, \Delta^{\prime}, \Delta^{\prime \prime}} \quad(\otimes)$, assuming $\Gamma=F \otimes$ $G, \Gamma^{\prime}, \Gamma^{\prime \prime}$.
By induction hypothesis, there are interpolants $C^{\prime}, C^{\prime \prime}$, as well as interpolating proofs (i) $\pi_{1}^{\prime} \vdash F, \Gamma^{\prime}, C^{\prime}$, (ii) $\pi_{2}^{\prime} \vdash C^{\prime \perp}, \Delta^{\prime}$, (iii) $\pi_{1}^{\prime \prime} \vdash G, \Gamma^{\prime \prime}, C^{\prime \prime}$ and (iv) $\pi_{2}^{\prime \prime} \vdash C^{\prime \prime \perp}, \Delta^{\prime \prime}$ such that

$$
\frac{\pi_{1}^{\prime} \pi_{2}^{\prime}}{\vdash F, \Gamma^{\prime}, \Delta^{\prime}}(\text { Cut }) \longrightarrow \longrightarrow_{\text {cut }}^{\star} \pi^{\prime} \quad \frac{\pi_{1}^{\prime \prime} \pi_{2}^{\prime \prime}}{\vdash G, \Gamma^{\prime \prime}, \Delta^{\prime \prime}} \text { (Cut) } \longrightarrow_{\text {cut }}^{\star} \pi^{\prime}
$$

Let $C=C^{\prime} 8 C^{\prime \prime}$ and let $\pi_{1}=\frac{\pi_{1}^{\prime} \pi_{1}^{\prime \prime}}{\frac{\vdash F \otimes G, \Gamma^{\prime}, \Gamma^{\prime \prime}, C^{\prime}, C^{\prime \prime}}{\vdash F \otimes G, \Gamma^{\prime}, \Gamma^{\prime \prime}, C^{\prime} 8 C^{\prime \prime}} \quad(\otimes) \quad \text { (8) }}$ and $\pi_{2}=\frac{\pi_{2}^{\prime} \quad \pi_{2}^{\prime \prime}}{\vdash C^{\prime \perp} \otimes C^{\prime \prime \perp}, \Delta^{\prime}, \Delta^{\prime \prime}}$
One observes that


If the last rule is (४), that is $\pi=\frac{\frac{\pi^{\prime}}{\vdash F, G, \Gamma^{\prime}, \Delta}}{\vdash F \gamma G, \Gamma^{\prime}, \Delta}$ (४) , assuming $\Gamma=F 8 G, \Gamma^{\prime}$. By induction hypothesis, there is an interpolant $C^{\prime}$ such that $\mathcal{L}\left(C^{\prime}\right) \subseteq \mathcal{L}\left(F, G, \Gamma^{\prime}\right) \cap \mathcal{L}(\Delta)$ as well as proofs $\pi_{1}^{\prime} \vdash F, G, \Gamma^{\prime}, C^{\prime}$ and $\pi_{2}^{\prime} \vdash C^{\prime \perp}, \Delta$ such that

$$
\frac{\frac{\pi_{1}^{\prime}}{\vdash F, G, \Gamma^{\prime}, C^{\prime}} \quad \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta^{\prime}}}{\vdash F, G, \Gamma^{\prime}, \Delta^{\prime}}(\mathrm{Cut}) \quad \longrightarrow_{\mathrm{cut}}^{\star} \pi^{\prime} .
$$

$$
\begin{aligned}
& \text { Setting } C=C^{\prime}, \pi_{1}=\frac{\pi_{1}^{\prime}}{\vdash F \gamma G, \Gamma^{\prime}, C} \text { ( }\left(\text { ) and } \pi_{2}=\pi_{2}^{\prime}\right. \text { we get: } \\
& \frac{\frac{\pi_{1}}{\vdash F \gamma G, \Gamma^{\prime}, C} \quad \frac{\pi_{2}}{\vdash F \gamma G, \Gamma^{\prime}, \Delta^{\prime}}{ }^{\vdash C^{\perp}, \Delta^{\prime}}}{(\text { Cut })} \quad \rightarrow \mathrm{cut}
\end{aligned}
$$

If the last rule is $(\perp)$, that is $\pi=\frac{\frac{\pi^{\prime}}{\vdash \Gamma^{\prime}, \Delta}}{\vdash \perp, \Gamma^{\prime}, \Delta}$
hypothesis, there is an interpolant $C^{\prime}$ and proof $\pi_{1}^{\prime} \vdash \Gamma^{\prime}, C^{\prime}$ and $\pi_{2}^{\prime} \vdash C^{\prime \perp}, \Delta$ such that

$$
{\frac{\pi_{1}^{\prime}}{\vdash \Gamma^{\prime}, C^{\prime}} \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta^{\prime}}}_{\vdash \Gamma^{\prime}, \Delta^{\prime}}^{\text {(Cut) }} \rightarrow_{\mathrm{cut}}^{\star} \pi^{\prime}
$$

By setting $C=C^{\prime}, \pi_{1}=\frac{\pi_{1}^{\prime}}{\vdash \perp, \Gamma^{\prime}, C^{\prime}}(\perp)$ and $\pi_{2}=\pi_{2}^{\prime}$, one gets

$$
\frac{\pi_{1}}{\frac{\vdash \perp, \Gamma^{\prime}, C}{\vdash \perp, \Gamma^{\prime}, \Delta^{\prime}} \frac{\pi_{2}}{\vdash C^{\perp}, \Delta^{\prime}}}\left(\text { Cut) } \longrightarrow_{\mathrm{cut}} \frac{\frac{\pi_{1}^{\prime}}{\vdash \Gamma^{\prime}, C} \frac{\pi_{2}}{\vdash C^{\perp}, \Delta^{\prime}}}{\frac{\vdash \Gamma^{\prime}, \Delta^{\prime}}{\vdash \perp, \Gamma^{\prime}, \Delta^{\prime}}(\perp)}(\mathrm{Cut}) \longrightarrow_{\mathrm{cut}}^{\star} \pi\right.
$$

If the last rule is $(T)$, that is $\pi=\overline{\vdash T, \Gamma^{\prime}, \Delta^{\prime}}{ }^{(T)}$, assuming $\Gamma=T, \Gamma^{\prime}$. Set $C=0$, $\pi_{1}^{\prime}=\overline{\vdash \mathrm{T}, \Gamma^{\prime}, 0}{ }^{(\mathrm{T})}$ and $\pi_{2}^{\prime}={\overline{\vdash \mathrm{T}, \Delta^{\prime}}}^{(\mathrm{T})}$ In such a case

$$
\frac{{\overline{\vdash \mathrm{T}, \Gamma^{\prime}, 0}}^{(\mathrm{T})}{\overline{\vdash \mathrm{T}, \Delta^{\prime}}}^{(\mathrm{T})}{ }_{(\mathrm{Cut})} \longrightarrow_{\mathrm{cut}}^{\star} \pi}{\Gamma^{\prime}, \Delta^{\prime}}
$$

If the last rule is $(\&)$, that is $\pi=\frac{\frac{\pi^{\prime}}{\vdash F, \Gamma^{\prime}, \Delta^{\prime}} \frac{\pi^{\prime \prime}}{\vdash F G G, \Gamma^{\prime}, \Delta^{\prime}}}{\vdash F \&, \Delta^{\prime}}$ (\&) , assuming $\Gamma=F \& G, \Gamma^{\prime}$.

By induction hypothesis, there are interpolants $C^{\prime}, C^{\prime \prime}$, as well as interpolating proofs (i) $\pi_{1}^{\prime} \vdash F, \Gamma^{\prime}, C^{\prime}$, (ii) $\pi_{2}^{\prime} \vdash C^{\prime \perp}, \Delta^{\prime}$, (iii) $\pi_{1}^{\prime \prime} \vdash G, \Gamma^{\prime}, C^{\prime \prime}$ and (iv) $\pi_{2}^{\prime \prime} \vdash C^{\prime \prime \perp}, \Delta^{\prime}$ such that

$$
\frac{\pi_{1}^{\prime} \pi_{2}^{\prime}}{\vdash F, \Gamma^{\prime}, \Delta^{\prime}}(\mathrm{Cut}) \longrightarrow_{\mathrm{cut}}^{\star} \pi^{\prime} \quad \frac{\pi_{1}^{\prime \prime} \pi_{2}^{\prime \prime}}{\vdash G, \Gamma^{\prime}, \Delta^{\prime}}(\mathrm{Cut}) \longrightarrow_{\mathrm{cut}}^{\star} \pi^{\prime}
$$

Let $C=C^{\prime} \oplus C^{\prime \prime}$ and let $\pi_{1}=\frac{\frac{\pi_{1}^{\prime}}{\vdash F, \Gamma^{\prime}, C^{\prime} \oplus C^{\prime \prime}}{ }^{\left(\oplus^{1}\right)} \frac{\pi_{1}^{\prime \prime}}{\vdash F \& G, \Gamma^{\prime}, C^{\prime} \oplus C^{\prime \prime}, C^{\prime} \oplus C^{\prime \prime}} \quad\left(\oplus^{2}\right)}{(\&)}$ and $\pi_{2}=\frac{\pi_{2}^{\prime} \pi_{2}^{\prime \prime}}{\vdash C^{\prime \perp} \& C^{\prime \prime \perp}, \Delta^{\prime}} \quad(\&)$.
One observes that

$$
\begin{aligned}
& \frac{\frac{\pi_{1}}{\vdash \Gamma, C} \quad \frac{\pi_{2}}{\vdash C^{\perp}, \Delta}}{\vdash \Gamma, \Delta} \text { (Cut) } \\
& \longrightarrow_{\mathrm{cut}} \frac{\frac{\pi_{1}^{\prime}}{\vdash F, \Gamma^{\prime}, C}{ }^{\left(\oplus^{1}\right)} \pi_{2}}{\vdash F, \Gamma^{\prime}, \Delta^{\prime}}(\mathrm{Cut}) \frac{\frac{\pi_{1}^{\prime \prime}}{\vdash G, \Gamma^{\prime}, C}{ }^{\left(\oplus^{2}\right)} \pi_{2}}{\vdash G, \Gamma^{\prime}, \Delta^{\prime}} \quad \text { (\&) } \quad \text { (Cut) } \\
& \longrightarrow_{\text {cut }} \frac{\pi_{1}^{\prime} \pi_{2}^{\prime}}{\vdash F, \Gamma^{\prime}, \Delta^{\prime}} \text { (Cut) } \frac{\frac{\pi_{1}^{\prime \prime}}{\vdash G, \Gamma^{\prime}, C}{ }^{\left(\oplus^{2}\right)} \pi_{2}}{\vdash G, \Gamma^{\prime}, \Delta^{\prime}} \text { (\&) } \quad \text { (Cut) } \\
& \longrightarrow_{\text {cut }} \frac{\frac{\pi_{1}^{\prime} \pi_{2}^{\prime}}{\vdash F, \Gamma^{\prime}, \Delta^{\prime}} \text { (Cut) } \frac{\pi_{1}^{\prime \prime} \pi_{2}^{\prime \prime}}{\vdash G, \Gamma^{\prime}, \Delta^{\prime}}}{\vdash F \& G, \Gamma^{\prime}, \Delta^{\prime}} \text { ((Cut) } \\
& \longrightarrow_{\text {cut }}^{\star} \pi \quad \text { by IH. }
\end{aligned}
$$

If the last rule is $\left(\oplus^{\mathrm{i}}\right), i \in\{1,2\}$, that is $\pi=\frac{\frac{\pi^{\prime}}{\vdash F_{i}, \Gamma^{\prime}, \Delta}}{\vdash F_{1} \oplus F_{2}, \Gamma^{\prime}, \Delta} \quad\left(\oplus^{\mathrm{i}}\right)$, assuming $\Gamma=F_{1} \oplus$ $F_{2}, \Gamma^{\prime}$. By induction hypothesis, there is an interpolant $C^{\prime}$ such that $\mathcal{L}\left(C^{\prime}\right) \subseteq \mathcal{L}\left(F_{i}, \Gamma^{\prime}\right) \cap$ $\mathcal{L}(\Delta)$ as well as proofs $\pi_{1}^{\prime} \vdash F_{i}, \Gamma^{\prime}, C^{\prime}$ and $\pi_{2}^{\prime} \vdash C^{\prime \perp}, \Delta$ such that

$$
\frac{\frac{\pi_{1}^{\prime}}{\vdash F_{i}, \Gamma^{\prime}, C^{\prime}} \quad \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta^{\prime}}}{\vdash F_{i}, \Gamma^{\prime}, \Delta^{\prime}}(\mathrm{Cut}) \quad \longrightarrow_{\mathrm{cut}}^{\star} \pi^{\prime} .
$$

Setting $C=C^{\prime}, \pi_{1}=\frac{\pi_{1}^{\prime}}{\vdash F_{1} \oplus F_{2}, \Gamma^{\prime}, C} \quad\left(\oplus^{i}\right)$ and $\pi_{2}=\pi_{2}^{\prime}$ we get the following cutreduction starting with a cut-commutation case:

$$
\begin{array}{ccc}
\frac{\pi_{1}}{\vdash F_{1} \oplus F_{2}, \Gamma^{\prime}, C} \quad \frac{\pi_{2}}{\vdash C^{\perp}, \Delta^{\prime}} \\
\frac{\vdash F_{1} \oplus F_{2}, \Gamma^{\prime}, \Delta^{\prime}}{(\mathrm{Cut})} & \longrightarrow \mathrm{cut} \\
\frac{\pi_{1}^{\prime}}{\vdash F_{i}, \Gamma^{\prime}, C^{\prime}} & \frac{\pi_{2}^{\prime}}{\vdash \vdash^{\prime}, \Delta^{\prime}} \\
\frac{\vdash F, G, \Gamma^{\prime}, \Delta^{\prime}}{\vdash F_{1} \oplus F_{2}, \Gamma^{\prime}, \Delta^{\prime}} \quad\left(\oplus^{\mathrm{i}}\right) & & \\
\text { (Cut) } & \longrightarrow_{\mathrm{cut}}^{\star} & \pi .
\end{array}
$$

If the last rule is (?d), that is $\pi=\frac{\frac{\pi^{\prime}}{\vdash F, \Gamma^{\prime}, \Delta}}{\vdash ? F, \Gamma^{\prime}, \Delta} \quad$ (?d) $\quad$ assuming $\Gamma=? F, \Gamma^{\prime}$. By induction hypothesis, there is an interpolant $C^{\prime}$ such that $\mathcal{L}\left(C^{\prime}\right) \subseteq \mathcal{L}\left(F, \Gamma^{\prime}\right) \cap \mathcal{L}(\Delta)$ as well as proofs $\pi_{1}^{\prime} \vdash F, \Gamma^{\prime}, C^{\prime}$ and $\pi_{2}^{\prime} \vdash C^{\prime \perp}, \Delta$ such that

$$
\frac{\frac{\pi_{1}^{\prime}}{\vdash F, \Gamma^{\prime}, C^{\prime}} \quad \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta}}{\vdash F, \Gamma^{\prime}, \Delta} \text { (Cut) } \longrightarrow_{\text {cut }}^{\star} \pi^{\prime} .
$$

By setting $C=C^{\prime}, \pi_{1}=\frac{\pi_{1}^{\prime}}{\vdash ? F, \Gamma^{\prime}, C^{\prime}} \quad$ (?d), one gets:

$$
\frac{\pi_{1}}{\frac{\pi_{2}}{\vdash ? F, \Gamma^{\prime}, C} \stackrel{\pi^{\prime}}{\vdash ? F, \Gamma^{\prime}, \Delta}}(\text { Cut }) \longrightarrow_{\mathrm{cut}} \frac{\frac{\pi_{1}^{\prime}}{\vdash F, \Gamma^{\prime}, C^{\prime}} \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta}}{\frac{\vdash F, \Gamma^{\prime}, \Delta}{\vdash ? F, \Gamma^{\prime}, \Delta}} \text { (?d) }(\mathrm{Cut}) \longrightarrow_{\mathrm{cut}}^{\star} \pi .
$$

If the last rule is $(!\mathrm{p})$, that is $\pi=\frac{\frac{\pi^{\prime}}{\vdash F, ? \Gamma^{\prime}, ? \Delta^{\prime}}}{\vdash!F, ? \Gamma^{\prime}, ? \Delta^{\prime}} \quad$ (!p) $\quad$ assuming $\Gamma=!F, ? \Gamma^{\prime}$ and $\Delta=? \Delta^{\prime}$. By induction hypothesis, there is an interpolant $C^{\prime}$ such that $\mathcal{L}\left(C^{\prime}\right) \subseteq \mathcal{L}\left(F, ? \Gamma^{\prime}\right) \cap \mathcal{L}(\Delta)$ as well as proofs $\pi_{1}^{\prime} \vdash F, ? \Gamma^{\prime}, C^{\prime}$ and $\pi_{2}^{\prime} \vdash C^{\prime}, ? \Delta^{\prime}$ such that

$$
\frac{\frac{\pi_{1}^{\prime}}{\vdash F, ? \Gamma^{\prime}, C^{\prime}} \quad \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, ? \Delta^{\prime}}}{\vdash F, ? \Gamma^{\prime}, ? \Delta^{\prime}}(\mathrm{Cut}) \quad \longrightarrow_{\mathrm{cut}}^{\star} \pi^{\prime}
$$

By setting $C=? C^{\prime}, \pi_{1}=\frac{\pi_{1}^{\prime}}{\vdash F, ? \Gamma^{\prime}, ? C^{\prime}} \stackrel{(? \mathrm{~d})}{\vdash!(\mathrm{p})}$ and $\pi_{2}=\frac{\pi_{2}^{\prime}}{\vdash!\Gamma^{\prime}, ? C^{\prime}}, ? \Delta^{\prime} \quad(\mathrm{p})$, one gets:

$$
\begin{aligned}
& \longrightarrow_{\text {cut }} \frac{\pi_{1}^{\prime} \pi_{2}^{\prime}}{\frac{\vdash F, ? \Gamma^{\prime}, ? \Delta^{\prime}}{\vdash ? F, ? \Gamma^{\prime}, ? \Delta^{\prime}}}{ }^{(\text {Cut })} \\
& \longrightarrow{ }_{\text {cut }}^{\star} \pi \text {. }
\end{aligned}
$$

If the last rule is (?w), that is $\pi=\frac{\frac{\pi^{\prime}}{\vdash \Gamma^{\prime}, \Delta}}{\vdash ? F, \Gamma^{\prime}, \Delta} \quad$ (?w) $\quad$ assuming $\Gamma=? F, \Gamma^{\prime}$. By induction hypothesis, there is an interpolant $C^{\prime}$ such that $\mathcal{L}\left(C^{\prime}\right) \subseteq \mathcal{L}\left(\Gamma^{\prime}\right) \cap \mathcal{L}(\Delta)$ as well as proofs $\pi_{1}^{\prime} \vdash \Gamma^{\prime}, C^{\prime}$ and $\pi_{2}^{\prime} \vdash C^{\prime \perp}, \Delta$ such that

$$
\frac{\frac{\pi_{1}^{\prime}}{\vdash \Gamma^{\prime}, C^{\prime}} \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta}}{\vdash \Gamma^{\prime}, \Delta} \quad(\mathrm{Cut}) \quad \longrightarrow_{\mathrm{cut}}^{\star} \pi^{\prime} .
$$

By setting $C=C^{\prime}, \pi_{1}=\frac{\pi_{1}^{\prime}}{\vdash ? F, \Gamma^{\prime}, C} \quad$ (?w), one gets $\mathcal{L}(C) \subseteq \mathcal{L}(\Gamma) \cap \mathcal{L}(\Delta)$ and:

$$
\frac{\frac{\pi_{1}}{\vdash ? F, \Gamma^{\prime}, C} \quad \frac{\pi_{2}}{\vdash C^{\perp}, \Delta}}{\vdash ? F, \Gamma^{\prime}, \Delta}(\mathrm{Cut}) \longrightarrow_{\mathrm{cut}} \frac{\frac{\pi_{1}^{\prime}}{\vdash \Gamma^{\prime}, C^{\prime}} \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta}}{\frac{\vdash \Gamma^{\prime}, \Delta}{\vdash ? F, \Gamma^{\prime}, \Delta}}(\text { (?w) }) \longrightarrow_{\mathrm{cut}}^{\star} \pi .
$$

If the last rule is (?c), that is $\pi=\frac{\pi^{\prime}}{\vdash ? F, ? F, \Gamma^{\prime}, \Delta} \quad$ (?c) $\quad$ assuming $\Gamma=? F, \Gamma^{\prime}$. By induction hypothesis, there is an interpolant $C^{\prime}$ such that $\mathcal{L}\left(C^{\prime}\right) \subseteq \mathcal{L}\left(? F, ? F, \Gamma^{\prime}\right) \cap \mathcal{L}(\Delta)$ as well as proofs $\pi_{1}^{\prime} \vdash ? F, ? F, \Gamma^{\prime}, C^{\prime}$ and $\pi_{2}^{\prime} \vdash C^{\prime \perp}, \Delta$ such that

$$
\frac{\frac{\pi_{1}^{\prime}}{\vdash ? F, ? F, \Gamma^{\prime}, C^{\prime}} \quad \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta}}{\vdash ? F, ? F, \Gamma^{\prime}, \Delta} \quad(\mathrm{Cut}) \quad \longrightarrow_{\mathrm{cut}}^{\star} \pi^{\prime} .
$$

By setting $C=C^{\prime}, \pi_{1}=\frac{\pi_{1}^{\prime}}{\vdash ? F, \Gamma^{\prime}, C^{\prime}} \quad$ (?c) and $\pi_{2}=\pi_{2}^{\prime}$ one gets:

$$
\begin{gathered}
\frac{\pi_{1}}{\frac{\vdash ? F, \Gamma^{\prime}, C}{\vdash ? F, \Gamma^{\prime}, \Delta} \frac{\pi_{2}}{\vdash C^{\perp}, \Delta}} \\
\frac{\pi_{1}^{\prime}}{\vdash ? F, ? F, \Gamma^{\prime}, C^{\prime}} \quad \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta} \\
\frac{\vdash ? F, ? F, \Gamma^{\prime}, \Delta}{\vdash ? F, \Gamma^{\prime}, \Delta}
\end{gathered} \quad \longrightarrow_{\text {cut }} \quad(\mathrm{Cut}) \quad \longrightarrow_{\mathrm{cut}}^{\star} \quad \pi .
$$

If the last rule is $(\forall)$, that is $\pi=\frac{\pi^{\prime}}{\frac{F, \Gamma^{\prime}, \Delta}{\vdash \forall x F, \Gamma^{\prime}, \Delta}} \quad(\forall) \quad x \notin \mathrm{FV}\left(\Gamma^{\prime}, \Delta\right)$ assuming $\Gamma=\forall x F, \Gamma^{\prime}$. By induction hypothesis, there is an interpolant $C^{\prime}$ such that $\mathcal{L}\left(C^{\prime}\right) \subseteq \mathcal{L}\left(F, \Gamma^{\prime}\right) \cap \mathcal{L}(\Delta)$ as well as proofs $\pi_{1}^{\prime} \vdash F, \Gamma^{\prime}, C^{\prime}$ and $\pi_{2}^{\prime} \vdash C^{\prime \perp}, \Delta$ such that

$$
\frac{\frac{\pi_{1}^{\prime}}{\vdash F, \Gamma^{\prime}, C^{\prime}} \quad \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta}}{\vdash F, \Gamma^{\prime}, \Delta} \quad(\mathrm{Cut}) \quad \longrightarrow_{\mathrm{cut}}^{\star} \pi^{\prime} .
$$

$$
\begin{aligned}
& \text { By setting } \left.C=\exists x \cdot C^{\prime}, \pi_{1}=\frac{\pi_{1}^{\prime}}{\vdash F, \Gamma^{\prime}, \exists x \cdot C^{\prime}} \stackrel{(\exists)}{\vdash \forall x F, \Gamma^{\prime}, \exists x \cdot C^{\prime}} \quad \text { ( } \forall\right) \text { and } \pi_{2}=\frac{\pi_{2}^{\prime}}{\vdash \forall x C^{\prime \perp}, \Delta} \quad(\forall) \text { one gets: } \\
& \frac{\pi_{1} \pi_{2}}{\vdash \forall x F, \Gamma^{\prime}, \Delta} \text { (Cut) } \\
& \left.\longrightarrow_{\mathrm{cut}} \frac{\frac{\pi_{1}^{\prime}}{\vdash F, \Gamma^{\prime}, \exists x \cdot C^{\prime}} \quad \text { (ヨ) } \frac{\pi_{2}^{\prime}}{\vdash \forall x C^{\prime \perp}, \Delta}}{\frac{\vdash F, \Gamma^{\prime}, \Delta}{\vdash \forall x F, \Gamma^{\prime}, \Delta}} \text { ( } \text { ( } \text { ( }\right) \text { ) } \\
& \longrightarrow_{\mathrm{cut}} \frac{\pi_{1}^{\prime} \pi_{2}^{\prime}}{\frac{\vdash F, \Gamma^{\prime}, \Delta}{\vdash \forall x F, \Gamma^{\prime}, \Delta}} \text { ( } \mathrm{Cut} \text { ) } \\
& \longrightarrow{ }_{\text {cut }}^{\star} \pi \text {. }
\end{aligned}
$$

If the last rule is $(\exists)$ ，that is $\pi=\frac{\pi^{\prime}}{\vdash F\{y / x\}, \Gamma^{\prime}, \Delta} \quad$（ $\exists$ ）assuming $\Gamma=\exists x F, \Gamma^{\prime}$ ．
In this case，Note that we treat only the case of a FO language containing no function symbols．
By induction hypothesis，there is an interpolant $C^{\prime}$ such that $\mathcal{L}\left(C^{\prime}\right) \subseteq \mathcal{L}\left(F\{y / x\}, \Gamma^{\prime}\right) \cap$ $\mathcal{L}(\Delta)$ as well as proofs $\pi_{1}^{\prime} \vdash F\{y / x\}, \Gamma^{\prime}, C^{\prime}$ and $\pi_{2}^{\prime} \vdash C^{\prime \perp}, \Delta$ such that

$$
\frac{\pi_{1}^{\prime}}{\frac{F F\{y / x\}, \Gamma^{\prime}, C^{\prime}}{\vdash F\{y / x\}, \Gamma^{\prime}, \Delta}} \frac{\pi_{2}^{\prime}}{\vdash C^{\prime \perp}, \Delta}(\mathrm{Cut}) \quad \longrightarrow_{\mathrm{cut}}^{\star} \pi^{\prime} .
$$

In this case，we reason by case on whether $y$ occurs in $\Gamma^{\prime}, \Delta$ ：
－If $y$ occurs in both，then we simply take $C=C^{\prime}$ as interpolant，$\pi_{1}=\frac{\pi_{1}^{\prime}}{\vdash \exists x F, \Gamma^{\prime}}$
and $\pi_{2}=\pi_{2}^{\prime}$ ．and we have $\mathcal{L}(C)=\mathcal{L}\left(C^{\prime}\right) \subseteq \mathcal{L}\left(F, \Gamma^{\prime}\right) \cap \mathcal{L}(\Delta)=\mathcal{L}\left(\exists x F, \Gamma^{\prime}\right) \cap \mathcal{L}(\Delta)$
－If $y$ occurs in $\Gamma^{\prime}$ but not in $\Delta$ ，then we set $C=\exists y C^{\prime}, \pi_{1}=\frac{\frac{\pi_{1}^{\prime}}{\exists x F, \Gamma^{\prime}, C^{\prime}}}{\exists x F, \Gamma^{\prime}, \exists y C^{\prime}}$ and $\pi_{2}=\frac{\pi_{2}^{\prime}}{\forall y C^{\prime \perp}, \Delta} \quad(\forall)$ one gets：

$$
\begin{aligned}
& \frac{\pi_{1} \pi_{2}}{\vdash \exists x . F, \Gamma^{\prime}, \Delta} \text { (Cut) } \quad \longrightarrow_{\text {cut }} \frac{\pi_{1}^{\prime}}{\frac{\vdash \exists x F, \Gamma^{\prime}, C^{\prime}}{\vdash \exists x F, \Gamma^{\prime}, \Delta} \pi_{2}^{\prime}} \\
& \longrightarrow_{\text {cut }} \frac{\pi_{1}^{\prime} \pi_{2}^{\prime}}{\frac{\vdash\{y / x\}, \Gamma^{\prime}, \Delta}{\vdash \exists x F, \Gamma^{\prime}, C^{\prime}}} \text { (ヨut) }
\end{aligned}
$$

$\longrightarrow{ }_{\text {cut }}^{\star} \pi$ ．
－If $y$ occurs in $\Delta$ but not in $\Gamma^{\prime}$ ，then we set $C=\forall y C^{\prime}, \pi_{1}=\frac{\pi_{1}^{\prime}}{\frac{\vdash \exists x F, \Gamma^{\prime}, C^{\prime}}{\vdash \exists x F, \Gamma^{\prime}, \forall y \cdot C^{\prime}}}{ }^{(\exists)}$（ $\forall$ ） and

$$
\begin{aligned}
& \pi_{2}=\frac{\pi_{2}^{\prime}}{\vdash \exists y C^{\prime \perp}, \Delta} \quad \text { (ヨ). One gets: } \\
& \frac{\pi_{1} \pi_{2}}{\vdash \exists x F, \Gamma^{\prime}, \Delta} \text { (Cut) } \longrightarrow_{\text {cut }} \frac{\pi_{1}^{\prime}}{\frac{\vdash \exists x F, \Gamma^{\prime}, C^{\prime}}{\vdash \exists x F, \Gamma^{\prime}, \Delta} \quad \pi_{2}^{\prime}} \text { (Cut) }
\end{aligned}
$$

$$
\begin{aligned}
& \longrightarrow{ }_{\text {cut }}^{\star} \pi \text {. }
\end{aligned}
$$

## B Linear embeddings of LJ and LK .

In the present section, we (sketch how to) deduce our proof-relevant statement of interpolation for LJ and LK as a direct corollary of the usual linear embeddings of classical and intuitionistic logics in linear logic:

- First one needs to extend the results of the previous section on proof-relevant interpolation to two-sided LL, which is clear. The only additional care amount to dealing properly with the notion of positive/negative subformula in order to obtain the refined Lyndon inteprolation result, but there is no difficulty in doing so;
- Second, consider some provable sequents $\Gamma \vdash_{\mathrm{LK}} \Delta\left(\right.$ resp. $\Gamma \vdash_{\mathrm{LJ}} A$ ) and their respective cut-free proofs under consideration $\pi_{\mathrm{LK}}$ (resp. $\pi_{\mathrm{LJ}}$ ) and consider a splitting of the sequents: $\Gamma^{\prime} \vdash \Delta^{\prime}$ and $\Gamma^{\prime \prime} \vdash \Delta^{\prime \prime}\left(\right.$ resp. $\Gamma^{\prime} \vdash$ and $\left.\Gamma^{\prime \prime} \vdash A\right)$;
- Consider the linear sequents and proofs corresponding to those sequents via linear translations which do not introduce additional cuts (some are described in our previous work $[36,35]$ and the linear translations of the splittings considered above.
- Apply proof-relevant LL interpolation to obtain a formula $C_{k}$ (resp. $C_{j}$ ) in the common vocabulary and proofs $\pi_{1}^{l}, \pi_{2}^{l}$ interpolating wrt. $C_{k}$ (resp. $C_{j}$ );
- By erasing all the linear information of $C_{k}$ (resp. $C_{j}$ ) and $\pi_{1}, \pi_{2}$ (which is called their skeletons), this provides us with the expected solution.

More details are provided in the following paragraphs.

## B. 1 Skeletons and decorations

The following section is standard, from original results by Danos, Joinet and Schellinx [15].

- Definition 21 (Skeleton). For $A$ an LL formula, we define $\operatorname{Sk}(A)$ inductively:

| $\operatorname{Sk}(A \otimes B)$ | $=\operatorname{Sk}(A) \wedge \operatorname{Sk}(B)$ | $\operatorname{Sk}(A \not B B)$ | $=\operatorname{Sk}(A) \vee \operatorname{Sk}(B)$ | $\operatorname{Sk}(!A)$ | $=\operatorname{Sk}(A)$ |
| ---: | :--- | ---: | :--- | ---: | :--- |
| $\operatorname{Sk}(A \& B)$ | $=\operatorname{Sk}(A) \wedge \operatorname{Sk}(B)$ | $\operatorname{Sk}(A \oplus B)$ | $=\operatorname{Sk}(A) \vee \operatorname{Sk}(B)$ | $\operatorname{Sk}(? A)$ | $=\operatorname{Sk}(A)$ |
| $\operatorname{Sk}(1)$ | $=\operatorname{Sk}(\top)=\mathrm{T}$ | $\operatorname{Sk}(\perp)$ | $=\operatorname{Sk}(0)=\mathrm{F}$ | $\operatorname{Sk}(a)$ | $=a$ |

Let $\pi$ be a two-sided LL proof of $\Gamma \vdash \Delta$. $\operatorname{Sk}(\pi)$ is the LK proof of $\operatorname{Sk}(\Gamma) \vdash \operatorname{Sk}(\Delta)$ obtained by the following recursive process by case analysis on the last rule $r$ of $\pi$ : (i) if $r \in\{(!\mathrm{p}),(? \mathrm{~d})\}$, then $\operatorname{Sk}(\pi)$ is the skeleton of the premise of $\pi ;(i i)$ otherwise, apply the corresponding rule with, for premises, the skeletons of the premises of $\pi$.

- Proposition 22. For any LL proof $\pi$ of $s, \operatorname{Sk}(\pi)$ is a $\operatorname{LK}$ proof of $\operatorname{Sk}(s)$.

A standard result of LL proof theory, developed by Danos, Joinet and Schellinx [15], is that there exist linear decorations for LK:

- Proposition 23. For any LK sequent $s$ and any LK proof $\pi$, there is a linear decoration of $\pi$, that is a LL proof $\pi^{d}$ such that $\operatorname{Sk}\left(\pi^{d}\right)=\pi$.


## B. 2 Interpolation for $L K$ and $L J$

Moreover, the skeleton maps cut-related LL-proofs to cut-related LK proofs (resp. LJ proofs).
Proof relevant interpolation for LJ and LK is therefore a direct and simple corollary and the above theory of linear decorations.

## C Details on interpolation as cut-introduction

We recall and provide further details on the notions of decorated and coherent proof:

- Definition 24. - $A$ decorated proof is an LL proof st. each sequent is equipped with a splitting.
- A coherent decorated proof is a decorated proof such that for each node, the splitting of the conclusion and of its premises is coherent wrt the ancestor relation: a formula belonging to the left (resp. right) component of the splitting has all its ancestors belonging to the left (resp. right) component of the splitting. More precisely:
- that each ancestor of an auxiliary formula belonging to the left (resp. right) component of the splitting belongs to the left (resp. right) component of the splitting of the corresponding premise;
- that each immediate subformula of a principal formula which belongs to the left (resp. right) component of the paritition, itself belongs to the left (resp. right) component of the splitting of its premises.

In a two-sided calculus, the coherence condition can be refined as follows:

1. each ancestor of an auxiliary formula belonging to the antecedent (resp. succedent) of left component of the splitting belongs to the antecedent (resp. succedent) of the left component of the splitting of the corresponding premise;
2. that each ancestor of an auxiliary formula belonging to the antecedent (resp. succedent) of right component of the splitting belongs to the antecedent (resp. succedent) of the right component of the splitting of the corresponding premise;
3. that each positive immediate subformula of a principal formula which belongs to the antecedent (resp. succedent) of the left component of the splitting, itself belongs to the antecedent (resp. succedent) of the left component of the splitting of its premises;
4. that each positive immediate subformula of a principal formula which belongs to the antecedent (resp. succedent) of the right component of the splitting, itself belongs to the antecedent (resp. succedent) of the right component of the splitting of its premises;
5. that each negative immediate subformula of a principal formula which belongs to the antecedent (resp. succedent) of the left component of the splitting, itself belongs to the succedent (resp. antecedent) of the left component of the splitting of its premisses;
6. that each negative immediate subformula of a principal formula which belongs to the antecedent (resp. succedent) of the right component of the splitting, itself belongs to the succedent (resp. antecedent) of the right component of the splitting of its premisses.

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We provide a summary of the transformations applied to a invariant finite representation in 731 Figure 6
(

Figure 6 Description of the interpolation process for pre-proofs $(J=\mu X . I[X, \ldots, X])$.

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[^0]:    1 Matthes [28] extends Čubrić' results to a natural deduction with general elimination rules and a corresponding term calculus while Nakazawa [25] considers interpolation in purely implicational fragments of intuitionistic logic and fins workarounds for the lack of interpolation in this setting. On the way, he considers various a sequent calculus a proves a result which as some similarities with our result but is weaker both in terms of the logical language which is restricted to implicative LJ and of the characterization of the equivalence between the interpolants and the original proofs.

[^1]:    2 Even though the condition for termination is far from optimal...

