On the denotation of circular and non-wellfounded proofs

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ABSTRACT

This paper investigates the question of denotational invariants of non-wellfounded and circular proofs of the linear logic with least and greatest fixed-points. Indeed, while non-wellfounded and circular proof theory made significant progress in the last twenty years, the corresponding denotational semantics is still underdeveloped.

A categorical semantics for non-wellfounded proofs is provided, building on the categorical axiomatization in [25]. Several properties of the semantics are then studied: its soundness, and the semantical content of the translations from finitary proofs to circular proofs and strongly valid circular proofs to finitary proofs. Then we will capture the syntactic validity criterion by considering a orthogonality construction [33] on the given categorical model. We also study two concrete models. The first is based on the category of sets and relations, and the second one is based on a notion of totality [25]. Finally, the paper focuses on circular proofs, trying to benefit from their regularity in order to define inductively the interpretation function. It is argued why the usual validity condition is too general for that purpose, while a fragment of circular proofs, strongly valid proofs, constitutes a well-behaved class for such an inductive interpretation.

KEYWORDS

linear logic, fixed-point logics, circular and non-wellfounded proofs, denotational semantics, category theory, proof theory, (co)induction

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1 INTRODUCTION

In the framework of logics providing induction and potentially coinduction (such as the μ-calculus, logics with inductive definitions, logics for Kleene algebras, etc...), circular and non-wellfounded proofs have gained growing attention over the past twenty years. Different proof systems have been considered for various logics: for classical logic [13–15], for intuitionistic logic [17], for linear logic [8, 22, 27, 42] as well as for linear-time or branching-time temporal logics [2, 21, 24, 35, 50, 50].

Beside non-wellfounded proof systems, there are also finitary proof systems that allow us to do inductive and coinductive reasoning. For instance, in the case of linear logic, Baelde and Miller considered an extension μMALL of multiplicative additive linear logic with induction and coinduction principles [3, 9] in the form of Park's rules. It is worth mentioning that actually those finitary proofs systems predate the circular ones in general. It seems that it is generally accepted that if we want to have a cut-elimination theorem for the finitary proof systems with an induction principle, then the price to pay is to lose the sub-formula property [37]. There are basically two ways to solve this, by considering either infinitary logic in the sense of [44, 47], or non-wellfounded proofs as above.

The relationship between finitary and non-wellfounded proof systems is an important and often difficult question, which remains open for a number of systems. In particular, in the substructural versions of the μ-calculus, it is not known whether the regular fragment of non-wellfounded proofs coincides with the finitary fragment. Berardi and Tatsuta showed [12] that in general circular and inductive proofs are not equivalent for the system of inductive definitions in classical logic for the first-order language [15]. It is also shown, by Simpson [45] on the one hand and Berardi and Tatsuta [11] on the other hand, that circular and inductive proofs are equivalent for classical logic when both systems (inductive and circular) contain Peano arithmetic. This question is still open for linear logic, and what we only know till now is that the provability of $\mu\mathsf{LL}_\infty$ circular proofs is strictly included in the provability of arbitrary $\mu\mathsf{LL}_\infty$ proofs based on the recent result by Das et al. [19].

However, one inclusion is clear, it proceeds by "unfolding" the (co)inductive inferences using the ability to build circular reasonings. In the case of μMALL, the finitary version of Park's rule [51] (the (ν_{rec}) rule) will be transformed to the following circular proof:

$$\dfrac{\overset{\pi_1}{\vdash \Delta, A} \qquad \overset{\pi_2}{\vdash A^\perp, F[A/\zeta]}}{\vdash \Delta, \nu\zeta.F}\;(\nu_{\mathsf{rec}}) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\vdash A^\perp, \nu\zeta.F}{\vdash (F[A/\zeta])^\perp, F[\nu\zeta.F/\zeta]}\;(\mathfrak{F}_F) \qquad \overset{\pi_2}{\vdash A^\perp, F[A/\zeta]}}{\vdash A^\perp, F[\nu\zeta.F/\zeta]}\;(\mathsf{cut})}{\vdash A^\perp, \nu\zeta.F}\;(\nu) \qquad \overset{\pi_1}{\vdash \Delta, A}}{\vdash \Delta, \nu\zeta.F}\;(\mathsf{cut})$$

Such translations are known to preserve provability. The present paper aims at clarifying the situation on operational properties of such translations to provide an evidence for the correctness of these translations from a Curry–Howard correspondence perspective. Hence we first need to a develop denotational semantics of $\mu\mathsf{LL}_\infty$.

Indeed, while the proof theory of circular proofs made progress in the last twenty years, their denotational semantics is still underdeveloped. Santocanale considered circular proofs in the framework of purely Additive linear logic, and he provided a categorical interpretation of circular proofs in μ-bicomplete categories [27, 42]. In this paper we will consider the full linear logic.

A categorical model of $\mu\mathsf{LL}$ is provided in [25] which is based on the standard notion of Seely category of classical linear logic and on strong functors acting on them. Baelde et al. [6] provided a denotational semantics for μMALL (finitary) proofs in the setting of Girard's ludics [32]: while considering finitary proofs, the interpretation considers an interpretation space made of infinitary objects (ludics' designs are sorts of Böhm trees) and the interpretation relies on both the unfolding of μMALL proofs and, when they prove a completeness result, a finitization process for designs.

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Contributions. This paper first revisits the syntactical relationship between the finitary and circular proofs in section 3. The translation from μMALL to circular proofs is extended to μLL. In the other direction, there is a translation from circular proofs to finitary ones under a condition so-called strongly valid proofs, and we have obtained a weaker condition to do this finitization. Then we investigate the question of denotational invariants of non-wellfounded proofs of linear logic with least and greatest fixed points.

A categorical semantics for non-wellfounded proofs is provided in section 4, by considering a Cpo structure on the categorical model of μLL in [25]. Then we have provided a refinement of our categorical model in section 5, based on the focused orthogonality construction given [33], to capture the syntactic validity criterion. We exhibit two simple instances of our settings. The first one is based on the category **Rel** of sets and relations, and the second one is an enrichment of **Rel** by considering sets equipped with an additional structure of *totality*: a non-uniform totality space (NUTS) is a pair $X = (|X|, \mathcal{T}(X))$ where $|X|$ is a set and $\mathcal{T}(X)$ is a set of subsets such that it coincides with its bidual for a duality expressed in terms of non-empty intersections.

This semantics is used to investigate the denotational content of the standard translation from finitary proofs to non-wellfounded ones: it is shown that the above mentioned translation from finitary proofs to circular ones is denotationally transparent (preserving semantics), suggesting that it is *the* correct translation (section 4.4). Moreover, the paper studies some properties of this semantics:

- the semantics is indeed sound in the sense that each proof of an infinite cut-reduction sequence of proofs converging to a cut-free valid proof has the same interpretation as its limit (section 4.2);
- it is also shown that valid proofs are interpreted as morphism in the focused orthogonality category, and hence as total elements of the semantics in **Nuts** (section 5.3).

In the case of the concrete model **Nuts**, although it is not true in general that the totality of the interpretation of a proof implies its validity, the notion of totality in NUTS provides a sort of maximal notion for validity as, intuitively, $\mathcal{T}(X)$ represents the total, that is, terminating computations of type X.

Even though it is still not clear if all circular proofs can be finitized, there is a proper fragment of circular proofs, called *strongly valid* proofs, that can be transformed to finite proofs [7, 22]. We will show that our semantics is transparent in the sense that the interpretation of the proofs are preserved via this transformation (section 6.1). Based on this result, we relax our assumption of having a Cpo-enriched category, and we provide a parameterized interpretation of strongly valid circular proofs by benefiting from their finitely presentable structure (section 6.2).

Slogan. This paper widens the ever-present "circular" certification between syntax and semantics to non-wellfounded proof theory in linear logic. Our semantics approves the design of the syntax by proving that the syntactical finitization does not change the semantics, and the syntax approves the design of the semantics by proving that the interpretations can be defined via syntactical unfolding.

2 BACKGROUND

Linear logic (LL) was introduced by Jean-Yves Girard in his seminal work [30]. LL is a refinement of both classical and intuitionistic logic taking its roots in the analysis of the denotational interpretation of System F in coherence spaces [29]. Contrary to classical logic LK, LL is a substructural logic: one does not have free access to the structural rules of weakening and contraction. More precisely, we can only weaken and contract formulas if they have been marked with the so-called exponential modalities.

The remainder of this section recalls how one can extend LL with least and greatest fixed points operators.

2.1 Syntax of formulas of linear logic with fixpoints of types

We assume to be given an infinite set of propositional variables $\mathcal{V}$ (ranged over by Greek letters $\zeta, \xi \dots$). We introduce a language of propositional LL formulas with least and greatest fixed points, called μLL pre-formulas:

$$A, B, \dots ::= 1 \mid \bot \mid A \otimes B \mid A \parr B \mid 0 \mid \top \mid A \oplus B \mid A \mathbin{\&} B \mid\ !A \mid\ ?A \mid \zeta \mid \mu\zeta\, A \mid \nu\zeta\, A$$

The notion of closed types is defined as usual, the two last constructions being the only binders. We refer to closed pre-formulas as μLL *formulas*. We can define three basic operations on formulas.

- *Substitution*: $A\,[B/\zeta]$, taking care of avoiding the capture of free variables (using α-conversion).
- *Negation* or *dualization*: defined by induction on formulas as usual for LL formulas plus $(\zeta)^\perp = \zeta$, $(\mu\zeta\, A)^\perp = \nu\zeta\,(A)^\perp$ and $(\nu\zeta\, A)^\perp = \mu\zeta\,(A)^\perp$. Clearly $A^{\perp\perp} = A$ for any formula.
- *Sub-formula*: We consider two notions of sub-formulas, the usual one and a notion of sub-formula which is specific to the μ-calculus, the Fischer-Ladner subformulas. Those are defined in Appendix A.

In the following sections, we shall consider two proof systems for deriving judgments concerning μLL formulas, a finitary proof system and a non-wellfounded one. Those proof systems derive *sequents* $\vdash \Gamma$ where Γ is an *ordered list* of μLL formulas.

Remark 1. *Using sequents as lists allows us to distinguish two different* occurrences *of the same formula in a sequent, by referring to their respective position in the sequent. The ability to distinguish occurrences is crucial to give a computational content to proofs and, in the following, it will even be required to define what is a* valid *non-wellfounded proof, using the notion of* threads.

The inference rule to be introduced in the following subsections will be equipped with a (pretty standard [16]) notion of formula ancestor, *relating for each inference, occurrences of formulas in the conclusion to occurrences of formulas in the premisses. The ancestry relation will be defined graphically in the proof system (as colored links) and will usually be kept implicit on examples unless useful, such as when exhibiting a validating thread. When we depict a line linking a list in the conclusion to the same list in the premise, we mean that each formula of the list is in relation with the formula in the same position in the other list.*

The identity and structural fragments:

$$\frac{}{\vdash F, F^\perp}\,(\mathrm{Ax}) \qquad \frac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta}\,(\mathrm{Cut}) \qquad \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\,(\mathrm{X})$$

The multiplicative fragment:

$$\frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\,(\parr) \qquad \frac{\vdash F, \Gamma \qquad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\,(\otimes) \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\,(\bot) \qquad \frac{}{\vdash 1}\,(1)$$

The additive fragment:

$$\frac{\vdash F, \Gamma \qquad \vdash G, \Gamma}{\vdash F \,\&\, G, \Gamma}\,(\&) \qquad \frac{\vdash A_i, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\,(\oplus_i) \qquad \frac{}{\vdash \top, \Gamma}\,(\top)$$

(no rule for 0)

The exponential fragment:

$$\frac{\vdash F, \Gamma}{\vdash ?F, \Gamma}\,(?\mathrm{d}) \qquad \frac{\vdash F, ?\Gamma}{\vdash !F, ?\Gamma}\,(!\mathrm{p}) \qquad \frac{\vdash \Gamma}{\vdash ?F, \Gamma}\,(?\mathrm{w}) \qquad \frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma}\,(?\mathrm{c})$$

Figure 1: Inference rules of LL

$$\frac{\vdash \Gamma, F\,[\mu\zeta\,.F/\zeta]}{\vdash \Gamma, \mu\zeta\,.F}\,(\mu) \qquad \frac{\vdash \Delta, A \qquad \vdash ?\Gamma, A^\perp, F\,[A/\zeta]}{\vdash \Delta, ?\Gamma, \nu\zeta\,.F}\,(\nu_{\mathrm{rec}})$$

Figure 2: Fixed-point inference rules of μLL

$$\frac{\vdash F[\mu\zeta.F/\zeta], \Gamma}{\vdash \mu\zeta.F, \Gamma}\,(\mu) \qquad \frac{\vdash F[\nu\zeta.F/\zeta], \Gamma}{\vdash \nu\zeta.F, \Gamma}\,(\nu)$$

Figure 3: Fixed-point inference rules of $\mu\mathsf{LL}_\infty$

2.2 Finitary μLL

In the present section, we will briefly describe the syntax of proofs of μLL [3]. The proof system of μLL, extends the usual one-sided sequent calculus of classical propositional LL [30], which are recalled in Figure 1, with the (μ) and (ν_{rec}) rules, given in Figure 2.

Example 2.1. As an example, consider the type of natural numbers $\mathsf{nat} = \mu\zeta\,.(1 \oplus \zeta)$ and its dual $\mathsf{nat}^\perp = \nu\zeta\,.(\bot \,\&\, \zeta)$. The following μLL proofs correspond respectively to the encoding of the natural numbers and of the successor function:

$$\pi_0 = \dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1 \oplus \mathsf{nat}}\,(\oplus_1)}{\vdash \mathsf{nat}}\,(\mu) \qquad \pi_{k+1} = \dfrac{\dfrac{\pi_k}{\vdash 1 \oplus \mathsf{nat}}\,(\oplus_2)}{\vdash \mathsf{nat}}\,(\mu) \qquad \pi_{succ} = \dfrac{\dfrac{\dfrac{}{\vdash \mathsf{nat}^\perp, \mathsf{nat}}\,(\mathrm{Ax})}{\vdash \mathsf{nat}^\perp, 1 \oplus \mathsf{nat}}\,(\oplus_2)}{\vdash \mathsf{nat}^\perp, \mathsf{nat}}\,(\mu)$$

2.3 Non-well-founded LL with fixed points ($\mu\mathsf{LL}_\infty$)

The syntax of $\mu\mathsf{LL}_\infty$ formulas is exactly the same as the one for μLL in 2.2. The inference rules of $\mu\mathsf{LL}_\infty$ is the extension of rules of [8, 22] with exponential rules of LL. In other words, the inference rules of $\mu\mathsf{LL}_\infty$ are the rules of LL [30] (see Figure 1) plus the two fixed-point inferences given in Figure 3.

Definition 2.2 ($\mu\mathsf{LL}_\infty$ pre-proofs). A $\mu\mathsf{LL}_\infty$ pre-proof is a possibly infinite tree, generated by the inference rules of $\mu\mathsf{LL}_\infty$. Among all $\mu\mathsf{LL}_\infty$ pre-proofs, the regular (or circular, or cyclic) proofs are the ones that have finitely many sub-trees.

Those circular proofs can be represented with finite proof-trees having back-edges or labels (see example below). Such a finite graph (ie a finite tree with back-edges) R can be unfolded to a $\mu\mathsf{LL}_\infty$ pre-proof $\mathsf{Unfold}(R)$. This unfolding is of course non-injective and given a pre-proof π, any R such that $\mathsf{Unfold}(R) = \pi$ is called a *finite representation* of π. The necessary technical apparatus for reasoning on those finite representations is detailed in Doumane's thesis [22]: we shall follow her definition and only recall the most important definitions useful for stating our results.

Example 2.3. The following proof corresponds to the function from nat to nat which returns the double of its input:

$$\pi_{double} = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1 \oplus \mathsf{nat}}\,(\oplus_1)}{\vdash \mathsf{nat}}\,(\mu)}{\vdash \bot, \mathsf{nat}}\,(\bot) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathsf{nat}^\perp, \mathsf{nat}}{\vdash \mathsf{nat}^\perp, 1 \oplus \mathsf{nat}}\,(\oplus_2)}{\vdash \mathsf{nat}^\perp, \mathsf{nat}}\,(\mu)}{\vdash \mathsf{nat}^\perp, 1 \oplus \mathsf{nat}}\,(\oplus_2)}{\vdash \mathsf{nat}^\perp, \mathsf{nat}}\,(\mu)}{\vdash \bot \,\&\, \mathsf{nat}^\perp, \mathsf{nat}}\,(\&)}{\vdash \mathsf{nat}^\perp, \mathsf{nat}}\,(\nu)$$

However, in general the pre-proofs can be unsound. For instance one can provide a pre-proof for any sequent $\vdash \Gamma$ (and in particular a pre-proof of the empty sequent $\vdash$) as follows:

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \nu\zeta\,.\zeta}\,(\nu)}{\vdash \nu\zeta\,.\zeta}\,(\nu) \qquad \dfrac{\dfrac{\vdots}{\vdash \Gamma, \mu\zeta\,.\zeta}\,(\mu)}{\vdash \Gamma, \mu\zeta\,.\zeta}\,(\mu)}{\vdash \Gamma}\,(\mathrm{cut})$$

In [8, 22], a criterion, called *validity condition*, is provided in order to distinguish proper proofs from pre-proofs. We only sum up this criterion here and provide some examples, and we refer to [8, 22, 39] for more details.

Definition 2.4. An infinite branch of a pre-proof π is a sequence $(\Gamma_i, j_i)_{i \in \omega}$ of pairs of sequents and indices, for $j_i \in \{1, 2\}$, such that Γ_0 is the root of π, Γ_{i+1} is the j_ith premises of Γ_i in the proof tree for each $i \in \omega$.

Definition 2.5. A *thread* on an infinite branch $\beta = (\Gamma_i, j_i)_{i \in \omega}$ is an infinite sequence of formula occurrences $t = (F_i)_{k \leq i \in \omega}$ such that for any $i \geq k$, $F_i \in \Gamma_i$ and F_{i+1} is an immediate ancestor of F_i.

A thread t is *stationary* if only finitely many of the F_i are principal[1] in Γ_i. We denote by $\mathsf{Inf}(t)$ the set of *recurring formulas*, that occur infinitely often in t.

With each infinite branch is associated a set of threads. Notice that there is not a unique thread in general (and there may be none).

For instance, the following proof has two threads: $\dfrac{\dfrac{\vdots}{\vdash \mu\zeta\,.\zeta, \nu\xi\,.\xi}\,(\nu)}{\vdash \mu\zeta\,.\zeta, \nu\xi\,.\xi}\,(\nu)$.

The threads are $t_1 = \mu\zeta\,.\zeta, \mu\zeta\,.\zeta, \mu\zeta\,.\zeta, \cdots$ and $t_2 = \nu\xi\,.\xi, \nu\xi\,.\xi, \nu\xi\,.\xi, \cdots$. Since the only rule applied in the proof is the (ν) rule, the formulas $\mu\zeta\,.\zeta$ are never principal, and the thread corresponding to the $\mu\zeta\,.\zeta$ is called *stationary*.

Now, we have all the required material to define the notion of valid threads and then valid proofs.

[1] By principal formula, we mean the one that the inference rule is applied on.

(a)

$$\cfrac{\cfrac{\cfrac{\vdash F \qquad \vdash G}{\vdash F \otimes G}\,(\otimes)}{\vdash G}\,(\nu)}{\vdash F}\,(\mu)$$

(b)

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\vdash F, G}{\vdash F, \bot, G}\,(\bot)}{\vdash F, G \oplus \bot, G}\,(\oplus_2)}{\vdash (F \parr (G \oplus \bot)), G}\,(\parr)}{\vdash 1 \oplus (F \parr (G \oplus \bot)), G}\,(\oplus_2)}{\vdash G, G}\,(\mu)}{\vdash F, G}\,(\nu)$$

Figure 4: Examples of valid and non valid pre-proofs.

Definition 2.6. A *valid thread* t is a non-stationary thread such that $\mathsf{Inf}(t)$ has as minimum (with respect to the usual sub-formula ordering) a ν-formula.

Appendix A provides details on $\mu\mathsf{LL}_\infty$ subformulas and the minimality invoked above.

Definition 2.7. A *valid* $\mu\mathsf{LL}_\infty$ *proof* (or $\mu\mathsf{LL}_\infty$ proof, for short) π is a pre-proof π such that any infinite branch contains a valid thread.

Remark 2 (On validity of finite representations). *The above notions of infinite branch, thread and validity on non-wellfounded proofs directly applies for the regular fragment and can naturally be adapted to* finite representations *of circular proofs. In that case, we shall, as in [22], consider respectively an* infinite path *in a finite representation (corresponding to an infinite branch), a* trace *(corresponding to a thread, ie. a sequence of ancestor-related formula occurrences of a finite representation) for an infinite path and of a* valid trace.

Doumane [22] proved the expected correspondence between valid traces and valid threads in the unfolding as well as validity of a finite representation R and of its unfolding $\mathsf{Unfold}(R)$, *which holds in the same way whether one considers locative occurrences as in [5, 8, 22] or sequents as ordered lists as in [39, 43], see [40] for details.*

On the other hand, using finite representations allows us to state some additional definitions, taking benefit from the circular structure, such as that of strongly connected component of a finite representation (which can be seen as corresponding to a class of infinite branches). While the above notion of validity of finite representation is invariant by unfolding, we shall see later in this paper a notion of validity, strong validity*, that is sensitive to the choice of a finite representation.*

We now examine some valid and non-valid pre-proofs.

Example 2.8. Let us consider Figure 4.(a) presenting a derivation of formula $F = \mu\zeta.(\nu\xi.(\zeta \otimes \xi))$ where $G = \nu\xi.(F \otimes \xi)$. The leftmost branch has a single thread that is $t = F, G, (F \otimes G), F, \cdots$, so, $\min(\mathsf{Inf}(t)) = F$. Hence this thread is not a valid thread, and there is no more thread on this branch. Hence this proof is not valid.

Let us consider another example in Figure 4.(b), with $F = \nu\zeta.\mu\xi.(1 \oplus (\zeta \parr (\xi \oplus \bot)))$ and $G = \mu\xi.(1 \oplus (F \parr (\xi \oplus \bot)))$. For the thread $t_2 = F, G, (1 \oplus (F \parr (G \oplus \bot))), (F \parr (G \oplus \bot)), F, \cdots$ we have $\min(\mathsf{Inf}(t_2)) = F$, since $F \leqslant_{\mathsf{sub}} G$. Hence t_2 is a valid thread and this proof is valid.

The set of primitive (single step) reduction rules of $\mu\mathsf{LL}_\infty$ are the ones for LL plus the following one together with the corresponding commutation rules (Figure 3.2 of [22]).

The proof
$$\cfrac{\cfrac{\cfrac{\pi}{\vdash \Gamma, F[\mu\zeta.F/\zeta]}}{\vdash \Gamma, \mu\zeta.F}\,(\mu) \qquad \cfrac{\cfrac{\pi'}{\vdash \Delta, F^\perp[\nu\zeta.F^\perp/\zeta]}}{\vdash \Delta, \nu\zeta.F^\perp}\,(\nu)}{\vdash \Gamma, \Delta}\,(\mathrm{cut})$$
reduces to
$$\cfrac{\cfrac{\pi}{\vdash \Gamma, F[\mu\zeta.F/\zeta]} \qquad \cfrac{\pi'}{\vdash \Delta, F^\perp[\nu\zeta.F^\perp/\zeta]}}{\vdash \Gamma, \Delta}\,(\mathrm{cut}).$$

Various cut-elimination theorems on non-wellfounded proofs are proved in [4, 8, 22] and especially of $\mu\mathsf{LL}_\infty$ itself [43] but the rest of the paper does not rely on those normalization results, so that they can safely be ignored. We end this section by stating the functoriality of $\mu\mathsf{LL}_\infty$ which we will use in Section 4.4:

Proposition 2.9. *Let $(\zeta, \xi_1, \ldots, \xi_k)$ be a list of pairwise distinct propositional variables containing all the free variables of a formula F and let $\overrightarrow{C} = (C_1, \ldots, C_k)$ be a sequence of closed formulas. Then the following rule is admissible in $\mu\mathsf{LL}_\infty$:*

$$\cfrac{\vdash ?\Gamma, A^\perp, B}{\vdash ?\Gamma, (F[A/\zeta, \overrightarrow{C}/\overrightarrow{\xi}])^\perp, F[B/\zeta, \overrightarrow{C}/\overrightarrow{\xi}]}\,(\mathfrak{F}_F)$$

Proof. The proof is done by induction on the formula F, and we refer to Definition 2.38 of [22] for details. The presence of exponentials does not modify the proof in any non trivial way [34]. ◻

3 CIRCULAR VS FINITARY PROOFS

Relating finitary proof and regular non-wellfounded proofs is notouriously difficult[11–14, 22, 45]. In this section, we will study the *syntactic relation* between the circular $\mu\mathsf{LL}_\infty$ proofs and $\mu\mathsf{LL}$ proofs by reviewing and extending known results about translations finitary and circular proof in $\mu\mathsf{LL}$. Their *semantical relation* will be studied in Sections 4.4 and 6.1.

3.1 Unfolding $\mu\mathsf{LL}$ proofs to circular proofs

As it is discussed in [22] for a wide class of fixed-point sequent calculi, provability of a sequent in a finitary sequent calculus with Park's rule entails its provability in the associated non-wellfounded sequent calculus. This can be done by translating a proof π of the finitary proof system to a circular proof, $\mathsf{Trans}(\pi)$, in the non-wellfounded proof system.

Here, we straightforwardly adapt the result from Doumane's thesis to $\mu\mathsf{LL}$. (Our version of $\mu\mathsf{LL}$ differs from the calculus considered in [22] as we use a more powerful (ν_{rec}) rule.)

Definition 3.1 ($\mathsf{Trans}(\pi)$). For any $\mu\mathsf{LL}$ proof π, we define by induction on the structure of π a $\mu\mathsf{LL}_\infty$ pre-proof $\mathsf{Trans}(\pi)$ deriving the same sequent as π. We just show the case of the (ν_{rec}) rule as the other ones are trivially defined homomorphically (See Appendix D for details):

$$\mathsf{Trans}\left(\cfrac{\cfrac{\pi_1}{\vdash \Delta, A} \qquad \cfrac{\pi_2}{\vdash ?\Gamma, A^\perp, F[A/\zeta]}}{\vdash \Delta, ?\Gamma, \nu\zeta.F}\,(\nu_{\mathrm{rec}})\right)$$
is the following circular pre-proof using the functoriality of formulas given in Section 2.3:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A^{\perp}, \nu\zeta\,.F}{\vdash ?\Gamma, (F\,[A/\zeta])^{\perp}, F\,[\nu\zeta\,.F/\zeta]}\,(\mathfrak{T}_F) \qquad \dfrac{\mathsf{Trans}\,(\pi_2)}{\vdash ?\Gamma, A^{\perp}, F\,[A/\zeta]}}{\vdash ?\Gamma, ?\Gamma, A^{\perp}, F\,[\nu\zeta\,.F/\zeta]}\,(\mathrm{cut})}{\vdash ?\Gamma, ?\Gamma, A^{\perp}, \nu\zeta\,.F}\,(\nu)}{\vdash ?\Gamma, A^{\perp}, \nu\zeta\,.F}\,(\mathrm{c}) \qquad \dfrac{\mathsf{Trans}\,(\pi_1)}{\vdash \Delta, A}}{\vdash \Delta, ?\Gamma, \nu\zeta\,.F}\,(\mathrm{cut})
$$

The following is proved similarly to Proposition 2.14 of [22].

Proposition 3.2. *For any μLL proof π, $\mathsf{Trans}\,(\pi)$ is a valid $\mu\mathsf{LL}_\infty$ proof of the same sequent.*

3.2 From circular to finitary proofs

The converse translation, that is going from circular proofs to finitary ones, is much more involved since one must find the appropriate co-inductive invariants for Park's rule (ν_{rec}) by looking at the circular proofs which does not contain such invariants.

We do not know of any general translation from $\mu\mathsf{LL}_\infty$ circular proofs into finitary ones (while we know, by [20] that there are non-wellfounded, non circular proofs of $\mu\mathsf{LL}_\infty$ proofs which cannot be finintized). However, there are some proper fragments of $\mu\mathsf{LL}_\infty$ for which the structure of validity conditions is simple enough so as to allow us to extract invariants. The simplest such fragment corresponds to Santocanale's original setting for circular proofs, that is $\mu\mathsf{ALL}_\infty$ in which any circular proof can be finitized to a $\mu\mathsf{ALL}$ proof. Such a finitization has been partially extended to the $\mu\mathsf{LL}_\infty$ as well as shown by Doumane [22]. She considers a fragment of *translatable circular proofs* characterized by the condition that each translatable proof has a circular representation such that along every infinite path (that is any infinite branch of its unfolding), there exists a *strongly valid trace* t, that is a trace such that exactly one formula of each recurring sequent of the *circular representation* is visited by t. Such a finitization was first considered in the study of an interpretation of $\mu\mathsf{MALL}$ in Ludics [7], and then used for the linear-time μ-calculus [23] and finally stated in a general way in Doumane's PhD [22] as the translatability condition.

In the following, we shall relax Doumane's condition and obtain a weaker but still sufficient condition to finitize $\mu\mathsf{LL}_\infty$ proofs: we therefore finitize more circular $\mu\mathsf{LL}_\infty$ proofs.

3.2.1 An extended notion of strong validity. In what follows we define a new notion of strongly valid trace (we keep the same terminology, referring to the condition of [22] as Doumane's strong validity) and a corresponding notion of strongly valid proof for which we prove a finitization result.

Definition 3.3 (trace-recurring formulas in a sequent). Given a circular representation R, a sequent s in R and a trace t on R, we define $\mathrm{Rec}(t, s)$ as the set of formula occurrences of s which are visited infinitely often by t: a formula occurrence F of s belongs to $\mathrm{Rec}(t, s)$ if there are infinitely many i such that $t_i = (s, j)$, with F being the occurrence $s(j)$.

Definition 3.4 (strong validity). Let R be a circular representation and p an infinite path on R. A trace $t = (s_i, j_i)_{i \in \omega}$ is said to be *strongly valid* if t is valid and if for every sequent s of R where t makes a progress (that is s is conclusion of a ν-rule unfolding the minimal recurring formula of t), $\mathrm{Rec}(t, s)$ is a *singleton*.

A finite representation R is *strongly valid*, if every infinite path in R admits a strongly valid trace. A circular pre-proof π is *strongly valid* is it admits a strongly valid finite representation.

Remark 3. *Let us stress the difference between the present notion of strong validity and Doumane's strong validity, stressing the impact of the relaxation we allow in terms of the circular proofs that are captures by our result. Doumane's definition of strong validity indeed imposes quite strong restrictions on the geometry of threads. For instance, it forces that if an $A \parr B$ formula is contributing to validity and is principal only one of A and B can contribute to validity (ie can support the trace), not both at the same time. Similarly, if $?A$ is visited by a trace and contracted, only one of the two copies of $?A$ can be visited by the trace and therefore can contributes to validity. More fundamentally, the type sort of multiplicative branching that can induce an interaction between two back-edges (that is they live in the same connected component) in Doumane's framework corresponds to multiplicative branching induced by the validating formula itself: no multiplicative branching can be caused by another formula of the sequent, which is a strong restriction on the multiplicative behaviour captured by the finitization result of [22].*

On the contrary, none of those restrictions are imposed by our notion of strong validity since only the progress *require to have a single validating formula. Of course this is still a restriction compared to the non restricted validity.*

3.2.2 Strong validity is a sufficient condition for finitization.

Proposition 3.5. *Let π be a circular pre-proof of $\vdash \Gamma$. If π is strongly valid, then there is a finite proof π^{fin} of $\vdash \Gamma$ in μLL.*

In the following, we will show how π^{fin} can be built: we first recall some results by Doumane that apply independently of the extension of the criterion and that are used for finitizing π and then explain our finitization process which generalize Doumane translatability criterion. Note that, while our construction extends very significantly her previous results [22], our proof follows the same ideas and does not present much difficulties.

Proposition 3.6 ([22], Proposition 2.1). *Let $\nu\zeta\ F$ be a μLL formula. Then, for any context Γ there is a formula I^F_Γ, called the invariant formula such that the following rules are derivable in μLL.*

$$
\dfrac{\vdash \Delta\,[\nu\zeta\,.F/\zeta]}{\vdash \Delta\,[I^F_\Gamma/\zeta]}\,(\mathrm{subst}) \qquad \dfrac{}{\vdash I^F_\Gamma, \Gamma}\,(\mathrm{close}) \qquad \dfrac{\vdash \Delta, F\,[I^F_\Gamma/\zeta]}{\vdash \Delta, I^F_\Gamma}\,(\mathrm{unfold})
$$

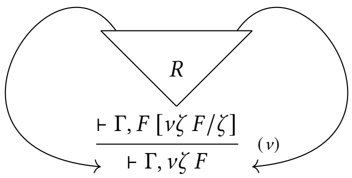
Moreover, (subst) *and* (unfold) *are derivable* circularly *in $\mu\mathsf{LL}_\infty$.*

The formula I^A_Γ in the proposition above is called the *invariant formula* and is defined as $\nu\zeta\,.(A \oplus (\parr\Gamma)^{\perp})$. In order to show the finitization results, we adopt the same measures on finite representations $\mathsf{size}(R)$ as it is done Definition 2.45 of [22]:

Definition 3.7. Let $\mathsf{nax}(\pi)$ and $\mathsf{elc}(\pi)$ be the numbers of the non-axiom rules in π and the numbers of the elementary cycles in π respectively. $\mathsf{size}(\pi)$ is defined as the pair $(\mathsf{elc}(\pi), \mathsf{nax}(\pi))$, ordered lexicographically.

The finitization process will consists in propagating the invariant formula in the circular proof in such a way that one can disable the back-edges, by using the derivable rule (close). The following proof pattern, $\Uparrow(R)$, will help us in that task and will serve later to interpret circular proofs:

Definition 3.8. Given a strongly valid proof π of $\vdash \Gamma, \nu\zeta\, F$ with a strongly connected finite representation as follows



(1)

and a strongly valid trace t of minimal recurring formula $\nu\zeta F$ that visits every sequent of R infinitely often, we define $\Uparrow(R)$, for R the premise of the finite representation of π, to be a finite representation of conclusion $\vdash \Gamma, F\left[I_\Gamma^F/\zeta\right]$ defined inductively on the structure of R (disregarding the back-edges from the inductive tree structure of course) and by case on the last rule as follows. The invariant that we will maintain is that if S has conclusion sequent $s = \vdash \Delta, \Sigma\,[\nu\zeta\, F/\zeta]$ where $\mathrm{Rec}(t, s) = \Sigma\,[\nu\zeta\, F/\zeta]$ (Σ may be empty), then $\Uparrow(S)$ has conclusion $\vdash \Delta, \Sigma\left[I_\Gamma^F/\zeta\right]$.

In the base case, S being reduced to a sequent $\vdash \Delta, \Sigma\,[\nu\zeta\, F/\zeta]$ which is the root of a back-edge, either the back-edge points to the root of R, in which case we know that $\Delta = \Gamma$ and $\Sigma = \zeta$ and we define $\Uparrow(S)$ to be $\dfrac{}{\vdash \Gamma, I_\Gamma^F}\,(\mathsf{close})$ or it points to some other node, in which case we do nothing but updating the sequent to $\vdash \Delta, \Sigma\left[I_\Gamma^F/\zeta\right]$.

Otherwise, assume $R = \dfrac{\dfrac{R'_l}{\vdash \Delta_l, \Sigma_l\,[\nu\zeta\, F/\zeta]\,, \Xi_l\,[\nu\zeta\, F/\zeta]} \quad l \in L}{\vdash \Delta, \Sigma\,[\nu\zeta\, F/\zeta]}\,(r)$

where the formulas of Δ_l are related to the formulas of Δ by the ancestor relation and the formulas of $\Sigma_l\,[\nu\zeta\, F/\zeta]\,, \Xi_l\,[\nu\zeta\, F/\zeta]$ are related to the formulas of $\Sigma\,[\nu\zeta\, F/\zeta]$ by the ancestor relation and if $\mathrm{Rec}(t, \vdash \Delta, \Sigma\,[\nu\zeta\, F/\zeta]) = \Sigma\,[\nu\zeta\, F/\zeta]$ and if we have that for any $l \in L$, $\mathrm{Rec}(t, \vdash \Delta_l, \Sigma_l\,[\nu\zeta\, F/\zeta]\,, \Xi_l\,[\nu\zeta\, F/\zeta]) = \Sigma_l\,[\nu\zeta\, F/\zeta]$.

- If r is the (ν) rule on some formula $F'\,[\nu\zeta\, F/\zeta]$ of $\Sigma\,[\nu\zeta\, F/\zeta]$, with $F' = \zeta$, then $\Sigma = F'$, L is a singleton and we write R' for the premise of R and we define $\Uparrow(R)$ as follows:

$$\dfrac{\dfrac{\Uparrow(R')}{\vdash \Gamma, F\left[I_\Gamma^F/\zeta\right]}}{\vdash \Gamma, I_\Gamma^F}\,(\mathsf{unfold})$$

- Otherwise, r is some $\mu\mathsf{LL}_\infty$ rule which does not correspond to a progress of t and we define $\Uparrow(R)$ as

$$\dfrac{\dfrac{\dfrac{\Uparrow(R'_l)}{\vdash \Delta_l, \Sigma_l\left[I_\Gamma^F/\zeta\right], \Xi_l\,[\nu\zeta\, F/\zeta]}}{\vdash \Delta_l, \Sigma_l\left[I_\Gamma^F/\zeta\right], \Xi_l\left[I_\Gamma^F/\zeta\right]}\,(\mathsf{subst}) \quad l \in L}{\vdash \Delta, \Sigma\left[I_\Gamma^F/\zeta\right]}\,(r)$$

Thanks to the previous definition, we can now easily prove Propositon 3.5.

Proof of Proposition 3.5. The proof goes by induction on $\mathrm{size}(\pi)$ with a base case when $\mathrm{elc}(\pi) = 0$: in that case, π has no back-edge as the finitization is the identity map. Otherwise, there are two cases: either the finite representation, R, associated to π is strongly connected as graph or it is not.

▹ Assuming that R is strongly connected. Then, there is an infinite path p that visits all the sequents of R and an associated strongly valid trace t of minimal formula $\nu\zeta\, A$. Wlog, assume that a sequent where the minimal formula of t has been unfolded, $\vdash \Gamma^\perp, \nu\zeta\, A$, is the conclusion of R. We are in the situation of Definition 3.8.

We can now consider the strongly valid finite representation $\Uparrow(R)$ of $\vdash \Gamma^\perp, A\left[I_\Gamma^A/\zeta\right]$. The complexity of the proof $\Uparrow(\pi)$ is strictly less than that of π, since $\mathrm{elc}(\Uparrow(\pi)) < \mathrm{elc}(\pi)$. So, by induction hypothesis, there is a $\mu\mathsf{LL}$ (finite) proof ρ of $\vdash \Gamma^\perp, A\left[I_\Gamma^A/\zeta\right]$. In this case, the π^{fin} is defined as follows where the rightmost proof of $\vdash I_\Gamma^A, \Gamma^\perp$ is the derived rule (close).

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp\left[(I_\Gamma^A)^\perp/\zeta\right], A\left[I_\Gamma^A/\zeta\right]}\,(\mathsf{ax}) \quad \dfrac{\rho}{\vdash \Gamma^\perp, A\left[I_\Gamma^A/\zeta\right]}}{\vdash A^\perp\left[(I_\Gamma^A)^\perp/\zeta\right] \,\&\, \Gamma^\perp, A\left[I_\Gamma^A/\zeta\right]}\,(\&)}{\vdash (I_\Gamma^A)^\perp, A\left[I_\Gamma^A/\zeta\right]}\,(\mu) \quad \dfrac{\dfrac{\dfrac{}{\vdash \Gamma, \Gamma^\perp}\,(\mathsf{ax})}{\vdash A\left[I_\Gamma^A/\zeta\right] \oplus \Gamma, \Gamma^\perp}\,(\oplus_2)}{\vdash I_\Gamma^A, \Gamma^\perp}\,(\nu)}{\vdash \Gamma^\perp, \nu\zeta\, A}\,(\nu_{\mathrm{rec}}) \tag{2}$$

▹ We now consider the case that R is not strongly connected, then there are two sequents $\vdash \Gamma$ and $\vdash \Delta$ such that there is no path from $\vdash \Gamma$ to $\vdash \Delta$. Let R_1 be the part of R which is reachable from $\vdash \Gamma$, and let R_2 be obtained from R by adding an auxiliary rule r on $\vdash \Gamma$ and taking the reachable part from the conclusion of R. R_1, R_2 respectively correspond to strongly valid circular proofs π_1 and π_2. Since R_1 does not have $\vdash \Gamma$ among its non-axiomatic rules, we have $\mathsf{nax}(R_1) < \mathsf{nax}(R)$, and then by induction hypothesis we have π_1^{fin} a finitization of π_1. By removing $\vdash \Gamma$ from R_2, we have $\mathsf{nax}(R_2) < \mathsf{nax}(R)$. Hence, by induction hypothesis, we have π_2^{fin}, a finitization of π_1. As π^{fin} is simply defined by plugging two proofs π_1^{fin} and π_2^{fin} at the assumption leaf introduced above. □

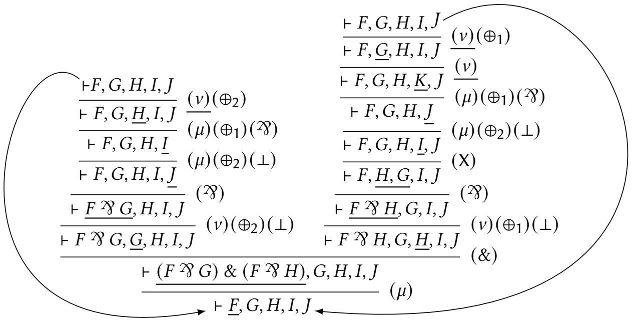
Remark 4. *Notice that the finite proof π^{fin} is not uniquely defined for a given strongly valid proof: it depends on a choice of a finite representation of π, of a set of strongly valid traces and, for each strongly connected component of R, of* one *strongly valid trace.*

Notice that our class of strongly valid proofs obviously contains all the unfoldings of finitary proofs (since they were already included in Doumane's translatable proofs that we extend).

3.2.3 *Beyond strong validity.* One of the main difficulties to extend Santocanale's approach to $\mu\mathsf{LL}_\infty$ can be seen with π_∞ defined in Figure 5. This is an example of a valid circular proof but having a quite involved validity structure, with three types of infinite branches (see Appendix H and [39] for more details). Santocanale's interpretation method relies on the possibility to identify a thread by a formula, π_∞ falls out of the scope of that method.

4 SEMANTICS OF NON-WELL-FOUNDED PRE-PROOFS

In this section we will show that by assuming a Cpo structure on the categorical model of $\mu\mathsf{LL}$ [25], one can obtain a categorical axiomatization of models of $\mu\mathsf{LL}_\infty$ pre-proofs. We first recall here the categorical model of $\mu\mathsf{LL}$.



with

$F = \mu X.((X \parr G) \& (X \parr H))$ $\qquad$ $I = \mu Z.((Z \parr J) \oplus \bot)$
$G = \nu X.(X \oplus \bot)$ $\qquad$ $J = \mu X.((K \parr X) \oplus \bot)$
$K = \nu Y.\mu Z.((Z \parr \mu X(Y \parr X) \oplus \bot) \oplus \bot)$ $\qquad$ $H = \nu X.(\bot \oplus X)$

Figure 5: Proof π_∞.

Definition 4.1 ([25]). A categorical model of μLL is a pair $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ where

- $\mathcal{L}$ is a model of linear logic, i.e. a Seely category [38].
- $\overrightarrow{\mathcal{L}} = (\mathcal{L}_n)_{n \in \mathbb{N}}$ where $\mathcal{L}_n$ is a class of strong functors $\mathcal{L}^n \to \mathcal{L}$, and $\mathcal{L}_0 = \mathrm{Obj}(\mathcal{L})$
- if $\mathbb{X} \in \mathcal{L}_n$ and $\mathbb{X}_i \in \mathcal{L}_k$ (for $i = 1, \ldots, n$) then $\mathbb{X} \circ \overrightarrow{\mathbb{X}} \in \mathcal{L}_k$
- the strong functors $\otimes$ and $\&$ belong to $\mathcal{L}_2$, the strong functor $!\_$ belongs to $\mathcal{L}_1$ and, if $\mathbb{X} \in \mathcal{L}_n$, then $(\mathbb{X})^\perp \in \mathcal{L}_n$
- and last, for all $\mathbb{X} \in \mathcal{L}_1$ the category $\mathbf{Coalg}_{\mathcal{L}}(\overline{\mathbb{X}})$ of coalgebras of the functor $\overline{\mathbb{X}}$ [2] has a final object. Moreover, for any $\mathbb{X} \in \mathcal{L}_{k+1}$, the associated strong functor $\nu\mathbb{X} : \mathcal{L}^k \to \mathcal{L}$ belongs to $\mathcal{L}_k$.

Although we refer to [25] for more details, we have brought some of the definitions in Appendix I.

Definition 4.2. A $\mu\mathsf{LL}_\infty$ *model* is a μLL model $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ where $\mathcal{L}$ is a bi-Cpo enriched category [3].

4.1 Interpreting formulas and proofs (outline)

The idea is to interpret a formula A with repetition-free sequence $\overrightarrow{\zeta} = (\zeta_1, \ldots, \zeta_k)$ of type variables containing all the free variables of A as an element in $\mathcal{L}^k$, and we denote it by $[\![A]\!]_{\overrightarrow{\zeta}}$. This interpretation is defined by induction on the formulas in the obvious way, for instance $[\![A \otimes B]\!]_{\overrightarrow{\zeta}} = \otimes \circ ([\![A]\!]_{\overrightarrow{\zeta}}, [\![B]\!]_{\overrightarrow{\zeta}})$ considering $\otimes \in \mathcal{L}_2$, and $[\![\nu\zeta\,.A]\!]_{\overrightarrow{\zeta}} = \nu([\![A]\!]_{\overrightarrow{\zeta},\zeta})$ [4]. Then one also has $[\![A^\perp]\!]_{\overrightarrow{\zeta}} = ([\![A]\!]_{\overrightarrow{\zeta}})^\perp$ up to a natural isomorphism which allows us to define other formula by De Morgan duality.

Let π be a $\mu\mathsf{LL}_\infty$ pre-proof of $\vdash \Gamma$. We want to interpret π as a morphsim in $\mathcal{L}(1, [\![\Gamma]\!])$. We first assume that in the inference rules of $\mu\mathsf{LL}_\infty$, we also have this rule: $\dfrac{}{\Gamma}\ (\Omega)$ for any sequence Γ, and we interpret this rule as the least element of $\mathcal{L}(1,1)$. We also assume that a finite $\mu\mathsf{LL}_\infty$ pre-proof can have the (Ω) rule. The interpretation of LL rules for instance in [38], and it is also given in Appendix B. So, we only need to say how we interpret the (ν) and (μ) rules:

$$\left[\!\!\left[\dfrac{\dfrac{\pi}{\vdash \Gamma, F\,[\mu\zeta\,.F/\zeta]}}{\vdash \Gamma, \mu\zeta\,.F}\ (\mu)\right]\!\!\right] = [\![\pi]\!] \qquad \left[\!\!\left[\dfrac{\dfrac{\pi}{\vdash \Gamma, F\,[\nu\zeta\,.F/\zeta]}}{\vdash \Gamma, \nu\zeta\,.F}\ (\nu)\right]\!\!\right] = [\![\pi]\!]$$

Then one can define the interpretation of a finite $\mu\mathsf{LL}_\infty$ pre-proof by induction on the structure of the proof.

Definition 4.3. Given a $\mu\mathsf{LL}_\infty$ pre-proof π of $\vdash \Gamma$, we define $[\![\pi]\!]$ as $\bigcup_{\rho \in \mathsf{fin}(\pi)} [\![\rho]\!]$ where $\mathsf{fin}(\pi)$ is the set of all finite sub-pre-proof of π (we are allowed to do this, since we added the (Ω) rule), and $\bigcup$ is the supremum of the directed subsets in $\mathcal{L}(1, [\![\Gamma]\!])$.

Remark 5. *The Cpo-enrichment gives us the unique fixed points, so Definition 4.3 is well defined.*

4.2 Soundness of the interpretation wrt cut-elimination

4.2.1 Soundness for one-step cut-elimination. We first prove that the semantic is preserved via the one-step cut reduction rules of $\mu\mathsf{LL}_\infty$.

Theorem 4.4. *Given two finite $\mu\mathsf{LL}_\infty$ proofs π and π' such that π' is obtained from π via an one-step cut-elimination rule, then $[\![\pi]\!] = [\![\pi']\!]$.*

See proof in Appendix F.1.

4.2.2 Soundness for Cauchy-sequences of cut-eliminations. One can define a natural metric d on the set of all finite $\mu\mathsf{LL}_\infty$ pre-proofs saying $d(\pi, \pi') = 0$ if two pre-proofs π and π' are identical, otherwise $d(\pi, \pi') = \frac{1}{2^k}$, where k is the length of the shortest position at which π and π' differ. Then we can see that indeed set of all infinite $\mu\mathsf{LL}_\infty$ pre-proofs is the metric completion of the finite proofs (Theorem C.5 of Appendix C). This is quite standard in the literature [10, 48, 49], however recorded the details of this development in Appendix C for the sake of self-containdness, and as it is nevertheless necessary for a precise definition of the semantics of non-wellfounded proofs.

Lemma 4.5. *Let (π_i) be a Cauchy sequence. Then $[\![\lim_{n\to\infty} \pi_i]\!] = \bigcup_i \bigcap_{j>i} [\![\pi_j]\!]$.*

See proof in Appendix F.2.

Theorem 4.6. *Let $(\pi_i)_{i \in \omega}$ be a Cauchy sequence such that $\forall i, j \in \omega$ we have $[\![\pi_i]\!] = [\![\pi_j]\!]$. Then $[\![\lim_{n\to\infty} \pi_i]\!]_{\mathbf{Rel}} = [\![\pi_0]\!]$.*

See proof in Appendix F.3.

And, we can now prove the soundness theorem for $\mu\mathsf{LL}_\infty$ as a direct conclusion of Theorem 4.4 and Theorem 4.6:

Corollary 4.7. *If π and π' are proofs of $\vdash \Gamma$ and π reduces to π' by the cut-elimination rules of $\mu\mathsf{LL}_\infty$, then $[\![\pi]\!] = [\![\pi']\!]$.*

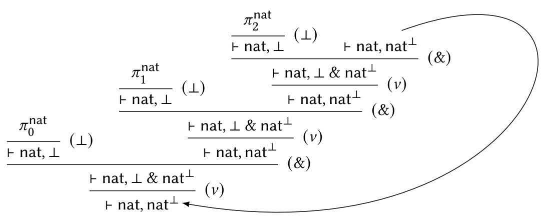
4.3 Rel as a concrete model of $\mu\mathsf{LL}_\infty$

Let $\mathbf{Rel}_n$ be the class of all n-ary strong functors F where $\overline{F}$ is a locally continuous and strict in the sense that it maps inclusions to inclusions, and for all $\overrightarrow{E}, \overrightarrow{F} \in \mathbf{Rel}^n$ and all directed set $D \subseteq \mathbf{Rel}^n(\overrightarrow{E}, \overrightarrow{F})$, one has $\mathbb{F}(\bigcup D) = \bigcup \{\mathbb{F}(\overrightarrow{s}) \mid \overrightarrow{s} \in D\}$. We know

[2] $\overline{\mathbb{X}}$ is the the underlying functor of the strong functor $\mathbb{X}$.
[3] A bi-Cpo is a Cpo that has infimum of directed subsets.
[4] We assume that the iso between νF and $F(\nu F)$ is always the identity as this holds in our concrete models. This assumption is highly debatable from the view point of category theory where the notion of equality of objects is not really meaningful. It will be dropped in a longer version of this paper.

that (**Rel**, $(\mathbf{Rel}_n)_{n\in\mathbb{N}}$) is a model of μLL [25]. Since **Rel** is a bi-cpo enriched category, (**Rel**, $(\mathbf{Rel}_n)_{n\in\mathbb{N}}$) is also a model of $\mu\mathsf{LL}_\infty$.

Let us look at an example. Consider the following circular proof $\pi_{\equiv_3}$ which correspond to the function on natural numbers which sends n to $n \mod 3$:



The interpretation of π_k^{nat} in **Rel** is, up to an iso, the natural number k, and we denote it by $\underline{k}$, i.e $[\![\pi_k^{\mathsf{nat}}]\!]_{\mathbf{Rel}} = \underline{k}$. To compute interpretation of $\pi_{\equiv_3}$, we need to take supremum of the interpretation of all finite sub pre-proofs. For example, imagine that in the proof $\pi_{\equiv_3}$ above, we do a Ω rule instead of the back-edge, and called this proof σ. Then we have

$$[\![\sigma]\!]_{\mathbf{Rel}} = \{(\underline{2}, (2, (2, (1, *))), (\underline{1}, (2, (1, *))), (\underline{0}, (1, *)))\}$$

That is to say, up to an iso, we have $[\![\sigma]\!]_{\mathbf{Rel}} = \{(\underline{2}, \underline{2}), (\underline{1}, \underline{1}), (\underline{0}, \underline{0})\}$. If we do one more step, we will see that $(\underline{0}, \underline{3}) \in [\![\pi_{\equiv_3}]\!]_{\mathbf{Rel}}$. So, one can see that $[\![\pi_{\equiv_3}]\!]_{\mathbf{Rel}} = \{(\underline{n}, \underline{m}) \mid \underline{n} = \underline{m} \mod 3\}$.

4.4 On the relation between the interpretation of finite proofs and their circular correspondent

Our main goal in the section is to prove that our semantics is preserved via the operation Trans () introduced in Section 3. Notice that if we associate a system of equations on the morphisms of the category $\mathcal{L}$ to a circular proof, then the interpretation given in Definition 4.3 is actually a solution of the corresponding system of equations. We will use the following lemma, which is well-known in the literature on fixed points of functors [1, 41], in the proof of Theorem 4.9.

Lemma 4.8. *Let A be an object of a category $\mathcal{A}$ and let $f_1, f_2 \in \mathcal{A}(A, \nu\mathcal{F})$. If there exists $l \in \mathcal{A}(A, \mathcal{F}(A))$ such that $\mathcal{F}(f_i)\, l = f_i$ for $i = 1, 2$, then $f_1 = f_2$.*

See proof in Appendix E.1.

Theorem 4.9. *Let $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ be a $\mu\mathsf{LL}_\infty$ model, and π be a μLL proof. Then we have $[\![\pi]\!] = [\![\mathsf{Trans}\,(\pi)]\!]$.*

See proof in Appendix E.2.

5 VALID PROOFS AS TOTAL ELEMENTS

In the previous section, we provided the interpretation of pre-proofs, and we did not consider whether a proof is valid. In this section, we will provide a refinement of our $\mu\mathsf{LL}_\infty$ model based on orthogonality construction given in [33], and we show that valid proofs will be interpreted as morphisms in the orthogonality category where the orthogonality relation satisfies a property called the focused orthogonality.

5.1 Preliminaries on orthogonality categories

We first recall some definitions here, and refer to [33] for more details. Let $\mathcal{L}$ be a *-autonomous category with monoidal units 1 and $\perp$. An orthogonality relation is a family of subsets

$$\perp_c \subseteq \mathcal{L}(1, c) \times \mathcal{L}(c, \perp)$$

indexed by objects $c \in \mathcal{L}$ and verifying some compatibility conditions with respect to the linear logic structure [33]. For a subset $X \subseteq \mathcal{L}(1, c)$, its orthogonal $X^\perp$ is $X^\perp := \{y : c \to \perp \mid \forall x \in X\, (x \perp_c y)\}$. And dually, for a subset $Y \subseteq \mathcal{L}(c, \perp)$, we have $Y^\perp := \{x : 1 \to c \mid \forall y \in Y\, (x \perp_c y)\}$. Finally we denoted by $\mathcal{D}(c)$ the set $\{X \subseteq \mathcal{L}(1, c) \mid X = X^{\perp\perp}\}$, and one can see that $\mathcal{D}(c)$ is a complete lattice. In this paper, we will restrict to the special case where the orthogonality relation arises from a distinguished subset $\perp\!\!\!\perp \subset \mathcal{L}(1, \perp)$, referred to as a pole, as follows:

$$\perp_c := \{(x, y) \in \mathcal{L}(1, c) \times \mathcal{L}(c, \perp) \mid y \circ x \in \perp\!\!\!\perp\}$$

Then we define the *focused orthogonality category* [33] as follows: The focused orthogonality category $O_{\perp\!\!\!\perp}(\mathcal{L})$ of a category $\mathcal{L}$ with $\perp\!\!\!\perp \subset \mathcal{L}(1, \perp)$ has objects given by pairs (c, X) with $c \in \mathcal{L}$ and $X \in \mathcal{D}(c)$, and morphisms $f : (c, X) \to (d, Y)$ given by morphisms $f : c \to d$ in $\mathcal{L}$ such that $\forall x \in X.\ f \circ x \in Y$.

5.2 Semantics of $\mu\mathsf{LL}_\infty$ in $O_{\perp\!\!\!\perp}(\mathcal{L})$

5.2.1 Interpretation of formulas. Given a closed $\mu\mathsf{LL}_\infty$ formula A, we denoted by $[\![A]\!]_{O_{\perp\!\!\!\perp}(\mathcal{L})}$ the interpretation of A in $O_{\perp\!\!\!\perp}(\mathcal{L})$. So, $[\![A]\!]_{O_{\perp\!\!\!\perp}(\mathcal{L})}$ is a pair $([\![A]\!]_{\mathcal{L}}, O([\![A]\!]_{O_{\perp\!\!\!\perp}(\mathcal{L})}))$ where $O([\![A]\!]_{O_{\perp\!\!\!\perp}(\mathcal{L})}) \in \mathcal{D}([\![A]\!]_{\mathcal{L}})$.

Let $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ be a $\mu\mathsf{LL}_\infty$ model with a pole $\perp\!\!\!\perp \subseteq \mathcal{L}(1, \perp)$. We know how to interpret the LL formulas in $O_{\perp\!\!\!\perp}(\mathcal{L})$ using Theorem 54 of [33]. For the fixpoints formulas $\mu\zeta\, A$ and $\nu\zeta\, A$, we know, by induction hypothesis, that $[\![A]\!]_{O_{\perp\!\!\!\perp}(\mathcal{L})}$ exists and it is lifting of the functor $[\![A]\!]_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$ in the following sense where U is the forgetfull functor:

$$\begin{tikzcd} O_{\perp\!\!\!\perp}(\mathcal{L}) \arrow[r, "{[\![A]\!]_{O_{\perp\!\!\!\perp}(\mathcal{L})}}"] \arrow[d, "U"'] & O_{\perp\!\!\!\perp}(\mathcal{L}) \arrow[d, "U"] \\ \mathcal{L} \arrow[r, "{[\![A]\!]_{\mathcal{L}}}"'] & \mathcal{L} \end{tikzcd}$$

Now using Corollary 3.4 of [26], we know that the initial algebra and final colalgebra of the endofucntor $[\![A]\!]_{O_{\perp\!\!\!\perp}(\mathcal{L})}$ exist, and we take them respectively as the interpretation of $\mu\zeta\, A$ and $\nu\zeta\, A$ [5].

As we are going to use the construction of $[\![\mu\zeta\, A]\!]_{O_{\perp\!\!\!\perp}(\mathcal{L})}$ and $[\![\mu\zeta\, A]\!]_{O_{\perp\!\!\!\perp}(\mathcal{L})}$ in section 5.2.2 (namely in the proof of Theorem 5.6), we summarize it here. Let (c, X) be an object of $O_{\perp\!\!\!\perp}(\mathcal{L})$. We denote by $([\![A]\!]_{\mathcal{L}}(c), \Phi_{[\![A]\!]}(X))$ the $[\![A]\!]_{O_{\perp\!\!\!\perp}(\mathcal{L})}((c, X))$ where $\Phi_{[\![A]\!]}(X) \in \mathcal{D}([\![A]\!]_{\mathcal{L}}(c))$. We consider a map $\Theta_{[\![A]\!]} : \mathcal{D}(\nu[\![A]\!]_{\mathcal{L}}) \to \mathcal{D}(\nu[\![A]\!]_{\mathcal{L}})$ as follows: if $X \in \mathcal{D}(\nu[\![A]\!]_{\mathcal{L}})$, then $O((\nu[\![A]\!]_{\mathcal{L}}, X)) \in \mathcal{D}([\![A]\!]_{\mathcal{L}}(\nu[\![A]\!]_{\mathcal{L}})) = \mathcal{D}(\nu[\![A]\!]_{\mathcal{L}})$, and we set $\Theta_{[\![A]\!]}(X) = O((\nu[\![A]\!]_{\mathcal{L}}, X))$. One can see that the map $\Theta_{[\![A]\!]}$ is a monotone operator, and hence by Knaster-Tarski's Theorem, it has least and greatest fixpoints. Hence we interpret $\nu\zeta\, A$ (resp. $\mu\zeta\, A$) as $(\nu[\![A]\!]_{\mathcal{L}}, \mathsf{gfp}(\Theta_{[\![A]\!]}))$ (resp. $(\mu[\![A]\!]_{\mathcal{L}}, \mathsf{lfp}(\Theta_{[\![A]\!]}))$).

More explicitly, one can see the interpretation of $\nu\zeta\, .A$ by a transfinite induction considering the sequence $(U_\alpha^A)_{\alpha\in\mathsf{Ord}}$ of the elements of $\mathcal{D}([\![\nu\zeta\, .A]\!])$ defined as follows: $U_0^A = \top(\mathcal{D}([\![\nu\zeta\, .A]\!]))$ where $\top(\mathcal{D}([\![\nu\zeta\, .A]\!]))$ is the largest element in the complete lattice $\mathcal{D}([\![\nu\zeta\, .A]\!])$, and $U_{\alpha+1}^A = O([\![A]\!]_{\mathcal{L}}([\![\nu\zeta\, .A]\!]_{\mathcal{L}}, U_\alpha^A))$ for the successor

[5] To have a more simple notation, we have only provided the interpretation of formulas with a single free variable. One can do it for any formulas in the obvious way.

ordinals, and $U^A_\delta = \bigcap_{\alpha<\delta} U^A_\alpha$ for the limit ordinals, and finally, there is an ordinal λ such that $U_\lambda = U_{\lambda+1}$, and we use λ_A for the least such ordinal.

To have simpler notation, we use the notation U_α (and U_λ) freely without mentioning the formula. One can find what the corresponding formula is from the context.

5.2.2 Interpretation of proofs. In section 4.1, we defined the interpretation of a pre-proof π of $\vdash \Gamma$ as a morphism $[\![\pi]\!]$ in $\mathcal{L}(1, [\![\Gamma]\!])$. In this section, we will prove that if the proof π is a valid proof, then $[\![\pi]\!]$ is a morphism in the orthogonality category $O_{\perp\!\!\!\perp}(\mathcal{L})$. In this section, when we write $[\![A]\!]$ for a $\mu\mathsf{LL}_\infty$ formula A, we mean $[\![A]\!]_{O_{\perp\!\!\!\perp}(\mathcal{L})}$.

The proof method is similar to the proof of soundness of LKID^ω in [13]. However the system of [13] is classical logic with inductive definitions, and their proof is for a Tarskian semantics. We need to adapt that proof in two aspects: considering $\mu\mathsf{LL}_\infty$ instead of LKID^ω, and trying to deal with a denotational semantics instead of a Tarskian semantics. The adaptation for $\mu\mathsf{LL}_\infty$ is somehow done in [22], since there is soundness theorem for $\mu\mathsf{MALL}_\infty$ with respect to the truncated truth semantics (a Tarskian semantics). So, basically, the main point of our proof is turning a Tarskian soundness theorem into a denotational soundness theorem.

We first borrowed the following definition from [22].

Definition 5.1. The marked formulas of $\mu\mathsf{LL}_\infty$ are defined as follows where α is an ordinal:

$$\begin{aligned} A, B, \ldots &:= 1 \mid 0 \mid \perp \mid \top \mid A \oplus B \mid A \otimes B \mid A \,\&\, B \mid A \parr B \mid \\ &\quad ?A \mid !B \mid \zeta \mid \mu\zeta.F \mid \nu^\alpha\zeta.F \end{aligned} \tag{3}$$

We denote by A° the label-stripped formula A. The interpretation of $\nu^\alpha\zeta.F$ in $O_{\perp\!\!\!\perp}(\mathcal{L})$ is $[\![\nu^\alpha\zeta.F]\!] = ([\![\nu\zeta.F]\!]_{\mathcal{L}}, U_\alpha)$, and the other marked formulas are interpreted as usual.

Proposition 5.2. *Let A be a $\mu\mathsf{LL}_\infty$ formula. Then we have $[\![\overline{A}]\!] = [\![A]\!]$ where $\overline{A}$ is the marked formula, obtained from A by marking every ν binder of A by the ordinal λ_A.*

The proof of this proposition is obvious.

Lemma 5.3. *If A is a $\mu\mathsf{LL}_\infty$ formula and $t \notin O([\![\nu^\alpha\zeta.F]\!])$, then there exists an ordinal $\gamma < \alpha$ such that $t \notin O([\![F[\nu^\gamma\zeta.F/\zeta]]\!])$.*

See proof in Appendix F.4.

Lemma 5.4. $O([\![F[\mu\zeta.F/\zeta]]\!]) = O([\![\mu\zeta.F]\!])$.

See proof in Appendix F.5

Lemma 5.5. *If π is a proof of $\vdash \Gamma$ and $[\![\pi]\!] \notin O(([\![\Gamma]\!]))$, then*

(1) π has an infinite branch $\gamma = (\vdash \Gamma_i)_{i\in\omega}$ such that $[\![\pi_i]\!] \notin O(([\![\Gamma_i]\!]))$ where π_i is the sub-proof of π rooted in $\vdash \Gamma_i$;

(2) and there exists a sequence of functions $(f_i)_{i\in\omega}$ where f_i maps all formulas D of Γ_i to a marked formula $f_i(D)$ such that

- *$(f_i(D))^\circ = D$,*
- *one can write $\Gamma_i = \Gamma'_i, C$,*
- *and there exists $x \in O([\![(f_i(\Gamma'_i))^\perp]\!])$ such that $[\![\pi_i]\!].x \notin O([\![f_i(C)]\!])$ where $\Gamma'_i = A^i_1, \cdots, A^i_{n_i}$ and $[\![(f_i(\Gamma'_i))^\perp]\!] = ([\![f_i(A^i_1)]\!])^\perp \otimes \cdots \otimes ([\![f_i(A^i_{n_i})]\!])^\perp$.*

See proof in Appendix F.6.

Proof. The idea is essentially to construct the infinite branch γ inductively using properties of orthogonality. $\square$

Now, we can state and prove our main result of this section.

Theorem 5.6. *If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!] \in O([\![\Gamma]\!])$.*

Proof. Let us assume $[\![\pi]\!] \notin O([\![\Gamma]\!])$. We can then apply Lemma 5.5 to obtain an infinite branch $(\vdash \Gamma_i)_{i\in\omega}$ and a sequence $(f_i)_{i\in\omega}$ satisfying properties 1 and 2 of Lemma 5.5. By the definition of valid proof (Definition 2.7), there exists a valid thread $t = (F_i)_{i\in\omega}$ for the infinite branch $(\vdash \Gamma_i)_{i\in\omega}$. Let $\nu\zeta\, F$ be the minimal formula formula of t. So, there are infinitely many times in t that we use a ν rule to unfold $\nu\zeta\, F$. Let $(i_k)_{k\in\omega}$ be the sequence of indices where $\nu\zeta\, F$ gets unfolded. Then $\nu\zeta\, F$ in the sequent Γ_{i_k} is sub-occurrence of $\nu\zeta\, F$ in the sequent $\Gamma_{i_{k'}}$ for $k \geqslant k'$. By the property 2 of Lemma 5.5, $f_{i_k}(\nu\zeta\, F) = \nu^{\alpha_k}\zeta.f_{i_k}(F)$. Therefore, by the property 2 of Lemma 5.5 and by the construction of the f_i in the proof of Lemma 5.5, the sequence $(\alpha_k)_{k\in\omega}$ is strictly decreasing. As this contradicts the well-foundedness property of the ordinals we obtain the required contradiction and conclude that $[\![\pi]\!] \in O([\![\Gamma]\!])$. $\square$

We denote by $1_{O_{\perp\!\!\!\perp}(\mathcal{L})}$ the unit $(1, O(1))$ of the tensor in the category $O_{\perp\!\!\!\perp}(\mathcal{L})$.

Corollary 5.7. *If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!] \in O_{\perp\!\!\!\perp}(\mathcal{L})(1_{O_{\perp\!\!\!\perp}(\mathcal{L})}, [\![\Gamma]\!])$.*

See proof in Appendix J

Remark 6. *The fact that we have considered focused orthogonality is important in our work, as we use it a lot in the proof of Lemma 5.5. This assumption is also essential in the construction of fixpoints in [26].*

Remark 7. *The category $O_{\perp\!\!\!\perp}(\mathcal{L})$ is not necessarily a $\mu\mathsf{LL}_\infty$ model in the sense of Definition 4.2, as it can be a non cpo-enriched category. We will see an example of this in Section 5.3. Nonetheless, the interpretation of $\mu\mathsf{LL}_\infty$ proofs are the same in both categories, i.e. $[\![\pi]\!]_{O_{\perp\!\!\!\perp}(\mathcal{L})} = [\![\pi]\!]_{\mathcal{L}}$.*

5.3 Nuts as a concrete model of $\mu\mathsf{LL}_\infty$

If we consider the category **Rel** and the pole $\perp\!\!\!\perp_{\mathbf{Rel}} = \{\{\mathrm{id}\}\}$, the category $O_{\perp\!\!\!\perp_{\mathbf{Rel}}}(\mathbf{Rel})$ is the category of non-uniform totality spaces (**Nuts**) studied in [25]. Explicitly, for a set A and a subset $X \subseteq \mathbf{Rel}(1, A) = \mathcal{P}(A)$, one has:

$$X^\perp = \{u' \subseteq A \mid \forall u \in \mathcal{T}\ u \cap u' \neq \emptyset\}$$

An object of **Nuts** is a a pair $X = (|X|, \mathcal{T}(X))$ where $|X|$ is a set, and $\mathcal{T}(X)$ is a totality candidate on $|X|$, that is, a $\uparrow$-closed subset of $\mathcal{P}(|X|)$ [25]. And we have $t \in \mathbf{Nuts}(X, Y)$ if $t \in \mathbf{Rel}(|X|, |Y|)$ and $\forall u \in \mathcal{T}(X)\ (\cdot tu \in \mathcal{T}(Y))$. As a direct conclusion of Theorem 5.6, we have the following corollary which says that the valid proofs will be interpreted as total elements.

Corollary 5.8. *If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!] \in \mathcal{T}([\![\Gamma]\!])$.*

One might think of the following statement as the converse of Corollary 5.8. If π is a pre-proof of the sequent $\vdash \Gamma$ such that $[\![\pi]\!] \in \mathcal{T}([\![\Gamma]\!])$, then π is a valid proof. This statement is not necessarily true, and there are many counterexamples indeed. For instance, take $F = \mu\zeta.(\perp \,\&\, (\zeta \parr \zeta))$ and $G = \nu\xi.(1 \oplus (\xi \parr \xi))$ and the pre-proofs π defined in Figure 6, where $\pi_{\Gamma;G}$ is defined (corecursively) on the right of the figure.

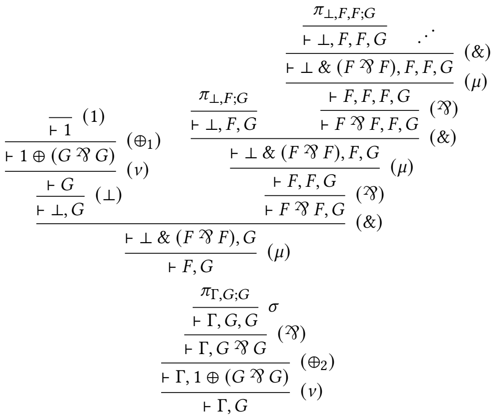


Figure 6: Proofs π and $\pi_{\Gamma;G}$

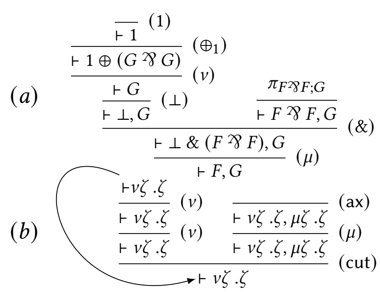


Figure 7: Non-valid proofs with total interpretations.

This pre-proof is not valid, since there is no valid thread in the rightmost branch. The interpretation of π in **Rel** is $[\![\pi]\!]_{\mathbf{Rel}} = \{((1,*),(1,*))\}$. However, $[\![\pi]\!]_{\mathbf{Rel}} \in \mathcal{T}([\![F \parr G]\!])$.

Notice that there are two ways to see that $[\![\pi]\!]_{\mathbf{Rel}} \in \mathcal{T}([\![F \parr G]\!])$. One can compute the interpretation of the formula $F \parr G$ in **Nuts**. And one can also provide a valid proof π' of $\vdash F, G$ such that $[\![\pi]\!]_{\mathbf{Rel}} = [\![\pi']\!]_{\mathbf{Rel}}$. Consider indeed the pre-proof π' of Figure 7 (a). This proof π' is a valid proof, since the thread $t = 1 \oplus (G \parr G), G \parr G, G, \cdots$ is a valid thread ($\min(\mathsf{Inf}(t)) = G$). We also have $[\![\pi']\!]_{\mathbf{Rel}} = \{((1,*),(1,*))\}$, and hence using Theorem 5.6, we know that $[\![\pi]\!] = [\![\pi']\!] \in \mathcal{T}([\![F \parr G]\!])$. The proof given in Figure 7 (b) is another example of non-valid proof whose interpretation is total. This examples differs however from π' (the proof given in Figure 7 (a)). It is true that this pre-proof does not respect the validity criterion with respect to the criterion of [8, 22]. However this proof is considered as a valid proof in a more recent work [5].

6 ON THE SEMANTICS OF CIRCULAR PROOFS

The semantics of the previous section allows us to interpret both general non-wellfounded and circular proofs, but it presents a drawback: in the case of circular proofs, the approximation semantics completely disregards the circularity of the proof objects.

In the present section, we will discuss what are the challenges and how to proceed to achieve those goals. We will also see that for a fragment of circular proofs, we can use the circularity of the proof tree to define the interpretation, following Santocanale's approach.

One of the main difficulties to extend Santocanale's approach to $\mu\mathsf{LL}_\infty$ can be seen in the example of π_∞ presented in Figure 5, page 7. Indeed, Santocanale's interpretation method strongly relies on the possibility to identify a thread by a formula, therefore π_∞ falls out of the scope of that method.

Two natural options are either (i) to disregard validity in interpreting circular proofs, as we did for non-well-founded proofs in previous sections, or (ii) to constrain the validity condition to make Santocanale's method usable. We discuss the second option below by considering strongly valid proofs introduced in Section 3.

6.1 Relating the interpretation of strongly valid proofs and their finitizations

We first want to show that the interpretation of the strongly valid circular proofs (section 3) are the same as the interpretation of their finitizations in any $\mu\mathsf{LL}_\infty$ model.

Lemma 6.1. *Let $\vdash \Gamma^\perp, \nu\zeta\, A$ be a $\mu\mathsf{LL}$ provable sequent. Then there is a unique morphism $\phi_A \in \mathcal{L}([\![\nu\zeta\, A]\!], [\![I^A_\Gamma]\!])$ such that it satisfies the following square:*

$$\begin{tikzcd}
{[\![\nu\zeta\, A]\!]} \arrow[r, "="] \arrow[dd, "\phi_A"'] & {[\![A]\!]([\![\nu\zeta\, A]\!])} \arrow[d, "{[\![A]\!](\phi_A)}"] \\
 & {[\![A]\!]([\![I^A_\Gamma]\!])} \arrow[d, "\mathsf{in}_1"] \\
{[\![I^A_\Gamma]\!]} \arrow[r, "="] & {[\![A]\!]([\![I^A_\Gamma]\!]) \oplus [\![\Gamma]\!]}
\end{tikzcd}$$

where I^A_Γ is the invariant formula (see Proposition 3.6), and in_1 is the first injection.

See proof in Appendix K.

Lemma 6.2. *Let π be a strongly connected and strongly valid proof of $\vdash \Gamma^\perp, \nu\zeta\, A$ where the last inference rule is the (ν) rule. Then $[\![\Uparrow(\pi)]\!]$ is the following morphism:*

$$[\![\Gamma]\!] \xrightarrow{[\![\pi]\!]} [\![\nu\zeta\, A]\!] \simeq [\![A]\!]([\![\nu\zeta\, A]\!]) \xrightarrow{[\![A]\!](\phi_A)} [\![A]\!]([\![I^A_\Gamma]\!])$$

See proof in Appendix L.

Theorem 6.3. *Let π be a strongly valid proof of $\vdash \Gamma^\perp, \nu\zeta\, A$. In any $\mu\mathsf{LL}_\infty$ model we have: $[\![\pi^{\mathsf{fin}}]\!] = [\![\pi]\!]$.*

Proof. The proof is by induction on $\mathsf{size}(\pi)$. We only provide here the case that π is strongly connected, and the full proof is provided in Appendix M.

▹ We first assume that π is strongly connected. Then, there is an infinite path p that visits all the sequents of π. Let t be a trace of p, and, without loss of generality, let $\vdash \Gamma^\perp, \nu\zeta\, A$ be the sequent where the minimal formula of t has been unfolded. Graphically, π is shown in Equation (1), page 6.

We now consider the $\mu\mathsf{LL}_\infty$ proof $\Uparrow(\pi)$ of $\vdash \Gamma^\perp, A\left[I^A_\Gamma/\zeta\right]$. The complexity of the proof $\Uparrow(\pi)$ is strictly less than that of π, since $\mathsf{elc}(\Uparrow(\pi)) < \mathsf{elc}(\pi)$. So, by induction hypothesis, there is a $\mu\mathsf{LL}$ (finite) proof ρ of $\vdash \Gamma^\perp, A\left[I^A_\Gamma/\zeta\right]$ such that $[\![\rho]\!] = [\![\Uparrow(\pi)]\!]$. In this case, π^{fin} is defined as in Equation (2), page 6.

Let f be the interpretation of the proof of $\vdash (I^A_\Gamma)^\perp, \nu\zeta\, A$. The morphism f satisfies the following universal property:

$$\begin{tikzcd}
\llbracket I^A_\Gamma \rrbracket = \llbracket A\left[I^A_\Gamma/\zeta\right]\rrbracket \oplus \Gamma \arrow[r, "\langle \mathsf{Id}, \llbracket\rho\rrbracket\rangle"] \arrow[d, "f"] & \llbracket A\rrbracket(\llbracket I^A_\Gamma\rrbracket) \arrow[dl, "\llbracket A\rrbracket(f)"] \\
\llbracket \nu\zeta\, A\rrbracket = \llbracket A\rrbracket(\llbracket\nu\zeta\, A\rrbracket) &
\end{tikzcd}$$

By Lemma 6.2, we have $\llbracket\rho\rrbracket = \llbracket A\rrbracket(\phi_A) \circ \llbracket\pi\rrbracket$, and hence

$$\begin{tikzcd}
\llbracket I^A_\Gamma \rrbracket \arrow[r, "f"] \arrow[d, "="] & \llbracket \nu\zeta\, A\rrbracket \arrow[dd, "="] \\
\llbracket A\rrbracket(\llbracket I^A_\Gamma\rrbracket) \oplus \llbracket\Gamma\rrbracket \arrow[d, "\langle \mathsf{Id}, \llbracket A\rrbracket(\phi_A)\circ\llbracket\pi\rrbracket\rangle"'] & \\
\llbracket A\rrbracket(\llbracket I^A_\Gamma\rrbracket) \arrow[r, "\llbracket A\rrbracket(f)"] & \llbracket A\rrbracket(\llbracket\nu\zeta\, A\rrbracket)
\end{tikzcd}$$

Moreover, we have the following diagram by Lemma 6.1:

$$\begin{tikzcd}
\llbracket \nu\zeta\, A\rrbracket \arrow[r, "\phi_A"] \arrow[dd, "="] & \llbracket I^A_\Gamma \rrbracket \arrow[d, "="] \\
& \llbracket A\rrbracket(\llbracket I^A_\Gamma\rrbracket) \oplus \llbracket\Gamma\rrbracket \arrow[d, "\langle \mathsf{Id}, \llbracket A\rrbracket(\phi_A)\circ\llbracket\pi\rrbracket\rangle"] \\
\llbracket A\rrbracket(\llbracket\nu\zeta\, A\rrbracket) \arrow[r, "\llbracket A\rrbracket(\phi_A)"] & \llbracket A\rrbracket(\llbracket I^A_\Gamma\rrbracket)
\end{tikzcd}$$

Hence, we have:

$$\begin{tikzcd}
\llbracket \nu\zeta\, A\rrbracket \arrow[r, "\phi_A"] \arrow[d, "="] & \llbracket I^A_\Gamma \rrbracket \arrow[r, "f"] & \llbracket \nu\zeta\, A\rrbracket \arrow[d, "="] \\
\llbracket A\rrbracket(\llbracket\nu\zeta\, A\rrbracket) \arrow[r, "\llbracket A\rrbracket(\phi_A)"] & \llbracket A\rrbracket(\llbracket I^A_\Gamma\rrbracket) \arrow[r, "\llbracket A\rrbracket(f)"] & \llbracket A\rrbracket(\llbracket\nu\zeta\, A\rrbracket)
\end{tikzcd}$$

So, we have $\llbracket A\rrbracket(f \circ \phi_A) = f \circ \phi_A$. By the universal property of $\llbracket\nu\zeta\, A\rrbracket$, we conclude that $f \circ \phi_A = \mathsf{Id}$.

Since $\llbracket\pi^{\mathsf{fin}}\rrbracket = f \circ \mathsf{in}_2$, we have the following using Lemma 6.2:

$$\begin{tikzcd}
\llbracket\Gamma\rrbracket \arrow[r, "\llbracket\pi^{\mathsf{fin}}\rrbracket"] \arrow[d, "\mathsf{in}_2"] & \llbracket \nu\zeta\, A\rrbracket \arrow[dd, "="] \\
\llbracket A\rrbracket(\llbracket I^A_\Gamma\rrbracket) \oplus \llbracket\Gamma\rrbracket = \llbracket I^A_\Gamma\rrbracket \arrow[d, "\langle \mathsf{Id}, \llbracket A\rrbracket(\phi_A)\circ\llbracket\pi\rrbracket\rangle"] & \\
\llbracket A\rrbracket(\llbracket I^A_\Gamma\rrbracket) \arrow[r, "\llbracket A\rrbracket(f)"] & \llbracket A\rrbracket(\llbracket\nu\zeta\, A\rrbracket)
\end{tikzcd}$$

As $(\langle \mathsf{Id}, \llbracket A\rrbracket(\phi_A) \circ \llbracket\pi\rrbracket\rangle) \circ \mathsf{in}_2 = \llbracket A\rrbracket(\phi_A) \circ \llbracket\pi\rrbracket$. Hence the following square commutes:

$$\begin{tikzcd}
\llbracket\Gamma\rrbracket \arrow[r, "\llbracket\pi^{\mathsf{fin}}\rrbracket"] \arrow[d, "\llbracket\pi\rrbracket"] & \llbracket \nu\zeta\, A\rrbracket \arrow[dd, "="] \\
\llbracket \nu\zeta\, A\rrbracket \arrow[d, "\llbracket A\rrbracket(\phi_A)"] & \\
\llbracket A\rrbracket(\llbracket I^A_\Gamma\rrbracket) \arrow[r, "\llbracket A\rrbracket(f)"] & \llbracket A\rrbracket(\llbracket\nu\zeta\, A\rrbracket)
\end{tikzcd}$$

We have $\llbracket A\rrbracket(f) \circ \llbracket A\rrbracket(\phi_A) = \llbracket A\rrbracket(f \circ \phi_A) = \mathsf{Id}$, since $f \circ \phi_A = \mathsf{Id}$. Therefore, we conclude that $\llbracket\pi\rrbracket = \llbracket\pi^{\mathsf{fin}}\rrbracket$.

□

Corollary 6.4. *Let π be a strongly valid $\mu\mathsf{LL}_\infty$ proof. Then $\llbracket\pi\rrbracket = \llbracket\pi^{\mathsf{fin}}\rrbracket$ where the interpretations of proofs are in any $\mu\mathsf{LL}_\infty$ model.*

Proof. We can always suppose wlog. that the conclusion of π is $\vdash \Gamma^\perp, \nu\zeta\, A$. So, by Theorem 6.3, we have $\llbracket\pi\rrbracket = \llbracket\pi^{\mathsf{fin}}\rrbracket$. □

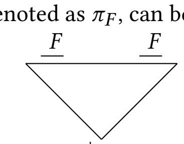
6.2 Interpreting strongly valid circular proofs

Let π be a strongly valid proof. Till now, we have two following ways to interpret π:

(1) In a $\mu\mathsf{LL}_\infty$ model: as we did in section 4.1.
(2) In a $\mu\mathsf{LL}$ model: By Proposition 3.5, one can first finitize π, and then will interpret the finitized proof π^{fin}.

By Corollary 6.4, we have seen that these two interpretations are the same. In this section, we will provide a direct way to interpret π in a $\mu\mathsf{LL}$ model.

Let $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ be a $\mu\mathsf{LL}$ model. We want to interpret a strongly valid proof π by induction on $\mathsf{size}(\pi)$ in $\mathcal{L}$. The general idea to interpret any valid circular proof π is first to consider two cases. If π is not strongly connected, we can always interpret it by induction on $\mathsf{size}(\pi)$. If π is strongly connected, we first choose a trace t for the infinite path p that visits all the sequents of π. Let $\nu\zeta\, A$ be the minimal formula of t. We then choose a sequent $\vdash \Gamma^\perp, \nu\zeta\, A$ such that the formula $\nu\zeta\, A$ has been unfolded. We suppose without loss of generality that the conclusion of π is $\vdash \Gamma^\perp, \nu\zeta\, A$. Graphically, π is what is described in Figure 1. We first discard all the backedges from the leaves of π to its root, and close them by a same assumption F. The resulting proof, denoted as π_F, can be shown as follows:



$\vdash \Gamma^\perp, \nu\zeta\, A$

If we take a morphism $f \in \mathcal{L}(\llbracket\Gamma\rrbracket, \llbracket\nu\zeta\, A\rrbracket)$ as the interpretation of F, we have, by induction hypothesis, the interpretation of π_F as a morphism in $\mathcal{L}(\llbracket\Gamma\rrbracket, \llbracket\nu\zeta\, A\rrbracket)$. So, considering F as a parameter, one can obtain from π a morphism f_π in $\mathcal{L}(C \otimes !(\llbracket\Gamma\rrbracket \multimap \llbracket\nu\zeta\, A\rrbracket), \llbracket\Gamma\rrbracket \multimap \llbracket\nu\zeta\, A\rrbracket)$ where we take C as the parameters coming from the assumptions of π. By analysing the proof of Theorem 6.3, we now want to show that the equation $f_\pi(C \otimes x) = x$ has a solution in $\mathcal{L}$, that is to say a morphism in $\mathcal{L}(\llbracket\Gamma\rrbracket, \llbracket\nu\zeta\, A\rrbracket)$, denoted by $\mathsf{fix}(f_\pi)$, such that $f_\pi(C \otimes \mathsf{fix}(f_\pi)) = \mathsf{fix}(f_\pi)$. To define $\mathsf{fix}(f_\pi)$, we first consider the $\mu\mathsf{LL}_\infty$ proof $\Uparrow(\pi)$ of $\vdash \Gamma^\perp, A\left[I^A_\Gamma/\zeta\right]$, and by induction hypothesis, we have $\llbracket\Uparrow(\pi)\rrbracket$. So, we have $\langle \mathsf{Id}, \llbracket\Uparrow(\pi)\rrbracket\rangle \in \mathcal{L}(\llbracket I^A_\Gamma\rrbracket, \llbracket A\rrbracket(\llbracket I^A_\Gamma\rrbracket))$. By the universal property of the final co-algebra $\llbracket\nu\zeta\, A\rrbracket$, there is a unique morphim $f \in \mathcal{L}(\llbracket I^A_\Gamma\rrbracket, \llbracket\nu\zeta\, A\rrbracket)$. Finally, we take $\mathsf{fix}(f_\pi)$ as $f \circ \mathsf{in}_2$ where $\mathsf{in}_2 \in \mathcal{L}(\llbracket\Gamma\rrbracket, \llbracket I^A_\Gamma\rrbracket)$.

As we saw, the interpretation of π, described above, depends on some choices such as choosing the validating trace t and choosing the sequent $\vdash \Gamma^\perp, \nu\zeta\, A$. We do not know whether changing those parameters, we obtain the same interpretation. Nevertheless, we can prove that if those $\mu\mathsf{LL}$ models are built on top of a $\mu\mathsf{LL}_\infty$ model, then the semantics does not depend on our choice of parameters:

Theorem 6.5. *Let* $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ *be a* $\mu\mathsf{LL}_\infty$ *model, and* π *be a strongly valid* $\mu\mathsf{LL}$ *proof. Then* $[\![\pi^{\mathsf{fin}}]\!] = [\![\pi]\!]$ *where the interpretations of proofs are in* $O_{\perp\!\!\perp}(\mathcal{L})$.

See proof in Appendix N.

Remark 8. *Notice that, as it is mentioned in Remark 7, there are focused orthogonality categories that are not a* $\mu\mathsf{LL}$ *model.*

7 CONCLUSION

In this paper, we studied the non-wellfounded proof system $\mu\mathsf{LL}_\infty$ from a Curry-Howard perspective, by providing a denotational semantics of $\mu\mathsf{LL}_\infty$. We also studied both syntactical and semantical relationships between finitary and circular proofs.

$\mu\mathsf{LL}_\infty$ models as a semantics of $\mu\mathsf{LL}_\infty$ pre-proofs. We first showed that any $\mu\mathsf{LL}$ model $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ such that $\mathcal{L}$ is a bi-cpo enriched category is a sound model of $\mu\mathsf{LL}_\infty$; **Rel** the category of sets and relations is such an example. More precisely, we interpret closed formulas and pre-proofs of $\mu\mathsf{LL}_\infty$ in $\mathcal{L}$, and prove that the semantics is preserved via a possibly infinite reduction sequence of cut-elimination rules.

Focused orthogonality categories as semantics of $\mu\mathsf{LL}_\infty$ valid proofs. We studied the focused orthogonality construction to capture the syntactic validity criterion and provided another concrete model of $\mu\mathsf{LL}_\infty$ based on the category **Nuts** of non-uniform totality spaces and relations preserving totality, which is not the case, in general, for pre-proofs. Although the interpretation of proofs in both models of **Rel** and **Nuts** are the same, one can obtain more information by looking at the interpretation in **Nuts** as we showed that any valid proof will be interpreted as a total element.

Semantics of strongly valid circular proofs. Benefiting from a finite representation of the circular proofs, we have provided a parameterized interpretation of strongly valid proofs in any $\mu\mathsf{LL}$ model (not only $\mu\mathsf{LL}_\infty$ models), and shown that the semantics is independent of the parameters for focused orthogonality categories.

Syntactical account of the relationship between finitary and circular proofs. We have extended the syntactical relation between finitary and circular proofs in the following sense. To go from finitary to circular proofs, we have considered the extension of full linear logic with fixed points from [25]. To go from circular to finitary proofs, we have relaxed Doumane's translatability condition [22], obtaining a weaker but still sufficient condition to finitize a proper fragment of circular proofs.

Semantical account of the relationship between finitary and circular proofs. We looked at the syntactical relationship between finitary and circular proofs through the magnifying glasses of Curry-Howard-Lambek correspondence. It is shown that the translation from finitary proofs to circular ones is sound (*i.e.* it preserves the semantics), bringing evidence of the computational soundness of this translation. Moreover, it is shown that the semantics is preserved via the finitization procedure.

Related works. Santocanale and Fortier considered circular proofs in the framework of purely Additive linear logic, and they provided a categorical interpretation of circular proofs in μ-bicomplete categories [27, 27, 42]. On the one hand, we have provided categorical axiomatizations to treat non-wellfounded proofs in full linear logic, but on the down side, we only benefit from the finitely presentable structure of strongly valid circular proofs and not all valid circular proofs, this is for future work.

Clairambault investigated in [17, 18] the game with totality semantics of an extension of intuitionistic logic with least and greatest fixed points in a finitary setting (independently of [36] and [28]). Although there is no infinitary logic in his paper, the interpretation of the fixed-point rules are very close to the interpretation of their unfolding in a infinitary system. First, one can ask how we can extend his semantics to deal with circular proofs, and then to see if the semantics is also preserved via both translation of from finitary to circular proofs and in the other direction. Another interesting question is to see if there is a Kleisli-like connection between our semantics and his work.

Along game-semantical models, Baelde et al. [6] provided a denotational semantics for $\mu\mathsf{MALL}$ (finitary) proofs in the setting of Girard's ludics [32]. One can as well see that their interpretations of fixed-point rules is built based on ludics designs which are infinitary objects. Moreover, the completeness result relies on finitization of infinitary objects. Although there is no infinitary logic in his paper, the finitization given by Doumane [22] is indeed generalizing finitization of designs. An essential limitation there is the difficulty to handle exponentials in ludics.

We conclude by mentioning some directions for future work.

One question could be seeking for a complete denotational model of $\mu\mathsf{LL}_\infty$ in the sense of Girard and Streicher [31, 46]. This could be useful to tackle the Brotherston-Simpson's conjecture for $\mu\mathsf{LL}$ (this conjecture says that inductive proofs and circular proofs have the same provability) as well as a *proof-relevant/denotational* version of the conjecture which would read as follows (the converse of this conjecture is Theorem 4.9):

Conjecture 7.1 (Semantical Brotherston-Simpson's conjecture). *Let* $\vdash \Gamma$ *be a* $\mu\mathsf{LL}$ *sequent and* π *be a circular* $\mu\mathsf{LL}_\infty$ *proof of* $\vdash \Gamma$. *There exists a* $\mu\mathsf{LL}$ *(finite) proof* π' *of* $\vdash \Gamma$ *such that* $[\![\pi]\!] = [\![\pi']\!]$.

As we saw in section 6.2, the semantics of strongly valid proofs are defined with respect to some parameters. We would like to investigate if our semantics depends on those parameters. So, the first question is how to define in a unique way the interpretation of strongly valid proofs in a $\mu\mathsf{LL}$ model. And secondly, we are wondering if we can strengthen Theorem 6.3 by considering $\mu\mathsf{LL}$ models instead of $\mu\mathsf{LL}_\infty$ models, namely the following conjecture:

Conjecture 7.2. *Let* π *be a strongly valid* $\mu\mathsf{LL}$ *proof. Then* $[\![\pi^{\mathsf{fin}}]\!] = [\![\pi]\!]$ *where the interpretations of proofs are in any* $\mu\mathsf{LL}$ *model.*

Some non-valid $\mu\mathsf{LL}_\infty$ have a total interpretation. A natural question is to understand what sort of information can be obtained from a total interpretation, if not syntactic validity. We saw in the paper that, for functions from nat to nat, the totality of nat $\multimap$ nat is all total relations on natural numbers; as a consequence it is not possible for a non-terminating program of type nat $\multimap$ nat to have a total interpretation in **Nuts**. A natural (but difficult) question is whether this can be lifted to all $\mu\mathsf{LL}_\infty$ types. The same was asked by Girard for 2nd-order types [29] and it is still an open problem.

There is a bigger notion of validity on proofs in [5], called *bouncing validity*. So, a natural question is whether the focused orthogonality categories captures this notion of validity, i.e extending Theorem 5.6 to bouncing validity.

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A ON THE TWO NOTIONS OF SUBFORMULAS

In the following, we shall consider two notions of sub-formulas, the usual one and a notion of sub-formula which is specific to the μ-calculus, the Fischer-Ladner subformulas.

Definition A.1. The sub-formula relation on $\mu\mathsf{LL}_\infty$ is defined as follows:

- $A * B \to_{\mathsf{sub}} A$ and $A * B \to_{\mathsf{sub}} B$ where $*$ is a binary LL connective.
- $@A \to_{\mathsf{sub}} A$ where @ is either ! or ?.
- $\sigma\zeta\, F \to_{\mathsf{sub}} F$ where σ is either ν or μ.

G is a subformula of F when $F \to^{\star}_{\mathsf{sub}} G$.

Notice that the usual sub-formula relation is an ordering, so, we write $A \leqslant_{\mathsf{sub}} B$ if A is sub-formula of B, i.e, we have $B \to^{\star}_{\mathsf{sub}} A$.

Definition A.2. We define the relation $\to_{\mathsf{FL}}$ on formulas as follows:

- $A * B \to_{\mathsf{FL}} A$ and $A * B \to_{\mathsf{FL}} B$ where $*$ is a binary LL connective.
- $@A \to_{\mathsf{FL}} A$ where @ is either ! or ?.
- $\sigma\zeta\, F \to_{\mathsf{FL}} F\,[\sigma\zeta\, F/\zeta]$ where σ is either ν or μ.

A formula G is a Fischer-Ladner sub-formula of F when $F \to^{\star}_{\mathsf{FL}} G$.

It is a well-known fact that the Fischer-Ladner closure of any formula (*ie.* the set of its Fischer-Ladner sub-formulas) is finite, see for instance Corollary 2.1 of [22].

Proposition A.3 (Proposition 2.7 of [22]). *If a thread t is coming from a branch of an $\mu\mathsf{LL}_\infty$ pre-proof, then $\mathsf{Inf}(t)$ admits a minimum with respect to the usual sub-formula ordering $\leqslant_{\mathsf{sub}}$ (see Definition A.1), denoted* $\min(\mathsf{Inf}(t))$.

Proof. Proposition 2.7 of [22]. The idea of the proof is based on the observation that $\mathsf{Inf}(t)$ forms a cycle, and roughly speaking, the minimum of $\mathsf{Inf}(t)$ is the nearest to the root in that cycle. □

B INTERPRETATION OF LL RULES

In this section, we have recall the interpretationof LL rules in a categorical model of LL. The idea is to interpret a sequent $\vdash \Gamma$ as a morphsim in $\mathcal{L}(1, \llbracket\Gamma\rrbracket)$. If the shape of our sequent is $\vdash \Delta, \Gamma$, we interpreted it as a morphism in $\mathcal{L}(\llbracket\Delta^\perp\rrbracket, \llbracket\Gamma\rrbracket)$.

$$\left\llbracket \dfrac{\begin{array}{cc} \vdots\,\pi_1 & \vdots\,\pi_2 \\ \vdash \Gamma, A & \vdash A^\perp, \Delta \end{array}}{\vdash \Gamma, \Delta}\ (\mathrm{cut}) \right\rrbracket = (\llbracket\Gamma\rrbracket)^\perp \xrightarrow{\llbracket\pi_1\rrbracket} \llbracket A\rrbracket \xrightarrow{\llbracket\pi_2\rrbracket} \llbracket\Delta\rrbracket$$

$$\left\llbracket \dfrac{}{\vdash A, A^\perp}\ (\mathsf{ax}) \right\rrbracket = \mathsf{Id}_A \qquad \left\llbracket \dfrac{}{\vdash 1}\ (1) \right\rrbracket = \mathsf{Id}_1$$

$$\left\llbracket \dfrac{\begin{array}{cc} \vdots\,\pi_1 & \vdots\,\pi_2 \\ \vdash \Gamma, A & \vdash \Delta, B \end{array}}{\vdash \Gamma, \Delta, A \otimes B}\ (\otimes) \right\rrbracket = (\llbracket\Gamma\rrbracket)^\perp \otimes (\llbracket\Delta\rrbracket)^\perp \xrightarrow{\llbracket\pi_1\rrbracket \otimes \llbracket\pi_2\rrbracket} \llbracket A\rrbracket \otimes \llbracket B\rrbracket$$

$$\left\llbracket \dfrac{\begin{array}{c} \vdots\,\pi \\ \vdash \Gamma \end{array}}{\vdash \Gamma, \perp}\ (\perp) \right\rrbracket = \mathsf{cur}(\llbracket\pi\rrbracket) \qquad \left\llbracket \dfrac{\begin{array}{c} \vdots\,\pi \\ \vdash \Gamma, A, B \end{array}}{\vdash \Gamma, A \parr B}\ (\parr) \right\rrbracket = \llbracket\pi\rrbracket$$

$$\left\llbracket \dfrac{}{\vdash \Gamma, \top}\ (\top) \right\rrbracket = \mathsf{t}_{(\llbracket\Gamma\rrbracket)^\perp}$$

$$\left\llbracket \dfrac{\begin{array}{c} \vdots\,\pi \\ \vdash \Gamma, A_i \end{array}}{\vdash \Gamma, A_1 \oplus A_2}\ (\oplus_i) \right\rrbracket = (\llbracket\Gamma\rrbracket)^\perp \xrightarrow{\llbracket\pi\rrbracket} \llbracket A_i\rrbracket \xrightarrow{\mathsf{in}_i} \llbracket A_1\rrbracket \oplus \llbracket A_2\rrbracket$$

$$\left\llbracket \dfrac{\begin{array}{cc} \vdots\,\pi_1 & \vdots\,\pi_2 \\ \vdash \Gamma, A & \vdash \Gamma, B \end{array}}{\vdash \Gamma, A \,\&\, B}\ (\&) \right\rrbracket = (\llbracket\Gamma\rrbracket)^\perp \xrightarrow{\langle\llbracket\pi_1\rrbracket, \llbracket\pi_2\rrbracket\rangle} \llbracket A\rrbracket \,\&\, \llbracket B\rrbracket$$

$$\left\llbracket \dfrac{\begin{array}{c} \vdots\,\pi \\ \vdash \Gamma \end{array}}{\vdash \Gamma, ?A}\ (\mathrm{w}) \right\rrbracket = \mathsf{cur}(f) \text{ where } f \text{ is as follows:}$$

$$\begin{tikzcd} (\llbracket\Gamma\rrbracket)^\perp \otimes \llbracket !A^\perp\rrbracket \arrow[r, "\mathsf{Id} \otimes \mathsf{w}_{A^\perp}"] & (\llbracket\Gamma\rrbracket)^\perp \otimes 1 \arrow[r, "\simeq"] & (\llbracket\Gamma\rrbracket)^\perp \arrow[d, "\llbracket\pi\rrbracket"] \\ & & \perp \end{tikzcd}$$

$$\left\llbracket \dfrac{\begin{array}{c} \vdots\,\pi \\ \vdash \Gamma, ?A, ?A \end{array}}{\vdash \Gamma, ?A}\ (\mathrm{c}) \right\rrbracket = \mathsf{cur}(f) \text{ where } f \text{ is}$$

$$(\llbracket\Gamma\rrbracket)^\perp \otimes (!A^\perp \otimes !A^\perp) \xrightarrow{\mathsf{Id} \otimes \mathsf{contr}_{A^\perp}} (\llbracket\Gamma\rrbracket)^\perp \otimes !A^\perp \xrightarrow{\llbracket\pi\rrbracket} \perp$$

$$\left\llbracket \dfrac{\begin{array}{c} \vdots\,\pi \\ \vdash \Gamma, A \end{array}}{\vdash \Gamma, ?A}\ (\mathrm{d}) \right\rrbracket = \mathsf{cur}(f) \text{ where } f \text{ is}$$

$$(\llbracket\Gamma\rrbracket)^\perp \otimes \llbracket !A^\perp\rrbracket \xrightarrow{\mathsf{Id} \otimes \mathsf{der}_{A^\perp}} (\llbracket\Gamma\rrbracket)^\perp \otimes A^\perp \xrightarrow{\llbracket\pi\rrbracket} \perp$$

$$\left\llbracket \dfrac{\begin{array}{c} \vdots\,\pi \\ \vdash ?\Gamma, A \end{array}}{\vdash ?\Gamma, !A}\ (\mathrm{p}) \right\rrbracket =$$

$$\begin{tikzcd} \llbracket\bigotimes_{B_i^\perp \in \Gamma} (!B_i)\rrbracket \arrow[r, "\bigotimes \mathsf{dig}_{B_i}"] & \llbracket\bigotimes_{B_i^\perp \in \Gamma} !!B_i\rrbracket \arrow[r, "\mu^n"] & !(\llbracket\bigotimes_{B_i^\perp \in \Gamma} !B_i\rrbracket) \arrow[d, "!\llbracket\pi\rrbracket"] \\ & & !\llbracket A\rrbracket \end{tikzcd}$$

C METRIC COMPLETION OF FINITE PROOFS

The purpose of this section is to develop a precise characterization of non-wellfounded proofs as the completion of a space of finite proof with a notion of approximant, much in the same way a Böhm trees for the λ-calculus. As such, the material in this section should not surprise the reader in its technical development but it is nevertheless necessary for a precise definition of the semantics of non-wellfounded proofs that we consider in the latter sections.

We consider the proof system of $\mu\mathsf{LL}_\infty$ extended with the following rule: $\dfrac{}{\vdash \Gamma}\ (\Omega)$ for any sequent Γ. The reason why we consider this assumption will be clear later, for instance in Definition C.3. Here, we can say that we are using this auxiliary rule in order to cut the infinite proofs at different levels and consider all its finite approximation.

Definition C.1. Given a $\mu\mathsf{LL}_\infty$ pre-proof π, we associate a set $\mathsf{Pos}(\pi)$ of positions corresponding to each sequent of π as follows:

- $\langle 0 \rangle \in \mathrm{Pos}(\pi)$
- Let r be an occurrence of an inference rule in π and that $\langle x \rangle$, which belongs to $\mathrm{Pos}(\pi)$, is the location of this occurrence in π
  - If $r \in \{(\otimes), (\&), (\mathrm{cut})\}$, then both $\langle x0 \rangle$ and $\langle x1 \rangle$ are in $\mathrm{Pos}(\pi)$;
  - Otherwise $\langle x0 \rangle \in \mathrm{Pos}(\pi)$.

The elements of $\mathrm{Pos}(\pi)$ are finite sequences of 0 and 1.

Definition C.2. Given a pre-proof π and $p \in \mathrm{Pos}(\pi)$, we denote by $\mathrm{Proof}(\pi, p)$ the last sequent of the sub-pre-proof of π rooted at position p.

As an example, consider the following proof π:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(\mathrm{ax})}{\vdash ?A^\perp, A}\,(\mathrm{d})}{\vdash ?A^\perp, ?B^\perp, A}\,(\mathrm{w}) \qquad \dfrac{\dfrac{\dfrac{}{\vdash B^\perp, B}\,(\mathrm{ax})}{\vdash ?B^\perp, B}\,(\mathrm{d})}{\vdash ?A^\perp, ?B^\perp, B}\,(\mathrm{w})}{\vdash ?A^\perp, ?B^\perp, A \,\&\, B}\,(\&)}{\vdash ?A^\perp, ?B^\perp, !(A \,\&\, B)}\,(\mathrm{p})$$

Then one can represent it by the $\mathrm{Pos}(\pi)$ as follows which is also annotated by the sequents. One can also label the edges by the inference rules.

$$\begin{array}{ccc}
 & \vdash ?A^\perp, ?B^\perp, !(A \,\&\, B) \;\; \langle 0 \rangle & \\
 & | & \\
 & \vdash ?A^\perp, ?B^\perp, A \,\&\, B \;\; \langle 00 \rangle & \\
 \swarrow & & \searrow \\
\vdash ?A^\perp, ?B^\perp, A \;\; \langle 000 \rangle & & \vdash ?A^\perp, ?B^\perp, B \;\; \langle 001 \rangle \\
| & & | \\
\vdash ?A^\perp, A \;\; \langle 0000 \rangle & & \vdash ?B^\perp, B \;\; \langle 0010 \rangle \\
| & & | \\
\vdash A^\perp, A \;\; \langle 00000 \rangle & & \vdash B^\perp, B \;\; \langle 00100 \rangle
\end{array}$$

Definition C.3. Let π be a pre-proof and P be a prefix-closed subset of $\mathrm{Pos}(\pi)$. We denote by $\pi(P)$ the sub-pre-proof of π whose set of positions is P, i.e, $\mathrm{Pos}(\pi(P)) = P$.

Notice that if we do not assume having the (Ω) rule, then $\pi(P)$ might not exist.

Definition C.4. If π is a pre-proof we denote by $\mathrm{Pos}_i(\pi)$ the subset of $\mathrm{Pos}(\pi)$ that contains only all position of length i, i.e, $\mathrm{Pos}_i(\pi) = \pi(\mathrm{Pos}(\pi) \cap \{0,1\}^i)$.

Let $\mathcal{X}$ be the set of all $\mu\mathsf{LL}_\infty$ finite proofs. One can define a distance $d : \mathcal{X} \times \mathcal{X} \to [0,1]$: $d(\pi, \pi') = 0$ if two proofs π and π' are identical, otherwise $d(\pi, \pi') = \frac{1}{2^k}$, where k is the length of the shortest position at which π and π' differ.

Denote by $C[\mathcal{X}]$ the collection of all Cauchy sequences in $\mathcal{X}$. Define a relation $\sim$ on $C[\mathcal{X}]$ by

$$(\pi_n) \sim (\pi'_n) \Leftrightarrow \lim_{n\to\infty} d(\pi_n, \pi'_n) = 0$$

It is easy to see that this is an equivalence relation on $C[\mathcal{X}]$. This definition does not depend on the choice of representatives in the two equivalence classes. Let $\mathcal{X}^*$ be the set of all equivalence classes for $\sim$. One can define the metric d^* on $\mathcal{X}^*$ as follows where $[(\pi_n)]$ is an equivalence class:

$$d^*([(\pi_n)], [(\pi'_n)]) = \lim_{n\to\infty} d(\pi_n, \pi'_n)$$

The metric space $(\mathcal{X}^*, d^*)$ is called *metric completion* of $\mathcal{X}$, and there is standard result showing that this is a complete space.

PROPOSITION C.5. *Let $\mathcal{X}_\infty$ be set of all (potentially infinite) $\mu\mathsf{LL}_\infty$ proofs. Then the metric space $(\mathcal{X}^*, d^*)$ is isomorphic to $\mathcal{X}_\infty$.*

PROOF. Since the completion of a metric space is unique up to isometry, it is enough to show that $(\mathcal{X}_\infty, d')$ is the completion of $\mathcal{X}$ for a metric d'. That is to show $\mathcal{X}$ is dense in $\mathcal{X}_\infty$ for taking d' same as d.

Take $\pi \in \mathcal{X}_\infty$. Consider the sequence (π_n) where $\pi_n = \pi(\bigcup_{i<n} \mathrm{Pos}_i(\pi))$. We have now $d(\pi, \pi_n) = \frac{1}{2^n}$, so, π is the limit of the sequence (π_n) of finite proofs. □

As the direct conclusion of C.5, the metric space $(\mathcal{X}_\infty, d)$ is complete, that is to say every Cauchy sequence of proofs in $\mathcal{X}_\infty$ has a limit inside of $\mathcal{X}_\infty$.

REMARK 9. *In the cut-elimination process of $\mu\mathsf{LL}_\infty$, for any natural number n, the number of steps of the sequence which reduces a* (cut) *rule at depth less that n is finite [22]. So, the cut-elimination reduction has countable length.*

We saw that the metric space $(\mathcal{X}_\infty, d)$ is a complete space, but this was a result of the proposition C.5. Here we show the completeness of this metric space directly.

PROPOSITION C.6. *The metric space $(\mathcal{X}_\infty, d)$ is complete.*

PROOF. Take a Cauchy sequence (π_n). First, we define the set P as $\bigcup_i \bigcap_{j>i} \mathrm{Pos}(\pi_i)$. And we also provide a function f that sends a $p \in P$ to a sequent as follows: Since $p \in P$, $\exists i \forall j > i (p \in \mathrm{Pos}(\pi_j) \wedge (\mathrm{Proof}(\pi_i, p) = \mathrm{Proof}(\pi_j, p))$. So, we define $f(p)$ as $\mathrm{Proof}(\pi_i, p)$ (this does not depend on the choice of i). Now since the sequence (π_n) is a Cauchy sequence, we have $\forall k, \exists N \forall i, j > N(d(\pi_i, \pi_j) < \frac{1}{2^k})$, and therefore $d(\Pi(P, f), \pi_i) < \frac{1}{2^k}$ where $\Pi(P, f)$ is the pre-proof tree that has P as set of its positions and it is labeled by element of $f(P)$ (one can deduce it by the contradiction). Hence the proof $\Pi(P, f)$ is the limit of the (π_n). □

We will use this direct proof later in proof of Theorem 4.4.

D DEFINITION OF Trans ()

- We have the following for the rule r in the following set: $\{(1), (\mathrm{ax}), (\perp), (\invamp), (\top), (\oplus_1), (\oplus_2), (\mathrm{w}), (\mathrm{c}), (\mathrm{d}), (\mathrm{p}), (\mu)\}$:

$$\mathsf{Trans}\left(\dfrac{\begin{array}{c}\pi \\ \vdash \Delta\end{array}}{\vdash \Gamma}\,\mathsf{r} \right) = \dfrac{\begin{array}{c}\mathsf{Trans}\,(\pi) \\ \vdash \Delta\end{array}}{\vdash \Gamma}\,\mathsf{r}$$

- We have the following for $r \in \{(\mathrm{cut}), (\otimes), (\&)\}$:

$$\mathsf{Trans}\left(\dfrac{\begin{array}{c}\pi_1 \\ \vdash \Delta_1\end{array} \qquad \begin{array}{c}\pi_2 \\ \vdash \Delta_2\end{array}}{\vdash \Gamma}\,\mathsf{r} \right) = \dfrac{\begin{array}{c}\mathsf{Trans}\,(\pi_1) \\ \vdash \Delta_1\end{array} \qquad \begin{array}{c}\mathsf{Trans}\,(\pi_2) \\ \vdash \Delta_2\end{array}}{\vdash \Gamma}\,\mathsf{r}$$

- And finally $\mathsf{Trans}\left(\dfrac{\begin{array}{c}\pi \\ \vdash ?\Gamma, A^\perp, F\,[A/\zeta]\end{array}}{\vdash ?\Gamma, A^\perp, \nu\zeta.F}\,(\nu'_{\mathrm{rec}}) \right)$ is the following circular proof using the functoriality of formulas given in

Section 2.3:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A^\perp, \nu\zeta.F}{\vdash ?\Gamma, (F[A/\zeta])^\perp, F[\nu\zeta.F/\zeta]}\,(\mathfrak{F}_F)}{\vdash ?\Gamma, (F[A/\zeta])^\perp, \nu\zeta.F}\,(\nu) \qquad \begin{array}{c}\pi\\ \vdash A^\perp, F[A/\zeta]\end{array}}{\vdash ?\Gamma, ?\Gamma, A^\perp, \nu\zeta.F}\,(\mathrm{cut})}{\vdash ?\Gamma, A^\perp, \nu\zeta.F}\,(\mathrm{c})$$

E PROOFS OF SECTION 4

E.1 Proof of Lemma 4.8

Lemma E.1. *Let A be an onject of a category $\mathcal{A}$ and let $f_1, f_2 \in \mathcal{A}(A, \nu\mathcal{F})$. If there exists $l \in \mathcal{A}(A, \mathcal{F}(A))$ such that $\mathcal{F}(f_i)\, l = f_i$ for $i = 1, 2$, then $f_1 = f_2$.*

Proof. Since $\mathcal{F}(f_i)\, l = f_i$ for $i = 1, 2$, we have $f_i \in \mathbf{Coalg}_{\mathcal{A}}(\mathcal{F})((A, l), (\nu\mathcal{F}, \mathsf{Id}))$ for $i = 1, 2$. $(\nu\mathcal{F}, \mathsf{Id})$ is the final object in $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F})((A, l), (\nu\mathcal{F}, \mathsf{Id}))$, so there is a unique morphism from (A, l) to $(\nu\mathcal{F}, \mathsf{Id})$. Hence $f_1 = f_2$. ◻

Remark 10. *In the proof of Lemma 4.8, we refer to the identity for the coalgebra morphism of $\nu\mathcal{F}$ but never use any of its property and the proof would go through using any iso instead of* Id*: it is just a consequence of the universal property of a final coalgebra.*

E.2 Proof of Theorem 4.9

The interpretation of a μLL formulas F that contains n free variable is an n-ary strong functor $\llbracket F\rrbracket$ [25]. We use the notations $\overline{\llbracket F\rrbracket}$ and $\widehat{\llbracket F\rrbracket}$ for the underlying functor and strength of the strong functor $\llbracket F\rrbracket$ respectively.

Theorem E.2. *Let $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ be a $\mu\mathsf{LL}_\infty$ model, and π be a μLL proof. Then we have $\llbracket\pi\rrbracket = \llbracket\mathsf{Trans}\,(\pi)\rrbracket$.*

Proof. The proof is by induction on π. Let us assume that the last inference rule is a (ν) rule so that π is the following proof:

$$\dfrac{\begin{array}{c}\pi'\\ \vdash ?\Gamma, A^\perp, F[A/\zeta]\end{array}}{\vdash ?\Gamma, A^\perp, \nu\zeta\, F}\,(\nu'_{\mathrm{rec}})$$

Let $f = \llbracket\mathsf{Trans}\,(\pi)\rrbracket$. By definition of $\mathsf{Trans}\,(\pi)$ given above, f should satisfy the following diagram:

$$\begin{tikzcd}
!\llbracket\Gamma^\perp\rrbracket\otimes A \arrow[r,"f"] \arrow[d,"\mathsf{C}_{!\llbracket\Gamma^\perp\rrbracket}\otimes\mathsf{Id}"] & \llbracket\nu\zeta\, F\rrbracket \\
!\llbracket\Gamma^\perp\rrbracket\otimes !\llbracket\Gamma^\perp\rrbracket\otimes A \arrow[d,"\mathsf{Id}\otimes\llbracket\pi'\rrbracket"] & \overline{\llbracket F\rrbracket}(\llbracket\nu\zeta\, F\rrbracket) \arrow[u,"\simeq"'] \\
!\llbracket\Gamma^\perp\rrbracket\otimes \overline{\llbracket F\rrbracket}(\llbracket A\rrbracket) \arrow[r,"\widehat{\llbracket F\rrbracket}"] & \overline{\llbracket F\rrbracket}(!\llbracket\Gamma^\perp\rrbracket\otimes\llbracket A\rrbracket) \arrow[u,"\overline{\llbracket F\rrbracket}(f)"']
\end{tikzcd}$$

By the construction given in [25] to interpret formulas and proofs of μLL, the interpretation of π is the unique morphism $\llbracket\pi\rrbracket \in \mathcal{L}(!\llbracket\Gamma^\perp\rrbracket\otimes A, \llbracket\nu\zeta\, F\rrbracket)$ satisfying the following diagram:

$$\begin{tikzcd}
!\llbracket\Gamma^\perp\rrbracket\otimes A \arrow[r,"\llbracket\pi\rrbracket"] \arrow[d,"\mathsf{C}_{!\llbracket\Gamma^\perp\rrbracket}\otimes\mathsf{Id}"] & \llbracket\nu\zeta\, F\rrbracket \\
!\llbracket\Gamma^\perp\rrbracket\otimes !\llbracket\Gamma^\perp\rrbracket\otimes A \arrow[d,"\mathsf{Id}\otimes\llbracket\pi'\rrbracket"] & \overline{\llbracket F\rrbracket}(\llbracket\nu\zeta\, F\rrbracket) \arrow[u,"\simeq"'] \\
!\llbracket\Gamma^\perp\rrbracket\otimes \overline{\llbracket F\rrbracket}(\llbracket A\rrbracket) \arrow[r,"\widehat{\llbracket F\rrbracket}"] & \overline{\llbracket F\rrbracket}(!\llbracket\Gamma^\perp\rrbracket\otimes\llbracket A\rrbracket) \arrow[u,"\overline{\llbracket F\rrbracket}(\llbracket\pi\rrbracket)"']
\end{tikzcd}$$

Hence, by Lemma 4.8, we have $\llbracket\pi\rrbracket = \llbracket\mathsf{Trans}\,(\pi)\rrbracket$. ◻

F PROOFS OF SECTION 4

F.1 Proof of Theorem 4.4

Theorem F.1. *Given two finite $\mu\mathsf{LL}_\infty$ proofs π and π' such that π' is obtained from π via an one-step cut-elimination rule, then $\llbracket\pi\rrbracket = \llbracket\pi'\rrbracket$.*

Proof. We only need to check the reduction of $(\mu) - (\nu)$ given in Section 2.3. And this is trivial, as both (μ) and (ν) rules have no effect on the interpretation by definition. ◻

F.2 Proof of Lemma 4.5

Lemma F.2. *Let (π_i) be a Cauchy sequence. Then $\llbracket\lim_{n\to\infty}\pi_i\rrbracket = \bigcup_i\bigcap_{j>i}\llbracket\pi_j\rrbracket$.*

Proof. By Proposition C.6, $\lim_{n\to\infty}\pi_i = \Pi(P, f)$ (we are using a notation introduced in the proof of Proposition C.6). By definition, $\llbracket\Pi(P, f)\rrbracket = \bigcup_{\pi\in\mathsf{fin}(\Pi(P,f))}\llbracket\pi\rrbracket$. Take a $\pi' \in \mathsf{fin}(\Pi(P, f))$. For each $p \in \mathsf{fin}(\Pi(P, f))$, we have $\exists i_p \forall j > i_p (p \in \mathsf{Pos}(\pi_j) \wedge (\mathsf{Proof}(\pi_j, p) = \mathsf{Proof}(\pi', p))$, by definition. Let i be the maximum among all i_p's (The set $\mathsf{Pos}(\pi')$ is finite). Then for all $j > i$ we have $\pi' \in \pi_j$. Hence we have the following:

$$\forall\pi' \in \mathsf{fin}(\Pi(P, f))\ \forall p \in \pi'\ \exists i\ \forall j > i\ (p \in \pi_j \wedge\ (\mathsf{Proof}(\pi_j, p) = \mathsf{Proof}(\pi', p)))$$

And that is to say for each $\pi' \in \mathsf{fin}(\Pi(P, f))$, there exists an i such that for all $j > i$, π' is a finite sub-pre-proof of all π_j. Hence $\llbracket\pi'\rrbracket$ is less than $\llbracket\pi_j\rrbracket$ for all $j > i$, so, $\llbracket\pi'\rrbracket \subseteq \bigcap_{j>i}\llbracket\pi_j\rrbracket$ (we use the notation $\subseteq$ for the partial order relation on the hom-set). ◻

F.3 Proof of Theorem 4.6

Theorem F.3. *Let $(\pi_i)_{i\in\omega}$ be a Cauchy sequence such that $\forall i, j \in \omega$ we have $\llbracket\pi_i\rrbracket = \llbracket\pi_j\rrbracket$. Then $\llbracket\lim_{n\to\infty}\pi_i\rrbracket = \llbracket\pi_0\rrbracket$.*

Proof.

$$\begin{aligned}
\llbracket\lim_{n\to\infty}\pi_i\rrbracket &= \bigcup_i\bigcap_{j>i}\llbracket\pi_j\rrbracket && \text{By Lemma 4.5}\\
&= \bigcup\bigcap\llbracket\pi_0\rrbracket\\
&= \llbracket\pi_0\rrbracket
\end{aligned}$$

◻

F.4 Proof of Lemma 5.3

Lemma F.4. *If A is a $\mu\mathsf{LL}_\infty$ formula and $t \notin O([\![\nu^\alpha\zeta.F]\!])$, then there exists an ordinal $\gamma < \alpha$ such that $t \notin O([\![F\,[\nu^\gamma\zeta.F/\zeta]]\!])$.*

Proof. If α is a successor ordinal $\delta+1$ then $U_\alpha = O([\![F]\!])([\![\nu\zeta\,F]\!]_{\mathcal{L}}, U_\delta)$ by definition, and obvioulsy $t \notin O([\![F]\!])(([\![\nu\zeta\,F]\!]_{\mathcal{L}}, U_\delta))$. And so $t \notin O([\![F\,[\nu^\gamma\zeta.F/\zeta]]\!])$ for $\gamma = \delta$.

If α is a limit ordinal, then: $U_\alpha = \bigcap_{\gamma<\alpha} U_\gamma$, and $t \notin \bigcap_{\gamma<\alpha} U_\gamma = \bigcap_{\delta+1<\alpha} U_{\delta+1}$. So, there exists an ordinal $\delta+1 < \alpha$ such that $t \notin U_{\delta+1}$ and we continue as before. □

F.5 Poof of Lemma 5.4

Lemma F.5. $O([\![F\,[\mu\zeta\,F/\zeta]]\!]) = O([\![\mu\zeta\,F]\!])$.

Proof. The interpretation of $\mu\zeta\,F$ is the least fixed-point of Θ_F. So, we have:

$$\begin{aligned}
O([\![\mu\zeta\,F]\!]) &= \Theta_F(O([\![\mu\zeta\,F]\!]))\\
&= O([\![F]\!])(([\![\mu\zeta\,F]\!]\mathcal{L}, O([\![\mu\zeta\,F]\!]))) && \text{by definition of } \Theta_F\\
&= O([\![F\,[\mu\zeta\,F/\zeta]]\!])
\end{aligned}$$

□

F.6 Proof of Lemma 5.5

Lemma F.6. *If π is a proof of $\vdash \Gamma$ and $[\![\pi]\!] \notin O(([\![\Gamma]\!]))$, then*

(1) π has an infinite branch $\gamma = (\vdash \Gamma_i)_{i\in\omega}$ such that $[\![\pi_i]\!] \notin O(([\![\Gamma_i]\!]))$ where π_i is the sub-proof of π rooted in $\vdash \Gamma_i$;

(2) and there exists a sequence of functions $(f_i)_{i\in\omega}$ where f_i maps all formulas D of Γ_i to a marked formula $f_i(D)$ such that
- *$(f_i(D))^\circ = D$,*
- *one can write $\Gamma_i = \Gamma'_i, C$,*
- *and there exists $x \in O([\![(f_i(\Gamma'_i))^\perp]\!])$ such that $[\![\pi_i]\!].x \notin O([\![f_i(C)]\!])$ where $\Gamma'_i = A^i_1, \cdots, A^i_{n_i}$ and $[\![(f_i(\Gamma'_i))^\perp]\!] = ([\![f_i(A^i_1)]\!])^\perp \otimes \cdots \otimes ([\![f_i(A^i_{n_i})]\!])^\perp$.*

Proof. We set $\Gamma_0 = \Gamma$, and $f_0(D) = \overline{D}$ for all $D \in \Gamma_0$:

- Since $\pi_0 = \pi$, $[\![\pi_0]\!] \notin \mathcal{T}([\![\Gamma_0]\!])$.
- Let C be the principal formula in Γ_0. The sequent $\vdash f_0(\Gamma_0)$ is denotationally the same as $\vdash (f_0(\Gamma'_0))^\perp \multimap f_0(C)$. By the proposition 5.2, $[\![f_0(D)]\!] = [\![D]\!]$ for all $D \in \Gamma_0$. So, $[\![\pi_0]\!] \notin O(f_0(\Gamma_0))$. That is to say $[\![\pi_0]\!] \notin O([\![(f_0(\Gamma'_0))^\perp \multimap f_0(C)]\!])$. Therefore, by definition, there exists $x \in O([\![(f_0(\Gamma'_0))^\perp]\!])$ such that $[\![\pi_0]\!].x \notin O([\![f_0(C)]\!])$.

Suppose that we have provided Γ_i and f_i for $i \in \omega$. We then define Γ_{i+1} and f_{i+1} depending on the rule applied on $\vdash \Gamma_i$ in π. Let us assume that the formula C is the principal in Γ_i:

- If $C = C_1 \parr C_2$, then Γ_{i+1} is the unique premise of $\vdash \Gamma_i$. $f_i(C) = B^1_C \parr B^2_C$ where B^1_C and B^2_C are two marked formulas, so, we set $f_{i+1}(C_1) = B^1_C$, $f_{i+1}(C_2) = B^2_C$, and $f_{i+1}(F) = f_i(F)$ for the other $F \in \Gamma_{i+1}$:
  - Since Γ_i is obtained by applying the $\parr$ rule on Γ_{i+1}, we have $[\![\pi_{i+1}]\!] = [\![\pi_i]\!]$, and $[\![\Gamma_{i+1}]\!] = [\![\Gamma_i]\!]$. By induction hypothesis, $[\![\pi_{i+1}]\!] \notin O([\![\Gamma_{i+1}]\!])$.
  - By induction hypothesis, there exists $x \in O([\![(f_i(\Gamma'_i))^\perp]\!])$ such that $[\![\pi_i]\!].x \notin O([\![f_i(C)]\!])$. So, $[\![\pi_{i+1}]\!].x = [\![\pi_i]\!].x \notin O([\![B^1_C \parr B^2_C]\!]) = ((O(([\![((B^1_C))^\perp]\!] \otimes [\![((B^2_C))^\perp]\!]))))^\perp$. So, there is a $y \in O(([\![((B^1_C))^\perp]\!] \otimes [\![((B^2_C))^\perp]\!]))$ such that $[\![\pi_{i+1}]\!].x \cap y \neq \emptyset$. Since $y \in O(([\![(B^1_C)^\perp]\!] \otimes [\![(B^2_C)^\perp]\!]))$, there is $u' \in O([\![(B^1_C)^\perp]\!])$ and $v' \in O([\![(B^1_C)^\perp]\!])$ such that $u' \times v' \subseteq y$. So, $[\![\pi_{i+1}]\!].x \cap (u' \times v') = \emptyset$. This statement is equivalent to $([\![\pi_{i+1}]\!].x).u' \cap v' \neq \emptyset$. $[\![\pi_{i+1}]\!].x \in$, and this is equivalent to $[\![\pi_{i+1}]\!].(x \times u') \cap v' \neq \emptyset$. We have shown till now that there exists $v' \in O([\![(B^1_C)^\perp]\!])$ such that $[\![\pi_{i+1}]\!].x' \cap v' \neq \emptyset$ where $x' = x \times u'$. So, by definition, $[\![\pi_{i+1}]\!].x' \notin O([\![B^1_C]\!])$.
- If $C = C_1 \oplus C_2$, then we proceed as above.
- If $C = C_1 \otimes C_2$. Let us call Γ^1_{i+1} and Γ^2_{i+1} for the two premises of $\vdash \Gamma_i$. $f_i(C) = B^1_C \otimes B^2_C$ where B^1_C and B^2_C are two marked formulas. Since $[\![\pi_i]\!] \notin O([\![\Gamma_i]\!])$, we have $[\![\pi^j_{i+1}]\!] \notin O([\![\Gamma^j_{i+1}]\!])$ for either $j = 1$ or $j = 2$ where π^1_{i+1} (respectively π^2_{i+1}) is the left (respectively the right) subproof of π_i. Let us assume that it is true for $j = 1$ (the proof of the case $j = 2$ is identical to the case $j = 1$). So we set $\Gamma_{i+1} = \Gamma^1_{i+1}$, $f_{i+1}(C_1) = B^1_C$, and $f_{i+1}(D) = f_i(D)$ for the other $D \in \Gamma^1_{i+1}$.
  - By induction hypothesis, there exists $x' \in O([\![(f_i(\Gamma^1_{i+1} \parr \Gamma^2_{i+1}))^\perp]\!])$ such that $[\![\pi_i]\!].x' \notin O([\![B^1_C \otimes B^2_C]\!])$. Hence $[\![\pi_i]\!] \notin O([\![f_i(\Gamma_i)]\!])$ by definition. So, we have $[\![\pi^j_{i+1}]\!] \notin O([\![f_{i+1}(\Gamma_{i+1}j') \parr B^j_C]\!])$ for either $j = 1$ or $j = 2$. Let us assume that is true for $j = 1$ (the proof of the case $j = 2$ is identical to the case $j = 1$). So, $[\![\pi^1_{i+1}]\!] \notin O([\![(f_{i+1}(\Gamma_{i+1}j'))^\perp \multimap B^1_C]\!])$. And therefore, by definition, there is a $y \in [\![(f_{i+1}(\Gamma_{i+1}j'))^\perp]\!]$ such that $[\![\pi^1_{i+1}]\!].y \notin O([\![B^1_C]\!])$.
- If $C = C_1 \,\&\, C_2$, then we proceed as above.
- IF $C = \mu\zeta\,F$, then Γ_{i+1} is the unique premise of $\vdash \Gamma_i$. Wlog let us say $\Gamma_i = A^i_1, \cdots, A^i_{n_i}, \mu\zeta\,F$. $f_i(C) = \mu\zeta\,B_C$ where B_C is a marked formula. By induction hypothesis, there exists $x \in O([\![(f_i(\Gamma'_i))^\perp]\!])$ such that $[\![\pi_i]\!].x \notin O([\![\mu\zeta\,B_C]\!])$ where $\Gamma'_i = A^i_1, \cdots, A^i_{n_i}$. So, $[\![\pi_{i+1}]\!].x \notin O([\![B_C\,[\mu\zeta\,B_C/\zeta]]\!])$, since $[\![\pi_{i+1}]\!] = [\![\pi_i]\!]$ and lemma 5.4. Then we set $f_{i+1}(F\,[C/\zeta]) = B_C\,[\mu\zeta\,B_C/\zeta]$ and $f_{i+1}(D) = f_i(D)$ for all the other formula $D \in \Gamma_{i+1}$ in order to have the second property of the lemma 5.5.
- If $C = \nu\zeta\,F$, then Γ_{i+1} is the unique premise of $\vdash \Gamma_i$. Wlog, let us say $\Gamma_i = A^i_1, \cdots, A^i_{n_i}, \nu\zeta\,F$. $f_i(C) = \nu^\theta\zeta.B_C$ where B_C is a marked formula. By induction hypothesis, there exists $x \in O([\![(f_i(\Gamma'_i))^\perp]\!])$ such that $[\![\pi_i]\!].x \notin O([\![\nu^\theta\zeta.B_C]\!])$ where $\Gamma'_i = A^i_1, \cdots, A^i_{n_i}$. By Lemma 5.3, there is an ordinal $\delta < \theta$ such that $[\![\pi_{i+1}]\!].x \notin O([\![B_C\left[\nu^\delta\zeta.B_C/\zeta\right]]\!])$, since $[\![\pi_{i+1}]\!] = [\![\pi_i]\!]$. So, we set $f_{i+1}(F\,[C/\zeta]) = f_i(F)\left[\nu^\delta\zeta.B_C/\zeta\right]$ and $f_{i+1}(D) = f_i(D)$ for all the other formula $D \in \Gamma_{i+1}$ in order to have the second property of the lemma.
- If the rule applied to $\vdash \Gamma_i$ is a (cut) rule on the C. Let us say Γ_i is Γ^1_i, Γ^2_i. By induction hypothesis, $[\![\pi_i]\!] \notin O([\![\Gamma_i]\!])$. So, we have either $[\![\pi_{i+1}]\!] \notin O([\![\Gamma^1_i \parr C]\!])$ or $[\![\pi_{i+1}]\!] \notin O([\![\Gamma^2_i \parr C^\perp]\!])$. Wlog let us say $[\![\pi_{i+1}]\!] \notin O([\![\Gamma^1_i \parr C]\!])$. Then we take $\Gamma_{i+1} = \Gamma^1_i, C$. And for the f_{i+1}, we define $f_{i+1}(D) = f_i(D)$ for all $D \in \Gamma^1_i$, and $f_i(C) = \overline{C}$.

  - By induction hypothesis, $[\![\pi_i]\!] \notin O([\![f_i(\Gamma_i)]\!])$. So, we have either $[\![\pi_{i+1}]\!] \notin O([\![f_i(\Gamma_i^1) \parr \overline{C}]\!])$ or $[\![\pi_{i+1}]\!] \notin O([\![f_i(\Gamma_i^1) \parr \overline{C^\perp}]\!])$. So, we can use definition of morphisms in the category $O_{\perp\!\!\!\perp}(\mathcal{L})$ to deduce the second property as we proceed as the case $C = C_1 \otimes C_2$.
- If the rule applied to $\vdash \Gamma_i$ is a (w) rule, then Γ_{i+1} is the unique premise of the (w) rule. And $f_{i+1}(D) = f_i(D)$ for all $D \in \Gamma_{i+1}$. We have $[\![\pi_{i+1}]\!] \notin O([\![f_i(\Gamma_{i+1})]\!]) = O([\![f_{i+1}(\Gamma_{i+1})]\!])$, since $[\![\pi_i]\!] \notin O([\![f_i(\Gamma_i)]\!])$ (here we are also using the soundness theorem of μLL in [25]).
- If the rule applied to $\vdash \Gamma_i$ is (c) rule on the formula $?C$, then we proceed as above.
- If the rule applied to $\vdash \Gamma_i$ is (d) rule on the formula $?C$. Let us say $\Gamma_i = \Gamma_i', ?C$. Then $\Gamma_{i+1} = \Gamma_i', C$. $f_{i+1}(D) = f_i(D)$ for all $D \in \Gamma_i'$. $f_i(?C) = ?B_C$ where B_C is a marked formula. Then we take $f_{i+1}(C) = B_C$. To show the second property, we can again use soundness theorem of of μLL [25].
- If the rule applied to $\vdash \Gamma_i$ is (p) rule on the formula $!C$, then we proceed as above.

□

F.7 Proof of Theorem 5.6

Theorem F.7. *If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!] \in O([\![\Gamma]\!])$.*

Proof. Let us assume $[\![\pi]\!] \notin O([\![\Gamma]\!])$. We can then apply Lemma 5.5 to obtain an infinite branch $(\vdash \Gamma_i)_{i\in\omega}$ and a sequence $(f_i)_{i\in\omega}$ satisfying properties 1 and 2 of Lemma 5.5. By the definition of valid proof (Definition 2.7), there exists a valid thread $t = (F_i)_{i\in\omega}$ for the infinite branch $(\vdash \Gamma_i)_{i\in\omega}$. Let $\nu\zeta\, F$ be the minimal formula formula of t. So, there are infinitely many times in t that we use a ν rule to unfold $\nu\zeta\, F$. Let $(i_k)_{k\in\omega}$ be the sequence of indices where $\nu\zeta\, F$ gets unfolded. Then $\nu\zeta\, F$ in the sequent Γ_{i_k} is sub-occurrence of $\nu\zeta\, F$ in the sequent $\Gamma_{i_{k'}}$ for $k \geqslant k'$. By the property 2 of Lemma 5.5, $f_{i_k}(\nu\zeta\, F) = \nu^{\alpha_k}\zeta.f_{i_k}(F)$. Therefore, by the property 2 of Lemma 5.5 and by the construction of the f_i in the proof of Lemma 5.5, the sequence $(\alpha_k)_{k\in\omega}$ is strictly decreasing. As this contradicts the well-foundedness property of the ordinals we obtain the required contradiction and conclude that $[\![\pi]\!] \in O([\![\Gamma]\!])$. □

G ON STRONG VALIDITY AND FINITIZATION OF CIRCULAR PROOFS

Definition G.1. Let π be a circular pre-proof and β an infinite branch. A thread $t = (F_i)_{k\leq i\in\omega}$ is said to be strongly valid if t is valid and if there is $k \in \omega$ such that $\forall i, j > k$, if $\beta_i = \beta_j$, then $F_i = F_j$. We say a circular pre-proof π is strongly valid, if every infinite branch of π has a strongly valid thread.

Proposition G.2. *Let $\nu\zeta\, A$ be a μLL formula. Then, for any context Γ there is a formula I such that the following rules are derivable in μLL$_\infty$.*

$$\frac{\vdash \Delta\,[\nu\zeta\,.F/\zeta]}{\vdash \Delta\,[I/\zeta]} \qquad \frac{}{\vdash I, \Gamma}$$

The formula I in the proposition above is called the invariant formula and is defined as $\nu\zeta\,.(A \oplus (\parr I)^\perp)$.

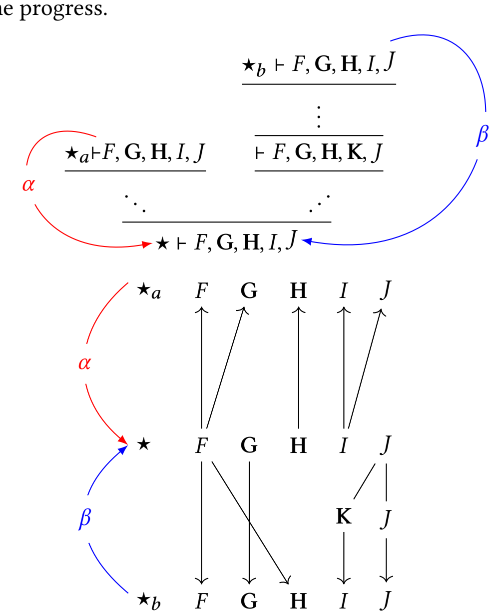
Proposition G.3. *Let π be a circular pre-proof of $\vdash \Gamma$. If π is strongly valid, then there is a finite proof $\tilde{\pi}$ of $\vdash \Gamma$ in μLL.*

H ON THE VALIDITY OF π_∞

In this appendix, we provide some additional details on the derivation π_∞ which is considered in Section 6 (and defined in Figure 5) and we discuss in details the structure of its validating threads.

We present below an abstracted version of the pre-proof π_∞ to outline its threading structure and its "validation modes". In what follows, coinductive formulas (namely G, H, K) are depicted in **bold face**. Note that K is a ν-subformula of I and J.

On the left, we only show four sequents of the proof, with the two back-edges. On the right, we show the threading structure of the pre-proof, showing the recreationg of fixed-point formulas as well as the progress.



The circular proof has two back-edges which induce, in its infinite unfolding, three types of infinite branches (or three types of infinite paths in the circular representation, which is equivalent):

(1) those branches which ultimately only visit the red back-edge, labeled α (visiting the blue back-edge only finitely many times);
(2) those branches which ultimately only visit the blue back-edge, labeled β (visiting the red back-edge only finitely many times);
(3) those branches which visit the red and blue back-edges, labeled α and β respectively, infinitely many times, that is such that in the "future", there will always be a change of direction.

Considering that validaty in non-wellfounded proofs is expressed in terms of recurring sequents, it is only a matter of its behaviour at the limit and one can neglect the transitory phase at the start and considering only the above three cases to classify all infinite branches.

(1) the infinite branches containing only the red back-edge, α, validate via a thread on H only: indeed, K is never principal

and formula G is erased and recreated at each iteration of the branch, making no progress.

(2) the infinite branches containing only the blue back-edge, β, validate via a thread on G only: indeed, K is principal on the branch but unfolds into I which is erased in the following iteration, no coinductive progrees is made there, while formula H is erased and recreated at each iteration of the branch, making no progress either.

(3) the infinite branches containing both infinitely many blue and red back-edges, α and β, validate via a thread on K only: indeed, G progresses on the left path but is erased next time the branches goes to the right while similarly, H progresses along the right path but is erased next time the branches goes to the left. On the other hand, K progresses infinitely: each time the branch **switches** from the right path to the left path, a coinductive progress is made on K, which is then stored in I and J until the next shift from a right path to a left path is made.

To sum up, one can then understand in the above example the complexity of the validation mode of π_∞: in each different class of branches, there is just one validating thread.

Moreover, π_∞ is not strongly valid since no unfolding of π_∞ to another circular proof can allow to synthesize the transition from a right path to a left path that is described above in order to ensure that one can specifically identify the occurrences of sequence on which K actually contributes to a coinductive progress.

I CATEGORICAL MODEL OF μLL

I.1 Strong functors on $\mathcal{L}$

Given $n \in \mathbb{N}$, an *n-ary strong functor* on $\mathcal{L}$ is a pair $\mathbb{F} = (\overline{\mathbb{F}}, \widehat{\mathbb{F}})$ where $\overline{\mathbb{F}} : \mathcal{L}^n \to \mathcal{L}$ is a functor and $\widehat{\mathbb{F}}_{X,\overrightarrow{Y}} \in \mathcal{L}(!X \otimes \overline{\mathbb{F}}(\overrightarrow{Y}), \overline{\mathbb{F}}(!X \otimes \overrightarrow{Y}))$ is a natural transformation, called the *strength* of $\mathbb{F}$. We use the notation $Z \otimes (Y_1, \dots, Y_n) = (Z \otimes Y_1, \dots, Z \otimes Y_n)$. It is assumed moreover that the diagrams of Figure 8 commute, expressing the monoidality of this strength as well as its compatibility with the comultiplication of !_.

I.1.1 Operations on strong functors. Let $\mathbb{F}$ be an n-ary strong functor and $\mathbb{G}_1, \dots, \mathbb{G}_n$ be k-ary strong functors. Then one defines a k-ary strong functor $\mathbb{H} = \mathbb{F} \circ (\mathbb{G}_1, \dots, \mathbb{G}_n)$: the functorial component $\overline{\mathbb{H}}$ is defined in the obvious compositional way. The strength is as follows:

$$!X \otimes \overline{\mathbb{H}}(\overrightarrow{Y}) \xrightarrow{\widehat{\mathbb{F}}} \overline{\mathbb{F}}((!X \otimes \overline{\mathbb{G}_i}(\overrightarrow{Y}))_{i=1}^n) \xrightarrow{\overline{\mathbb{F}}((\widehat{\mathbb{G}_i})_{i=1}^k)} \overline{\mathbb{F}}((\overline{\mathbb{G}_i}(!X \otimes \overrightarrow{Y}))_{i=1}^n)$$

Given an n-ary strong functor, we can define its *De Morgan dual* $(\mathbb{F})^\perp$ which is also an n-ary strong functor. On objects, we set $\overline{(\mathbb{F})^\perp}(\overrightarrow{Y}) = (\overline{\mathbb{F}}((\overrightarrow{Y})^\perp))^\perp$ and similarly for morphisms. The strength of $(\mathbb{F})^\perp$ is defined as the Curry transpose of the following morphism (remember that $!X \multimap (\overrightarrow{Y})^\perp = (!X \otimes \overrightarrow{Y})^\perp$ up to canonical iso):

$$\begin{tikzcd}
!X \otimes (\overline{\mathbb{F}}((\overrightarrow{Y})^\perp))^\perp \otimes \overline{\mathbb{F}}(!X \multimap (\overrightarrow{Y})^\perp) \arrow[r, phantom, "\cong"] & !X \otimes \overline{\mathbb{F}}(!X \multimap (\overrightarrow{Y})^\perp) \otimes (\overline{\mathbb{F}}((\overrightarrow{Y})^\perp))^\perp \arrow[d, "\widehat{\mathbb{F}} \otimes \mathsf{Id}"] \\
\overline{\mathbb{F}}((\overrightarrow{Y})^\perp) \otimes (\overline{\mathbb{F}}((\overrightarrow{Y})^\perp))^\perp \arrow[d, "\mathsf{ev}\ \gamma"'] & \overline{\mathbb{F}}(!X \otimes (!X \multimap (\overrightarrow{Y})^\perp)) \otimes (\overline{\mathbb{F}}((\overrightarrow{Y})^\perp))^\perp \arrow[l, "\overline{\mathbb{F}}(\mathsf{ev}) \otimes \mathsf{Id}"'] \\
\perp &
\end{tikzcd}$$

Fixed Points of strong functors.

Definition I.1. Let $\mathcal{A}$ be a category and $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ be a functor. A *coalgebra* of $\mathcal{F}$ is a pair (A, f) where A is an object of $\mathcal{A}$ and $f \in \mathcal{A}(A, \mathcal{F}(A))$. Given two coalgebras (A, f) and (A', f') of $\mathcal{F}$, a coalgebra morphism from (A, f) to (A', f') is an $h \in \mathcal{A}(A, A')$ such that $f'\, h = \mathcal{F}(h)\, f$. The category of coalgebras of the functor $\mathcal{F}$ will be denoted as $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F})$. The notion of algebra of an endofunctor is defined dually (reverse the directions of the arrows f and f') and the corresponding category is denoted as $\mathbf{Alg}_{\mathcal{A}}(\mathcal{F})$.

J PROOF OF COROLLARY 5.7

Corollary J.1. *If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!] \in \mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})(1_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}, [\![\Gamma]\!])$.*

Proof. Let us assume $[\![\pi]\!] \notin \mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})(1_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}, [\![\Gamma]\!])$. So, there is $x \in \mathcal{O}(1)$ such that $[\![\pi]\!] \circ x \notin \mathcal{O}([\![\Gamma]\!])$. We know that $\mathcal{D}(1) = \{X \subseteq \mathcal{L}(1,1) \mid X = X^{\perp\perp}\} = \{\{\mathsf{Id}_1\}\}$. So, $x = \mathsf{Id}_1$, and $[\![\pi]\!] \circ x = [\![\pi]\!] \notin \mathcal{O}(\Gamma)$ which contradicts Theorem 5.6. □

K PROOF OF LEMMA 6.1

Lemma K.1. *Let $\vdash \Gamma^\perp, \nu\zeta\, A$ be a μLL provable sequent. Then there is a unique morphism $\phi_A \in \mathcal{L}([\![\nu\zeta\, A]\!], [\![I_\Gamma^A]\!])$ such that it satisfies the following square:*

$$\begin{tikzcd}
{[\![\nu\zeta\, A]\!]} \arrow[r, "="] \arrow[dd, "\phi_A"] & {[\![A]\!]([\![\nu\zeta\, A]\!])} \arrow[d, "{[\![A]\!](\phi_A)}"] \\
 & {[\![A]\!]([\![I_\Gamma^A]\!])} \arrow[d, "\mathsf{in}_1"] \\
{[\![I_\Gamma^A]\!]} \arrow[r, "="] & {[\![A]\!]([\![I_\Gamma^A]\!]) \oplus [\![\Gamma]\!]}
\end{tikzcd}$$

where I_Γ^A is the invariant formula (see Proposition 3.6), and in_1 is the first injection.

Proof. Consider the following co-algebra morphism:

$$[\![\nu\zeta\, A]\!] \xrightarrow{\simeq} [\![A]\!]([\![\nu\zeta\, A]\!]) \xrightarrow{\mathsf{in}_1} [\![A]\!]([\![\nu\zeta\, A]\!]) \oplus [\![\Gamma]\!]$$

Since $[\![\nu\zeta\, A]\!]$ is the final co-algebra, there is a unique mrophism ϕ_A in $\mathcal{L}([\![\nu\zeta\, A]\!], [\![I_\Gamma^A]\!])$ such that it satisfies the following diagram:

$$\begin{tikzcd}
{[\![\nu\zeta\, A]\!]} \arrow[r, "\simeq"] \arrow[d, "\phi_A"] & {[\![A]\!]([\![\nu\zeta\, A]\!])} \arrow[r, "\mathsf{in}_1"] & {[\![A]\!]([\![\nu\zeta\, A]\!]) \oplus [\![\Gamma]\!]} \arrow[dl, "{\langle \mathsf{in}_1 \circ [\![A]\!](\phi_A), \mathsf{in}_2 \rangle}"] \\
{[\![I_\Gamma^A]\!]} \arrow[r, "="] & {[\![A]\!]([\![I_\Gamma^A]\!]) \oplus [\![\Gamma]\!]} &
\end{tikzcd}$$

Since the following diagram commutes, we obtain the required diagram.

$$\begin{tikzcd}
{[\![A]\!]([\![\nu\zeta\, A]\!])} \arrow[r, "\mathsf{in}_1"] \arrow[d, "{[\![A]\!](\phi_A)}"'] & {[\![A]\!]([\![\nu\zeta\, A]\!]) \oplus [\![\Gamma]\!]} \arrow[ddl, "{\langle \mathsf{in}_1 \circ [\![A]\!](\phi_A), \mathsf{in}_2 \rangle}"] \\
{[\![A]\!]([\![I_\Gamma^A]\!])} \arrow[d, "\mathsf{in}_1"'] & \\
{[\![A]\!]([\![I_\Gamma^A]\!]) \oplus [\![\Gamma]\!]} &
\end{tikzcd}$$

□

$$\begin{tikzcd}
(!X_1 \otimes !X_2) \otimes \overline{\mathbb{F}}(\overrightarrow{Y}) \arrow[r, "\mathsf{m}^2 \otimes \overline{\mathbb{F}}(\overrightarrow{Y})"] \arrow[d, "!X_1 \otimes \widehat{\mathbb{F}}_{X_2, \overrightarrow{Y}}"'] & !(X_1 \mathbin{\&} X_2) \otimes \overline{\mathbb{F}}(\overrightarrow{Y}) \arrow[dd, "\widehat{\mathbb{F}}_{X_1 \& X_2, \overrightarrow{Y}}"] \\
!X_1 \otimes \overline{\mathbb{F}}(!X_2 \otimes \overrightarrow{Y}) \arrow[d, "\widehat{\mathbb{F}}_{X_1, !X_2 \otimes \overrightarrow{Y}}"'] & \\
\overline{\mathbb{F}}(!X_1 \otimes !X_2 \otimes \overline{\mathbb{F}}(\overrightarrow{Y})) \arrow[r, "\overline{\mathbb{F}}(\mathsf{m}^2 \otimes \overrightarrow{Y})"] & \overline{\mathbb{F}}(!(X_1 \mathbin{\&} X_2) \otimes \overrightarrow{Y})
\end{tikzcd}$$

$$\begin{tikzcd}
1 \otimes \overline{\mathbb{F}}(\overrightarrow{Y}) \arrow[r, "\mathsf{m}^0 \otimes \overline{\mathbb{F}}(\overrightarrow{Y})"] \arrow[d, phantom, "\wr\|"] & !\top \otimes \overline{\mathbb{F}}(\overrightarrow{Y}) \arrow[d, "\widehat{\mathbb{F}}_{\top, \overrightarrow{Y}}"] \\
\overline{\mathbb{F}}(1 \otimes \overrightarrow{Y}) \arrow[r, "\overline{\mathbb{F}}(\mathsf{m}^0 \otimes \overrightarrow{Y})"] & \overline{\mathbb{F}}(!\top \otimes \overrightarrow{Y})
\end{tikzcd}
\qquad
\begin{tikzcd}
!X \otimes \overline{\mathbb{F}}(\overrightarrow{Y}) \arrow[r, "\mathsf{dig}_X \otimes \overline{\mathbb{F}}(\overrightarrow{Y})"] \arrow[d, "\widehat{\mathbb{F}}_{X, \overrightarrow{Y}}"'] & !!X \otimes \overline{\mathbb{F}}(\overrightarrow{Y}) \arrow[d, "\widehat{\mathbb{F}}_{!X, \overrightarrow{Y}}"] \\
\overline{\mathbb{F}}(!X \otimes \overrightarrow{Y}) \arrow[r, "\overline{\mathbb{F}}(\mathsf{dig}_X \otimes \overrightarrow{Y})"] & \overline{\mathbb{F}}(!!X \otimes \overrightarrow{Y})
\end{tikzcd}$$

Figure 8: Monoidality and dig **diagrams for strong functors**

L PROOF OF LEMMA 6.2

We first introduce a notation to make the proof of Lemma 6.2 simpler. Given a $\mu\mathsf{LL}_\infty$ connective r, we denote by $r((A_l)_{l\in L})$ the formula A which is obtained by the application of r on the formulas A_l. For example, we denote $A \otimes B$ as $\otimes(A_1, A_2)$. We also denote by $\llbracket r \rrbracket$ the functorial application of r, namely the functor $\llbracket r \rrbracket : \Pi_{l\in L}\mathcal{L} \to \mathcal{L}$ such that $\llbracket A \rrbracket = \llbracket r \rrbracket((\llbracket A_l \rrbracket)_{l\in L})$

Now, let π be the following proof such that r is a $\mu\mathsf{LL}_\infty$ rule and $A = r((A_l)_{l\in L})$:

$$\dfrac{\dfrac{(\rho_l)_{l\in L}}{(\vdash \Delta_l, A_l)_{l\in L}}}{\vdash \Gamma, A}\ (r)$$

Then one can easily (by a case analysis on r) see that $\llbracket \pi \rrbracket = \llbracket r \rrbracket((\llbracket \rho_l \rrbracket)_{l\in L})$.

Lemma L.1. *Let π be a strongly connected and strongly valid proof of $\vdash \Gamma^\perp, \nu\zeta\, A$ where the last inference rule is the (ν) rule. Then $\llbracket \Uparrow(\pi) \rrbracket$ is the following morphism:*

$$\llbracket \Gamma \rrbracket \xrightarrow{\llbracket \pi \rrbracket} \llbracket \nu\zeta\, A \rrbracket \simeq \llbracket A \rrbracket(\llbracket \nu\zeta\, A \rrbracket) \xrightarrow{\llbracket A \rrbracket(\phi_A)} \llbracket A \rrbracket(\llbracket I^A_\Gamma \rrbracket)$$

Proof. The proof is by induction on the structure of π, and by case analysis of the inference rule (r) in Definition 3.8. We will use the same notation as Definition 3.8.

▹ If r is some $\mu\mathsf{LL}_\infty$ rule which does not correspond to a progress of t, then we have the following situation:

$$\begin{aligned}
\llbracket \Uparrow(R) \rrbracket &= \llbracket r \rrbracket((\llbracket \Xi_l \rrbracket(\phi_A) \circ \llbracket \Uparrow(R'_l) \rrbracket)_{l\in L}) \\
&= \llbracket r \rrbracket((\llbracket \Xi_l \rrbracket(\phi_A) \circ (\llbracket \Sigma_l \rrbracket(\phi_A) \circ \llbracket R'_l \rrbracket))_{l\in L}) \quad \text{by IH} \\
&= \llbracket r \rrbracket((\llbracket \Xi_l \rrbracket(\phi_A) \circ \llbracket \Sigma_l \rrbracket(\phi_A)) \circ \llbracket R'_l \rrbracket)_{l\in L} \\
&= (\llbracket r \rrbracket(\llbracket \Xi_l \rrbracket(\phi_A) \circ \llbracket \Sigma_l \rrbracket(\phi_A)))_{l\in L} \circ (\llbracket r \rrbracket(\llbracket R'_l \rrbracket))_{l\in L} \quad \text{by functoriality of } \llbracket r \rrbracket \\
&= (\llbracket r \rrbracket(\llbracket \Xi_l \rrbracket(\phi_A) \circ \llbracket \Sigma_l \rrbracket(\phi_A)))_{l\in L} \circ \llbracket R \rrbracket \quad \text{by definition of } \llbracket r \rrbracket \\
&= \llbracket \Sigma \rrbracket(\phi_A) \circ \llbracket R \rrbracket \quad \text{by functoriality of } \llbracket \Xi_l \rrbracket \text{ and } \llbracket \Sigma_l \rrbracket
\end{aligned}$$

▹ If r is the (ν) rule on some formula $F'\,[\nu\zeta\, F/\zeta]$ of $\Sigma\,[\nu\zeta\, F/\zeta]$. Then we have

$$\begin{aligned}
\llbracket \Uparrow(R) \rrbracket &= \mathsf{in}_1 \circ \llbracket \Uparrow(R') \rrbracket \quad \text{by interpretation of the (unfold) rule} \\
&= \mathsf{in}_1 \circ (\llbracket F \rrbracket(\phi_A) \circ \llbracket R' \rrbracket) \quad \text{by IH} \\
&= (\mathsf{in}_1 \circ \llbracket \phi_A \rrbracket) \circ \llbracket R' \rrbracket \\
&= \phi_A \circ \llbracket R' \rrbracket \quad \text{by Lemma 6.1} \\
&= \phi_A \circ \llbracket R \rrbracket \quad \text{by interpretation of the } (\nu) \text{ rule} \\
&= \llbracket \Sigma \rrbracket(\phi_A) \circ \llbracket R \rrbracket \quad \text{since } \Sigma = \zeta
\end{aligned}$$

□

M PROOF OF THEOREM 6.3

Theorem M.1. *Let π be a strongly valid proof of $\vdash \Gamma^\perp, \nu\zeta\, A$. Then $\llbracket \pi^{\mathsf{fin}} \rrbracket = \llbracket \pi \rrbracket$ where the interpretations of proofs are in any $\mu\mathsf{LL}_\infty$ model.*

Proof. The proof is by induction on $\mathrm{size}(\pi)$ providing a base case if $\mathrm{elc}(\pi) = 0$. In the base case, we have $\pi^{\mathsf{fin}} = \pi$, since we have a finite proof (no cycle), so, it is obvious that $\llbracket \pi^{\mathsf{fin}} \rrbracket = \llbracket \pi \rrbracket$. Then we consider two cases. Either π is strongly connected as graph or not.

▹ We first assume that π is strongly connected. Then, there is an infinite path p that visits all the sequents of π. Let t be a trace of p, and, without loss of generality, let $\vdash \Gamma^\perp, \nu\zeta\, A$ be the sequent where the minimal formula of t has been unfolded. Graphically, π is shown in Figure 1, page 6.

We now consider the $\mu\mathsf{LL}_\infty$ proof $\Uparrow(\pi)$ of $\vdash \Gamma^\perp, A\left[I^A_\Gamma/\zeta\right]$. The complexity of the proof $\Uparrow(\pi)$ is strictly less than that of π, since $\mathrm{elc}(\Uparrow(\pi)) < \mathrm{elc}(\pi)$. So, by induction hypothesis, there is a $\mu\mathsf{LL}$ (finite) proof ρ of $\vdash \Gamma^\perp, A\left[I^A_\Gamma/\zeta\right]$ such that $\llbracket \rho \rrbracket = \llbracket \Uparrow(\pi) \rrbracket$. In this case, the π^{fin} is defined as follows where the righmost proof of $\vdash I^A_\Gamma, \Gamma^\perp$ is the derived rule (close).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp\left[(I^A_\Gamma)^\perp/\zeta\right], A\left[I^A_\Gamma/\zeta\right]}\ (\mathrm{ax}) \qquad \dfrac{\rho}{\vdash \Gamma^\perp, A\left[I^A_\Gamma/\zeta\right]}}{\vdash A^\perp\left[(I^A_\Gamma)^\perp/\zeta\right] \mathbin{\&} \Gamma^\perp, A\left[I^A_\Gamma/\zeta\right]}\ (\&)}{\vdash (I^A_\Gamma)^\perp, A\left[I^A_\Gamma/\zeta\right]}\ (\mu)}{\vdash (I^A_\Gamma)^\perp, \nu\zeta\, A}\ (\nu_{\mathrm{rec}}) \qquad \dfrac{\dfrac{\dfrac{}{\vdash \Gamma, \Gamma^\perp}\ (\mathrm{ax})}{\vdash A\left[I^A_\Gamma/\zeta\right] \oplus \Gamma, \Gamma^\perp}\ (\oplus_2)}{\vdash I^A_\Gamma, \Gamma^\perp}\ (\nu)}{\vdash \Gamma^\perp, \nu\zeta\, A}\ (\mathrm{cut})$$

Let f be the interpretation of the proof of $\vdash (I^A_\Gamma)^\perp, \nu\zeta\, A$. The morphism f satisfies the following universal property:

$$\begin{tikzcd}
\llbracket I^A_\Gamma \rrbracket = \llbracket A\left[I^A_\Gamma/\zeta\right] \rrbracket \oplus \Gamma \arrow[r, "\langle \mathsf{Id}, \llbracket \rho \rrbracket \rangle"] \arrow[d, "f"] & \llbracket A \rrbracket(\llbracket I^A_\Gamma \rrbracket) \arrow[dl, "\llbracket A \rrbracket(f)"] \\
\llbracket \nu\zeta\, A \rrbracket = \llbracket A \rrbracket(\llbracket \nu\zeta\, A \rrbracket) &
\end{tikzcd}$$

By Lemma 6.2, we have $\llbracket \rho \rrbracket = \llbracket A \rrbracket(\phi_A) \circ \llbracket \pi \rrbracket$, and hence

$$\begin{tikzcd}
\llbracket I^A_\Gamma \rrbracket \arrow[r, "f"] \arrow[d, "="] & \llbracket \nu\zeta\, A \rrbracket \arrow[dd, "="] \\
\llbracket A \rrbracket(\llbracket I^A_\Gamma \rrbracket) \oplus \llbracket \Gamma \rrbracket \arrow[d, "\langle \mathsf{Id}, \llbracket A \rrbracket(\phi_A) \circ \llbracket \pi \rrbracket \rangle"'] & \\
\llbracket A \rrbracket(\llbracket I^A_\Gamma \rrbracket) \arrow[r, "\llbracket A \rrbracket(f)"] & \llbracket A \rrbracket(\llbracket \nu\zeta\, A \rrbracket)
\end{tikzcd}$$

Moreover, we have the following diagram by Lemma 6.1:

$$\begin{tikzcd}
\llbracket \nu\zeta\, A \rrbracket \arrow[rr, "\phi_A"] \arrow[dd, "="] & & \llbracket I^A_\Gamma \rrbracket \arrow[d, "="] \\
& & \llbracket A \rrbracket(\llbracket I^A_\Gamma \rrbracket) \oplus \llbracket \Gamma \rrbracket \arrow[d, "\langle \mathsf{Id}, \llbracket A \rrbracket(\phi_A) \circ \llbracket \pi \rrbracket \rangle"] \\
\llbracket A \rrbracket(\llbracket \nu\zeta\, A \rrbracket) \arrow[rr, "\llbracket A \rrbracket(\phi_A)"] & & \llbracket A \rrbracket(\llbracket I^A_\Gamma \rrbracket)
\end{tikzcd}$$

Hence, we have:

$$\begin{tikzcd}
\llbracket \nu\zeta\, A \rrbracket \arrow[r, "\phi_A"] \arrow[d, "="] & \llbracket I^A_\Gamma \rrbracket \arrow[r, "f"] & \llbracket \nu\zeta\, A \rrbracket \arrow[d] \\
\llbracket A \rrbracket(\llbracket \nu\zeta\, A \rrbracket) \arrow[r, "\llbracket A \rrbracket(\phi_A)"] & \llbracket A \rrbracket(\llbracket I^A_\Gamma \rrbracket) \arrow[r, "\llbracket A \rrbracket(f)"] & \llbracket A \rrbracket(\llbracket \nu\zeta\, A \rrbracket)
\end{tikzcd}$$

So, we have $\llbracket A \rrbracket(f \circ \phi_A) = f \circ \phi_A$. By the universal property of $\llbracket \nu\zeta\, A \rrbracket$, we conclude that $f \circ \phi_A = \mathsf{Id}$.

Since $\llbracket \pi^{\mathsf{fin}} \rrbracket = f \circ \mathsf{in}_2$, we have the following using Lemma 6.2:

$$\begin{tikzcd}
\llbracket \Gamma \rrbracket \arrow[rr, "\llbracket \pi^{\mathsf{fin}} \rrbracket"] \arrow[d, "\mathsf{in}_2"] & & \llbracket \nu\zeta\, A \rrbracket \arrow[dd, "="] \\
\llbracket A \rrbracket(\llbracket I^A_\Gamma \rrbracket) \oplus \llbracket \Gamma \rrbracket = \llbracket I^A_\Gamma \rrbracket \arrow[d, "\langle \mathsf{Id}, \llbracket A \rrbracket(\phi_A) \circ \llbracket \pi \rrbracket \rangle"] & & \\
\llbracket A \rrbracket(\llbracket I^A_\Gamma \rrbracket) \arrow[rr, "\llbracket A \rrbracket(f)"] & & \llbracket A \rrbracket(\llbracket \nu\zeta\, A \rrbracket)
\end{tikzcd}$$

As $(\langle \mathsf{Id}, \llbracket A \rrbracket(\phi_A) \circ \llbracket \pi \rrbracket \rangle) \circ \mathsf{in}_2 = \llbracket A \rrbracket(\phi_A) \circ \llbracket \pi \rrbracket$. Hence the following square commutes:

$$\begin{tikzcd}
\llbracket \Gamma \rrbracket \arrow[r, "\llbracket \pi^{\mathsf{fin}} \rrbracket"] \arrow[d, "\llbracket \pi \rrbracket"] & \llbracket \nu\zeta\, A \rrbracket \arrow[dd, "="] \\
\llbracket \nu\zeta\, A \rrbracket \arrow[d, "\llbracket A \rrbracket(\phi_A)"] & \\
\llbracket A \rrbracket(\llbracket I^A_\Gamma \rrbracket) \arrow[r, "\llbracket A \rrbracket(f)"] & \llbracket A \rrbracket(\llbracket \nu\zeta\, A \rrbracket)
\end{tikzcd}$$

We have $\llbracket A \rrbracket(f) \circ \llbracket A \rrbracket(\phi_A) = \llbracket A \rrbracket(f \circ \phi_A) = \mathsf{Id}$, since $f \circ \phi_A = \mathsf{Id}$. Therefore, we conclude that $\llbracket \pi \rrbracket = \llbracket \pi^{\mathsf{fin}} \rrbracket$.

▹ We now consider the case that π is not strongly connected, then there are two sequents $\vdash \Gamma$ and $\vdash \Delta$ such that there is no path from $\vdash \Gamma$ to $\vdash \Delta$. Let π_1 be the proof tree which is the reachable part of π from $\vdash \Gamma$, and let π_2 be the proof tree obtained from π by adding an auxiliary rule r on $\vdash \Gamma$ and taking the reachable part from the conclusion of π. Since π_1 does not have $\vdash \Gamma$, we have $\mathsf{nax}(\pi_1) < \mathsf{nax}(\pi)$, and then by induction hypothesis we have $\llbracket \pi_1 \rrbracket = \llbracket \pi_1^{\mathsf{fin}} \rrbracket$. We now take the interpretation of the rule r with the conclusion $\vdash \Gamma$ as $\llbracket \pi_1 \rrbracket$. Then by removing $\vdash \Gamma$ from π_2, we have $\mathsf{nax}(\pi_2) < \mathsf{nax}(\pi)$. Hence, by induction hypothesis, we have $\llbracket \pi_2 \rrbracket = \llbracket \pi_2^{\mathsf{fin}} \rrbracket$. As π^{fin} is plugging two proofs π_1^{fin} and pi_2^{fin}, we can conclude that $\llbracket \pi \rrbracket = \llbracket \pi^{\mathsf{fin}} \rrbracket$. □

N PROOF OF THEOREM 6.5

Theorem N.1. *Let $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ be a $\mu\mathsf{LL}_\infty$ model, and π be a strongly valid $\mu\mathsf{LL}$ proof. Then $\llbracket \pi^{\mathsf{fin}} \rrbracket = \llbracket \pi \rrbracket$ where the interpretations of proofs are in $O_{\perp\!\!\perp}(\mathcal{L})$.*

Proof. We know that $\llbracket \pi \rrbracket_{O_{\perp\!\!\perp}(\mathcal{L})} = \llbracket \pi \rrbracket_{\mathcal{L}}$ and $\llbracket \pi^{\mathsf{fin}} \rrbracket_{O_{\perp\!\!\perp}(\mathcal{L})} = \llbracket \pi^{\mathsf{fin}} \rrbracket_{\mathcal{L}}$. By Corollary 6.4, we have $\llbracket \pi \rrbracket_{\mathcal{L}} = \llbracket \pi^{\mathsf{fin}} \rrbracket_{\mathcal{L}}$. Hence we have $\llbracket \pi^{\mathsf{fin}} \rrbracket = \llbracket \pi \rrbracket_{O_{\perp\!\!\perp}(\mathcal{L})}$. □

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