On denotations of circular and non-wellfounded proofs

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Abstract

This paper investigates the question of denotational invariants of non-wellfounded and circular proofs of the linear logic with least and greatest fixed-points. Indeed, while non-wellfounded and circular proof theory made significant progress in the last twenty years, the corresponding denotational semantics is still underdeveloped.

A denotational semantics for non-wellfounded proofs, based on a notion of totality, is provided, building on previous work by Ehrhard and Jafarrahmani. Several properties of the semantics are then studied: its soundness, the relation between totality and validity and the semantical content of the translation from finitary proofs to circular proofs. Finally, the paper focuses on circular proofs, trying to benefit from their regularity in order to define inductively the interpretation function. It is argued why the usual validity condition is too general for that purpose, while a fragment of circular proofs, strongly valid proofs, constitutes a well-behaved class for such an inductive interpretation.

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1 Introduction

In the framework of logics providing induction and potentially coinduction (such as the μ-calculus, logics with inductive definitions, logics for Kleene algebras, etc...), circular and non-wellfounded proofs have gained growing attention over the past twenty years. Regarding this circular and non-wellfounded proofs, different proof systems have been considered over different: for classical logic [12, 13, 11], for intuitionistic logic [15], for linear logic [6, 23, 32, 18] as well as for linear-time or branching-time temporal logics [40, 27, 40, 17, 20, 2].

Beside non-wellfounded proof systems, there are also finitary proof systems that allow us to do inductive and coinductive reasoning. For instance, in the case of linear logic, Baelde and Miller considered an extension μMALL of multiplicative additive linear logic with induction and coinduction principles [3, 7] in the form of Park's rules. It is worth mentioning that actually those finitary proofs systems predate the circular ones in general. It seems that it is generally accepted that if we want to have a cut-elimination theorem for the finitary proof systems with an induction principle, then the price to pay is to loose the sub-formula property (this is mentioned by Per Martin-Löf [28]). There are basically two ways to discard such a situation: either considering infinitary logic in the sense of [34, 37], or considering non-well founded proofs as mentioned above.

The relationship between finitary and non well-founded proof systems is an important and often difficult question, which remains open for a number of systems. In particular, in the substructural fragments of the μ-calculus, it is not known whether the regular fragment of

non-wellfounded proofs coincides with the finitary fragment. Berardi and Tatsuta showed [10] that in general circular and inductive proofs are not equivalent for the system of inductive definitions in classical logic for the first-order language [13]. It is also shown, by Simpson [35] on the one hand and Berardi and Tatsuta [9] on the other hand, that circular and inductive proofs are equivalent for classical logic when both systems (inductive and circular) contain Peano arithmetic. This question is still open for linear logic, and what we only know till now is that the provability of $\mu\mathsf{LL}_\infty$ circular proofs is strictly included in the provability of arbitrary $\mu\mathsf{LL}_\infty$ proofs based on the recent result by Das, De, and Saurin [16].

However, one inclusion is clear, it proceeds by "unfolding" the (co)inductive inferences using the ability to build circular reasonings. For instance, in the case of $\mu\mathsf{MALL}$, the finitary version of Park's rule [41] (the (ν_{rec}) rule) will be transformed to the following circular proof:

$$\dfrac{\overset{\pi_1}{\vdash \Delta, A} \qquad \overset{\pi_2}{\vdash ?\Gamma, A^\perp, F\,[A/\zeta]}}{\vdash \Delta, ?\Gamma, \nu\zeta\,.F}\;(\nu_{\mathsf{rec}}) \;\rightsquigarrow\; \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A^\perp, \nu\zeta\,.F}{\vdash ?\Gamma, (F\,[A/\zeta])^\perp, F\,[\nu\zeta\,.F/\zeta]}\;(\mathfrak{F}_F) \qquad \overset{\pi_2}{\vdash ?\Gamma, A^\perp, F\,[A/\zeta]}}{\vdash ?\Gamma, ?\Gamma, A^\perp, F\,[\nu\zeta\,.F/\zeta]}\;(\mathsf{cut})}{\vdash ?\Gamma, ?\Gamma, A^\perp, \nu\zeta\,.F}\;(\nu)}{\vdash ?\Gamma, A^\perp, \nu\zeta\,.F}\;(\mathsf{c}) \qquad \overset{\pi_1}{\vdash \Delta, A}}{\vdash \Delta, ?\Gamma, \nu\zeta\,.F}\;(\mathsf{cut})$$

Such translations are known to preserve provability. The present paper aims at clarifying the situation on operational properties of such translations, using techniques from denotational semantics.

This paper investigates the question of denotational invariants of non-wellfounded proofs of linear logic with least and greatest fixed points. Indeed, while the proof theory of circular proofs made progress in the last twenty years, their denotational semantics is still underdeveloped. Santocanale considered circular proofs in the framework of purely Additive linear logic, and he provided a categorical interpretation of circular proofs in μ-bicomplete categories [32, 23]. In this paper we will consider the full linear logic, whereas it is not clear how μ-bicomplete categories provide the monoidal and exponential structures required for interpreting $\mu\mathsf{LL}$. However, we only provide concrete models, and not categorical axiomatization.

A denotational semantics for non-wellfounded proofs is provided, building on previous work by the first and second authors [21] which consists in:

- a categorical semantics of $\mu\mathsf{LL}$ extending the standard notion of Seely category of classical linear logic;
- a simple instance of this setting based on the category **Rel** of sets and relations, which does not distinguish least from greatest fixed points;
- an enrichment of the previous model by considering sets equipped with an additional structure of *totality*: a non-uniform totality space (NUTS) is a pair $X = (|X|, \mathcal{T}(X))$ where $|X|$ is a set and $\mathcal{T}(X)$ is a set of subsets such that it coincides with its bidual for a duality expressed in terms of non-empty intersections.

This semantics is used to investigate the denotational content of the standard translation from finitary proofs to non-wellfounded ones: it is shown that the above mentioned translation from finitary proofs to circular ones is denotationally transparent (preserving semantics), suggesting that it is *the* correct translation. Moreover, the paper studies some properties of this semantics:

- the semantics is indeed sound in the sense that each proof of an infinite cut-reduction sequence of proofs converging to a cut-free valid proof has the same interpretation as its limit;
- it is also shown that valid proofs are interpreted as total elements of the semantics.

Although it is not true in general that the totality of the interpretation of a proof implies its validity, the notion of totality in NUTS provides a sort of maximal notion for validity as, intuitively, $\mathcal{T}(X)$ represents the total, that is, terminating computations of type X.

2 Background

Linear logic (LL) was introduced by Jean-Yves Girard in his seminal work [25]. LL is a refinement of both classical and intuitionistic logic taking its roots in the analysis of the denotational interpretation of System F in coherence spaces [24]. Contrary to classical logic LK, LL is a substructural logic: one does not have free access to the structural rules of weakening and contraction. More precisely, we can only weaken and contract formulas if they have been marked with the so-called exponential modalities.

The remainder of this section recalls how one can extend LL with least and greatest fixed points operators.

2.1 Syntax of formulas of linear logic with fixpoints of types

We assume to be given an infinite set of propositional variables $\mathcal{V}$ (ranged over by Greek letters $\zeta, \xi \dots$). We introduce a language of propositional LL formulas with least and greatest fixed points, called μLL pre-formulas:

$$
\begin{aligned}
A, B, \dots := & 1 \mid \bot \mid A \otimes B \mid A \parr B \mid 0 \mid \top \mid A \oplus B \mid A \,\&\, B \mid !A \mid ?A \\
& \mid \zeta \mid \mu\zeta\, A \mid \nu\zeta\, A
\end{aligned}
$$

The notion of closed types is defined as usual, the two last constructions being the only binders. Refer to closed pre-formulas as μLL *formulas*.

We can define two basic operations on formulas.

- *Substitution*: $A\,[B/\zeta]$, taking care of avoiding the capture of free variables (using α-conversion).
- *Negation* or *dualization*: defined by induction on formulas as usual for LL formulas plus $(\zeta)^{\bot} = \zeta$, $(\mu\zeta\, A)^{\bot} = \nu\zeta\,(A)^{\bot}$ and $(\nu\zeta\, A)^{\bot} = \mu\zeta\,(A)^{\bot}$. Obviously $A^{\bot\bot} = A$ for any formula A.
- *Sub-formula*: We consider two notions of sub-formulas, the usual one and a notion of sub-formula which is specific to the μ-calculus, the Fischer-Ladner subformulas. Those are defined in Appendix A.

In the following sections, we shall consider two proof systems for deriving judgments concerning μLL formulas, a finitary proof system and a non-wellfounded one. Those judgments which will be derived will be *sequents* $\vdash \Gamma$ where Γ is an *ordered list* of μLL formulas.

▶ Remark 1. Using sequents as lists allows us to distinguish two different *occurrences* of the same formula in a sequent, by referring to their respective position in the sequent. The ability to distinguish occurrences is crucial to give a computational content to proofs and, in the following, it will even be required to define what is a *valid* non-wellfounded proof, using the notion of *threads*.

The inference rule to be introduced in the following subsections will be equipped with a (pretty standard [14]) notion of *formula ancestor*, relating for each inference, occurrences of formulas in the conclusion to occurrences of formulas in the premisses. The ancestry relation will be defined graphically in the proof system (as colored links) and will usually be kept implicit on examples unless useful, such as when exhibiting a validating thread. When we

The identity and structural fragments:

$$\frac{}{\vdash F, F^{\perp}}\ (\mathsf{Ax}) \qquad \frac{\vdash \Gamma, F \quad \vdash F^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (\mathsf{Cut}) \qquad \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ (\mathsf{X})$$

The multiplicative fragment:

$$\frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ (\parr) \qquad \frac{\vdash F, \Gamma \quad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash \Gamma}{\vdash \perp, \Gamma}\ (\perp) \qquad \frac{}{\vdash 1}\ (1)$$

The additive fragment:

$$\frac{\vdash F, \Gamma \quad \vdash G, \Gamma}{\vdash F \,\&\, G, \Gamma}\ (\&) \qquad \frac{\vdash A_i, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ (\oplus_i) \qquad \frac{}{\vdash \top, \Gamma}\ (\top) \qquad \text{(no rule for 0)}$$

The exponential fragment:

$$\frac{\vdash F, \Gamma}{\vdash ?F, \Gamma}\ (?\mathsf{d}) \qquad \frac{\vdash F, ?\Gamma}{\vdash !F, ?\Gamma}\ (!\mathsf{p}) \qquad \frac{\vdash \Gamma}{\vdash ?F, \Gamma}\ (?\mathsf{w}) \qquad \frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma}\ (?\mathsf{c})$$

Figure 1 Inference rules of LL

$$\frac{\vdash \Gamma, F\,[\mu\zeta\,.F/\zeta]}{\vdash \Gamma, \mu\zeta\,.F}\ (\mu) \qquad \frac{\vdash \Delta, A \quad \vdash ?\Gamma, A^{\perp}, F\,[A/\zeta]}{\vdash \Delta, ?\Gamma, \nu\zeta\,.F}\ (\nu_{\mathsf{rec}})$$

Figure 2 Fixed-point inference rules of μLL

depict a line linking a list in the conclusion to the same list in the premise, we mean that each formula of the list is in relation with the formula in the same position in the other list.

2.2 Finitary μLL

In the present section, we will briefly describe the syntax of proofs of μLL [3]. The proof system of μLL, extends the usual one-sided sequent calculus of classical propositional LL [25], which are recalled in Figure 1, with the (μ) and (ν_{rec}) rules, given in Figure 2.

▶ **Example 2.** As an example, consider the type of natural numbers $\mathsf{nat} = \mu\zeta\,.(1 \oplus \zeta)$ and its dual $\mathsf{nat}^{\perp} = \nu\zeta\,.(\perp \,\&\, \zeta)$. The following μLL proofs correspond respectively to the encoding of the natural numbers and of the successor function:

$$\pi_0 = \dfrac{\dfrac{\dfrac{}{\vdash 1}\ (1)}{\vdash 1 \oplus \mathsf{nat}}\ (\oplus_1)}{\vdash \mathsf{nat}}\ (\mu) \qquad \pi_{\mathsf{k}+1} = \dfrac{\dfrac{\pi_{\mathsf{k}}}{\vdash 1 \oplus \mathsf{nat}}\ (\oplus_2)}{\vdash \mathsf{nat}}\ (\mu) \qquad \pi_{\mathsf{succ}} = \dfrac{\dfrac{\dfrac{}{\vdash \mathsf{nat}^{\perp}, \mathsf{nat}}\ (\mathsf{Ax})}{\vdash \mathsf{nat}^{\perp}, 1 \oplus \mathsf{nat}}\ (\oplus_2)}{\vdash \mathsf{nat}^{\perp}, \mathsf{nat}}\ (\mu)$$

2.3 Non-well-founded LL with fixed points ($\mu\mathsf{LL}_\infty$)

The syntax of $\mu\mathsf{LL}_\infty$ formulas is exactly the same as the one for μLL in 2.2. The inference rules of $\mu\mathsf{LL}_\infty$ is the extension of rules of [18, 6] with exponential rules of LL. In other words, the inference rules of $\mu\mathsf{LL}_\infty$ are the rules of LL [25] (see Figure 1) plus the two fixed-point inferences given in Figure 3.

$$\frac{\vdash F[\mu\zeta.F/\zeta], \Gamma}{\vdash \mu\zeta.F, \Gamma}\ (\mu) \qquad \frac{\vdash F[\nu\zeta.F/\zeta], \Gamma}{\vdash \nu\zeta.F, \Gamma}\ (\nu)$$

Figure 3 Fixed-point inference rules of $\mu\mathsf{LL}^{\infty}$

A $\mu\mathsf{LL}_\infty$ pre-proof is a possibly infinite tree, generated by the inference rules of $\mu\mathsf{LL}_\infty$. Among all $\mu\mathsf{LL}_\infty$ pre-proofs, the regular/circular proofs are the ones that have finitely many sub-trees. Those circular proofs can be represented with finite proof-trees having back-edges or labels.

▶ **Example 3.** The following proof corresponds to the function from nat to nat which returns the double of its input:

$$\pi_{double} = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1 \oplus \mathsf{nat}}\,(\oplus_1)}{\vdash \mathsf{nat}}\,(\mu)}{\vdash \bot, \mathsf{nat}}\,(\bot) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathsf{nat}^\perp, \mathsf{nat}}{\vdash \mathsf{nat}^\perp, 1 \oplus \mathsf{nat}}\,(\oplus_2)}{\vdash \mathsf{nat}^\perp, \mathsf{nat}}\,(\mu)}{\vdash \mathsf{nat}^\perp, 1 \oplus \mathsf{nat}}\,(\oplus_2)}{\vdash \mathsf{nat}^\perp, \mathsf{nat}}\,(\mu)}{\dfrac{\vdash \bot \mathbin{\&} \mathsf{nat}^\perp, \mathsf{nat}}{\vdash \mathsf{nat}^\perp, \mathsf{nat}}\,(\nu)}\,(\&)$$

However, in general the pre-proofs can be unsound. For instance one can provide a pre-proof for any sequent $\vdash \Gamma$ (and in particular a pre-proof of the empty sequent $\vdash$) as follows:

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \nu\zeta.\zeta}\,(\nu)}{\vdash \nu\zeta.\zeta}\,(\nu) \qquad \dfrac{\dfrac{\vdots}{\vdash \Gamma, \mu\zeta.\zeta}\,(\mu)}{\vdash \Gamma, \mu\zeta.\zeta}\,(\mu)}{\vdash \Gamma}\,(\mathsf{cut})$$

In [18, 6], a criterion, called *validity condition*, is provided in order to distinguish proper proofs from pre-proofs. We only sum up this criterion here and provide some examples, and we refer to [30] for more details.

▶ Remark 4. The variable ζ is a subformula of $A = (\nu\zeta.\zeta) \otimes \zeta$. However, there are two different ζ in the formula A, one is the bound ζ and the other is the free one. To distinguish them, one can use the notion of *occurrence* [30]. As we also need this notion for Definition 8, we provide it here.

▶ **Definition 5.** *An infinite branch of a pre-proof π is a sequence $(\Gamma_i, j_i)_{i\in\omega}$ of pairs of sequents and indices, for $j_i \in \{1, 2\}$, such that Γ_0 is the root of π, Γ_{i+1} is the j_ith premises of Γ_i in the proof tree for each $i \in \omega$.*

▶ **Definition 6.** *A* thread *on an infinite branch $\beta = (\Gamma_i, j_i)_{i\in\omega}$ is an infinite sequence of formula occurrences $t = (F_i)_{k\leq i\in\omega}$ such that for any $i \geq k$, $F_i \in \Gamma_i$ and F_{i+1} is an immediate ancestor of F_i.*

A thread t is stationary *if only finitely many of the F_i are principal in Γ_i. By principal formula, we mean the one that the inference rule is applied on.*

We denote by $\mathsf{Inf}(t)$ the set of formulas that occur infinitely often in t.

With each infinite branch is associated a set of threads. Notice that there is not a unique thread in general (and there may be none).

For instance, we have two threads in the following proof: $\dfrac{\dfrac{\vdots}{\vdash \mu\zeta.\zeta, \nu\xi.\xi}\,(\nu)}{\vdash \mu\zeta.\zeta, \nu\xi.\xi}\,(\nu)$. The threads are $t_1 = \mu\zeta.\zeta, \mu\zeta.\zeta, \mu\zeta.\zeta, \cdots$ and $t_2 = \nu\xi.\xi, \nu\xi.\xi, \nu\xi.\xi, \cdots$. Since the only rule applied in the proof is the (ν) rule, the formulas $\mu\zeta.\zeta$ are never principal, and the thread corresponding to the $\mu\zeta.\zeta$ is called *stationary*.

Now, we have all the required material to define the notion of valid threads and then valid proofs.

$$(a) \qquad \cfrac{\cfrac{\cfrac{\vdash F \qquad \vdash G}{\vdash F \otimes G}\,(\otimes)}{\vdash G}\,(\nu)}{\vdash F}\,(\mu) \qquad\qquad (b) \qquad \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\vdash F, G}{\vdash F, \bot, G}\,(\bot)}{\vdash F, G \oplus \bot, G}\,(\oplus_2)}{\vdash (F \parr (G \oplus \bot)), G}\,(\parr)}{\vdash 1 \oplus (F \parr (G \oplus \bot)), G}\,(\oplus_2)}{\vdash G, G}\,(\mu)}{\vdash F, G}\,(\nu)$$

Figure 4 Examples of valid and non valid pre-proofs.

▶ **Definition 7** (Valid thread, valid pre-proof). *A* valid thread t *is a non-stationary thread such that* $\mathsf{Inf}(t)$ *has a minimum (with respect to the usual sub-formula ordering) which is a* ν*-formula.*

Appendix A provides details on $\mu\mathsf{LL}_\infty$ subformulas and the minimality invoked above.

▶ **Definition 8.** *A* valid $\mu\mathsf{LL}_\infty$ proof *(or $\mu\mathsf{LL}_\infty$ proof, for short) π is a pre-proof π such that any infinite branch contains a valid thread.*

We now examine some valid and non-valid pre-proofs. Let us consider Figure 4.(a) presenting a derivation of formula $F = \mu\zeta\,.(\nu\xi\,.(\zeta \otimes \xi))$ where $G = \nu\xi\,.(F \otimes \xi)$. The only thread that we have for the leftmost branch is $t = F, G, (F \otimes G), F, \cdots$, so, $\mathsf{min}(\mathsf{Inf}(t)) = F$. Hence this thread is not a valid thread, and there is no more thread on this branch. Hence this proof is not valid. Let us consider another example in Figure 4.(b), with $F = \nu\zeta\,.\mu\xi\,.(1 \oplus (\zeta \parr (\xi \oplus \bot)))$ and $G = \mu\xi\,.(1 \oplus (F \parr (\xi \oplus \bot)))$. For the thread $t_2 = F, G, (1 \oplus (F \parr (G \oplus \bot))), (F \parr (G \oplus \bot)), F, \cdots$ we have $\mathsf{min}(\mathsf{Inf}(t_2)) = F$, since $F \leqslant_{sub} G$. Hence t_2 is a valid thread, and hence this proof is valid.

The set of primitive (single step) reduction rules of $\mu\mathsf{LL}_\infty$ are the ones for LL plus the following one together with the corresponding commutation rules (Figure 3.2 of [18]). The proof

$$\cfrac{\cfrac{\cfrac{\pi}{\vdash \Gamma, F\,[\mu\zeta\,.F/\zeta]}}{\vdash \Gamma, \mu\zeta\,.F}\,(\mu) \qquad \cfrac{\cfrac{\pi'}{\vdash \Delta, F^\perp\,[\nu\zeta\,.F^\perp/\zeta]}}{\vdash \Delta, \nu\zeta\,.F^\perp}\,(\nu)}{\vdash \Gamma, \Delta}\,(\mathsf{cut}) \quad \text{reduces to} \quad \cfrac{\cfrac{\pi}{\vdash \Gamma, F\,[\mu\zeta\,.F/\zeta]} \qquad \cfrac{\pi'}{\vdash \Delta, F^\perp\,[\nu\zeta\,.F^\perp/\zeta]}}{\vdash \Gamma, \Delta}\,(\mathsf{cut}).$$

Various cut-elimination theorems on non-wellfounded proofs are proved in [6, 18, 4] and especially of $\mu\mathsf{LL}_\infty$ itself [33] but the rest of the paper does not rely on those normalization results, so that they can safely be ignored.

We end this section by stating the functoriality of $\mu\mathsf{LL}_\infty$ which we will use the functoriality of $\mu\mathsf{LL}_\infty$ in Section 3.2:

▶ **Proposition 9.** *Let $(\zeta, \xi_1, \ldots, \xi_k)$ be a list of pairwise distinct propositional variables containing all the free variables of a formula F and let $\overrightarrow{C} = (C_1, \ldots, C_k)$ be a sequence of closed formulas. Then the following rule is admissible in $\mu\mathsf{LL}_\infty$:*

$$\cfrac{\vdash ?\Gamma, A^\perp, B}{\vdash ?\Gamma, (F[A/\zeta, \overrightarrow{C}/\overrightarrow{\xi}])^\perp, F[B/\zeta, \overrightarrow{C}/\overrightarrow{\xi}]}\,(\mathfrak{F}_F)$$

Proof. The proof is done by induction on the formula F, and we refer to [18] for the details (Definition 2.38 of [18]). The presence of exponential modalities does not modify the proof in any non trivial way. ◀

3 Semantics of non-well-founded proofs

In this section we will first provide interpretation of formulas and inferences rules of $\mu\mathsf{LL}_\infty$ in the category **Rel** of sets and relations (this is done in Section 3.1). We therefore interpret $\mu\mathsf{LL}_\infty$ pre-proofs in **Rel**. Then we will investigate the relationship between the interpretation of inductive proofs and their image as circular proofs in a more categorical setting, i.e categorical model of $\mu\mathsf{LL}$ in the sense of [21] (this is done in Section 3.2). Afterward, in Section 4.2, we will consider the category **Nuts** of non-uniform totality spaces, which can be seen as a refinement of **Rel**, as another concrete model of $\mu\mathsf{LL}_\infty$. The interpretation of proofs in both models of **Rel** and **Nuts** are the same. However, by looking at their interpretation in **Nuts**, we will obtain more information. For instance, we will show that the interpretations of the valid proofs are total elements (Theorem 21).

3.1 Interpreting non-well-founded proofs

We first recall briefly the interpretation of formulas of linear logic with fixpoints of types [21]. The idea is to interpret a formula A with repetition-free sequence $\overrightarrow{\zeta} = (\zeta_1, \ldots, \zeta_k)$ of type variables containing all the free variables of A as a k-ary CPO strong functor. We denote the interpretation of A by $[\![A]\!]_{\overrightarrow{\zeta}}$. Then the interpretation of formula is defined by induction on the formulas in the obvious way, for instance $[\![A \otimes B]\!]_{\overrightarrow{\zeta}} = \otimes \circ ([\![A]\!]_{\overrightarrow{\zeta}}, [\![B]\!]_{\overrightarrow{\zeta}})$ considering $\otimes$ as a 2-ary CPO strong functor, and $[\![\nu\zeta\,.A]\!]_{\overrightarrow{\zeta}} = \nu([\![A]\!]_{\overrightarrow{\zeta},\zeta})$ [1]. Then one also has $[\![A^\perp]\!]_{\overrightarrow{\zeta}} = ([\![A]\!]_{\overrightarrow{\zeta}})^\perp$ up to a natural isomorphism which allows us to define other formula by De morgan duality. By $\nu([\![A]\!]_{\overrightarrow{\zeta},\zeta})$, we mean the final coalgebra of the endofunctor $[\![A]\!]_{\overrightarrow{\zeta},\zeta}$, and we know this exists as **Rel** is a CPO-enriched category and $[\![A]\!]_{\overrightarrow{\zeta},\zeta}$ is a CPO functor [22].

We now provide our definition for interpretation of $\mu\mathsf{LL}_\infty$ pre-proof in **Rel**. First, one can define the interpretation of finite $\mu\mathsf{LL}_\infty$ proofs by induction on last inference rule. One can find the interpretation of LL rules for instance in [29], and it is also given in the appendix (Section B) for any categorical model of LL. We just provide explicitly two cases below as examples, as we will use those cases afterwards.

$$\left[\!\!\left[\dfrac{\dfrac{\pi}{\vdash \Gamma}}{\vdash \Gamma, \perp}\ \perp\right]\!\!\right] = \{(\gamma, *) \mid \gamma \in [\![\pi]\!]_{\mathbf{Rel}}\} \qquad \left[\!\!\left[\dfrac{\dfrac{\pi}{\vdash \Gamma, A_i}}{\vdash \Gamma, A_1 \oplus A_2}\ (\oplus_i)\right]\!\!\right] = \{(\gamma, (i, a)) \mid (\gamma, a) \in [\![\pi]\!]\}$$

$$\left[\!\!\left[\dfrac{\dfrac{\pi_1}{\vdash \Gamma, A} \quad \dfrac{\pi_2}{\vdash \Gamma, B}}{\vdash \Gamma, A \mathbin{\&} B}\ (\&)\right]\!\!\right] = \{(\gamma, (1, a)) \mid (\gamma, a) \in [\![\pi_1]\!]\} \cup \{(\gamma, (2, b)) \mid (\gamma, b) \in [\![\pi_2]\!]\}$$

Now, we only need to say how we interpret the (ν) and (μ) rules in **Rel**, and this is done, in an obvious way, as follows:

$$\left[\!\!\left[\dfrac{\dfrac{\pi}{\vdash \Gamma, F\,[\mu\zeta\,.F/\zeta]}}{\vdash \Gamma, \mu\zeta\,.F}\ (\mu)\right]\!\!\right] = [\![\pi]\!] \qquad \left[\!\!\left[\dfrac{\dfrac{\pi}{\vdash \Gamma, F\,[\nu\zeta\,.F/\zeta]}}{\vdash \Gamma, \nu\zeta\,.F}\ (\nu)\right]\!\!\right] = [\![\pi]\!]$$

We also take the empty set as the interpretation of the (Ω) rule. Finally, given a $\mu\mathsf{LL}_\infty$ proof π, we define $[\![\pi]\!]_{\mathbf{Rel}} = \bigcup_{\rho \in \mathsf{fin}(\pi)} [\![\rho]\!]_{\mathbf{Rel}}$ where $\mathsf{fin}(\pi)$ is the set of all finite sub-pre-proof of π (we are allowed to do this, since we added the (Ω) rule).

[1] We assume that the iso between νF and $F(\nu F)$ is always the identity as this holds in our concrete models. This assumption is highly debatable from the view point of category theory where the notion of equality of objects is not really meaningful. It will be dropped in a longer version of this paper.

Let us look at an example. Consider the following circular proof $\pi_{\equiv_3}$ which correspond to the function on natural numbers which sends n to $n \mod 3$:

$$
\begin{prooftree}
\AxiomC{π_0^{nat}}
\RightLabel{$(\bot)$}
\UnaryInfC{$\vdash \mathsf{nat}, \bot$}
\AxiomC{π_1^{nat}}
\RightLabel{$(\bot)$}
\UnaryInfC{$\vdash \mathsf{nat}, \bot$}
\AxiomC{π_2^{nat}}
\RightLabel{$(\bot)$}
\UnaryInfC{$\vdash \mathsf{nat}, \bot$}
\AxiomC{$\vdash \mathsf{nat}, \mathsf{nat}^\perp$}
\RightLabel{$(\&)$}
\BinaryInfC{$\vdash \mathsf{nat}, \bot \;\&\; \mathsf{nat}^\perp$}
\RightLabel{(ν)}
\UnaryInfC{$\vdash \mathsf{nat}, \mathsf{nat}^\perp$}
\RightLabel{$(\&)$}
\BinaryInfC{$\vdash \mathsf{nat}, \bot \;\&\; \mathsf{nat}^\perp$}
\RightLabel{(ν)}
\UnaryInfC{$\vdash \mathsf{nat}, \mathsf{nat}^\perp$}
\RightLabel{$(\&)$}
\BinaryInfC{$\vdash \mathsf{nat}, \bot \;\&\; \mathsf{nat}^\perp$}
\RightLabel{(ν)}
\UnaryInfC{$\vdash \mathsf{nat}, \mathsf{nat}^\perp$}
\end{prooftree}
$$

The interpretation of π_k^{nat} in $\mathbf{Rel}$ is, up to an iso, the natural number k, and we denote it by $\underline{k}$, i.e $[\![\pi_k^{\mathsf{nat}}]\!]_{\mathbf{Rel}} = \underline{k}$. To compute interpretation of $\pi_{\equiv_3}$, we need to take supremum of the interpretation of all finite sub pre-proofs. For example, imagine that in the proof $\pi_{\equiv_3}$ above, we do a Ω rule instead of the back-edge, and called this proof σ. Then we have

$$[\![\sigma]\!]_{\mathbf{Rel}} = \{(\underline{2}, (2, (2, (1, *))), (\underline{1}, (2, (1, *))), (\underline{0}, (1, *)))\}$$

That is to say, up to an iso, we have $[\![\sigma]\!]_{\mathbf{Rel}} = \{(\underline{2}, \underline{2}), (\underline{1}, \underline{1}), (\underline{0}, \underline{0})\}$. If we do one more step, we will see that $(\underline{0}, \underline{3}) \in [\![\pi_{\equiv_3}]\!]_{\mathbf{Rel}}$. So, one can see that $[\![\pi_{\equiv_3}]\!]_{\mathbf{Rel}} = \{(\underline{n}, \underline{m}) \mid \underline{n} = \underline{m} \mod 3\}$.

3.2 On the relation between the interpretation of finite proofs and their circular correspondent

In this section we will talk about the comparison between $\mu\mathsf{LL}$ proofs and $\mu\mathsf{LL}_\infty$ circular proofs. As it is mentioned in [18], if a sequent $\vdash \Gamma$ is provable in $\mu\mathsf{LL}$, then it is provable in $\mu\mathsf{LL}_\infty$. This can be done by translating a $\mu\mathsf{LL}$ proof π of $\vdash \Gamma$ into a circular $\mu\mathsf{LL}_\infty$ proof π' of $\vdash \Gamma$ that we will denote it by $\mathsf{Trans}\,(\pi)$. This translation can be done by induction on π. We just deal with the case of the (ν_{rec}) rule, and the other ones is done in a trivial way in Appendix D.

If the last inference rule is the (ν_{rec}) rule, then $\mathsf{Trans}\left(\dfrac{\overset{\pi_1}{\vdash \Delta, A} \qquad \overset{\pi_2}{\vdash ?\Gamma, A^\perp, F\,[A/\zeta]}}{\vdash \Delta, ?\Gamma, \nu\zeta\,.F}\,(\nu_{\mathsf{rec}})\right)$ is the following circular proof using the functoriality of formulas given in Section 2.3:

$$
\begin{prooftree}
\AxiomC{$\vdash ?\Gamma, A^\perp, \nu\zeta\,.F$}
\RightLabel{$(\mathfrak{F}_F)$}
\UnaryInfC{$\vdash ?\Gamma, (F\,[A/\zeta])^\perp, F\,[\nu\zeta\,.F/\zeta]$}
\AxiomC{π_2}
\noLine
\UnaryInfC{$\vdash ?\Gamma, A^\perp, F\,[A/\zeta]$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash ?\Gamma, ?\Gamma, A^\perp, F\,[\nu\zeta\,.F/\zeta]$}
\RightLabel{(ν)}
\UnaryInfC{$\vdash ?\Gamma, ?\Gamma, A^\perp, \nu\zeta\,.F$}
\RightLabel{(c)}
\UnaryInfC{$\vdash ?\Gamma, A^\perp, \nu\zeta\,.F$}
\AxiomC{π_1}
\noLine
\UnaryInfC{$\vdash \Delta, A$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash \Delta, ?\Gamma, \nu\zeta\,.F$}
\end{prooftree}
$$

And finally, Proposition 2.14 of [18] shows that $\mathsf{Trans}\,(\pi)$ is a valid $\mu\mathsf{LL}_\infty$ proof.

Our main goal in the section is to prove that the semantic is preserved via this operation $\mathsf{Trans}\,()$. To do so, first of all, we need to say what the interpretation of a $\mu\mathsf{LL}_\infty$ circular proof is in any categorical model of $\mu\mathsf{LL}$. The interpretation of each inference rule of $\mu\mathsf{LL}_\infty$ is given in Section 3.1. To interpret the $\mu\mathsf{LL}_\infty$ circular proofs, the general idea is to associate a system of equations on the morphisms of the given category to the proof, and then proving that it has a solution which we take it as the interpretation of the circular proof. This is done in the case of additive linear logic in [23, 32]. However, in this paper we only do this on the

circular proofs that are coming from the translation of an inductive proof, i.e, image of the operation $\mathsf{Trans}\,()$, and we leave this question for all $\mu\mathsf{LL}_\infty$ circular proofs to a future work.

We suppose that we have a categorical model of $\mu\mathsf{LL}$ in the sense of [21]. That is to say a pair $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ where $\mathcal{L}$ is a model of LL and $\overrightarrow{\mathcal{L}}$ is class of strong functors which is closed under composition and a $\mu\mathsf{LL}$ operations and it also satisfies some further conditions expressing the monoidality of as well as its compatibility with the comultiplication of $!\_$.

We will use the following lemma, which is well-known in the literature on fixed points of functors [31, 1], in the proof of Theorem 11.

See proof in Appendix E.1.

▶ **Lemma 10.** *Let A be an object of a category $\mathcal{A}$ and let $f_1, f_2 \in \mathcal{A}(A, \nu\mathcal{F})$. If there exists $l \in \mathcal{A}(A, \mathcal{F}(A))$ such that $\mathcal{F}(f_i)\, l = f_i$ for $i = 1, 2$, then $f_1 = f_2$.*

See proof in Appendix E.2.

▶ **Theorem 11.** *Let π be a $\mu\mathsf{LL}$ proof. Then we have $[\![\pi]\!] = [\![\mathsf{Trans}\,(\pi)]\!]$ where the interpretation is given in a model $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ of $\mu\mathsf{LL}$.*

4 Properties of the semantics

In this section, we will show essentially the soundness of our model, and then we will also show that if we interpret the $\mu\mathsf{LL}_\infty$ proofs in the model **Nuts** which is introduce in [21], then the valid proofs will be interpreted as a total element.

4.1 Soundness of the interpretation wrt cut-elimination

4.1.1 Soundness for one-step cut-elimination

We first prove that the semantic is preserved via the one-step cut reduction rules of $\mu\mathsf{LL}_\infty$.

See proof in Appendix F.1.

▶ **Theorem 12.** *Given two finite $\mu\mathsf{LL}_\infty$ proofs π and π' such that π' is obtained from π via an one-step cut-elimination rule, then $[\![\pi]\!] = [\![\pi']\!]$.*

4.1.2 Soundness for Cauchy-sequences of cut-eliminations

One can define a natural metric d on the set of all $\mu\mathsf{LL}_\infty$ finite proofs saying $d(\pi, \pi') = 0$ if two proofs π and π' are identical, otherwise $d(\pi, \pi') = \frac{1}{2^k}$, where k is the length of the shortest position at which π and π' differ. Then we can see that indeed set of all infinite $\mu\mathsf{LL}_\infty$ proofs is the metric completion of the finite proofs (Theorem 30 of Appendix C). This is quite standard in the literature [8, 38, 39], however we have just mentioned the details of this technical development in Appendix C for the sake of self-containdness, and as it is nevertheless necessary for a precise definition of the semantics of non-wellfounded proofs that we consider in the latter sections.

See proof in Appendix F.2.

▶ **Lemma 13.** *Let (π_i) be a Cauchy sequence. Then $[\![\lim_{n\to\infty} \pi_i]\!]_{\mathbf{Rel}} = \bigcup_i \bigcap_{j>i} [\![\pi_j]\!]_{\mathbf{Rel}}$.*

▶ **Theorem 14.** *Let $(\pi_i)_{i\in\omega}$ be a Cauchy sequence such that $\forall i, j \in \omega$ we have $[\![\pi_i]\!]_{\mathbf{Rel}} = [\![\pi_j]\!]_{\mathbf{Rel}}$. Then $[\![\lim_{n\to\infty} \pi_i]\!]_{\mathbf{Rel}} = [\![\pi_0]\!]$.*

Proof.
$$\begin{aligned} [\![\lim_{n\to\infty} \pi_i]\!]_{\mathbf{Rel}} &= \bigcup_i \bigcap_{j>i} [\![\pi_j]\!]_{\mathbf{Rel}} && \text{By Lemma 13} \\ &= \bigcup \bigcap [\![\pi_0]\!]_{\mathbf{Rel}} && = [\![\pi_0]\!]_{\mathbf{Rel}} \end{aligned}$$
◀

And, we can now prove the soundness theorem for $\mu\mathsf{LL}_\infty$ as a direct conclusion of Theorem 12 and Theorem 14:

▶ **Corollary 15.** *If π and π' are proofs of $\vdash \Gamma$ and π reduces to π' by the cut-elimination rules of $\mu\mathsf{LL}_\infty$, then $[\![\pi]\!]_{\mathbf{Rel}} = [\![\pi']\!]_{\mathbf{Rel}}$.*

4.2 Valid proofs are interpreted as total elements

What we have seen till now is the interpretation of $\mu\mathsf{LL}_\infty$ proofs in **Rel** and a soundness theorem for $\mu\mathsf{LL}_\infty$ with respect to **Rel**. However, as one might notice, we did not talk about valid proofs. Indeed, Lemma 13 is true in general for any Cauchy sequence of $\mu\mathsf{LL}_\infty$ pre-proofs (not necessary the valid ones). In this section, we provide a denotational account of the validity criterion using the model introduced in [21], i.e, **Nuts**. **Nuts** is the category which has sets equipped with a notion of totality on top of it as its objects, and relations preserving totality as its morphisms. More precisely, a NUTS X is is a pair $X = (|X|, \mathcal{T}(X))$ where $|X|$ is a set and $\mathcal{T}(X) \subseteq \mathcal{P}(|X|)$ satisfies $\mathcal{T}(X) = \mathcal{T}(X)^{\perp\perp}$ for the orthogonality $\_^{\perp}$ defined as follows [2]:

$$(\mathcal{T}(X))^{\perp} = \{u' \subseteq |X| \mid \forall u \in \mathcal{T}(X)\ u \cap u' \neq \varnothing\} .$$

The morphism $f \in \mathbf{Nuts}(X, Y)$ is a morphism $f \in \mathbf{Rel}(|X|, |Y|)$ such that $\forall u \in \mathcal{T}(X)\ f \cdot u \in \mathcal{T}(Y)$ where $f \cdot u$ is the image under relation f. Working with **Nuts** has also the benefit that distinguish interpretation of least and greatest fixpoint, whereas this is not the case in **Rel** (as $X^{\perp} = X$ for any object X in **Rel**).

We prove the main result of this section which says that the interpretation of any valid proof is a total element, i.e. Theorem 21. The proof method is similar to the proof of soundness of LKID^ω in [11]. However the system of [11] is classical logic with inductive definitions, and their proof is for a Tarskian semantics. We need to adapt that proof in two aspects: considering $\mu\mathsf{LL}_\infty$ instead of LKID^ω, and trying to deal with a denotational semantics instead of a Tarskian semantics. The adaptation for $\mu\mathsf{LL}_\infty$ is somehow done in [18], since there is soundness theorem for $\mu\mathsf{MALL}_\infty$ with respect to the truncated truth semantics (a Tarskian semantics). So, basically, the main point of our proof is turning a Tarskian soundness theorem into a denotational soundness theorem.

From now on, when we write the interpretation of formula as $[\![F]\!]$, we mean its interpretation in **Nuts** (definition of $\mathcal{T}([\![F]\!])$ is provided in [21]). And when we write the interpretation of proof as $[\![\pi]\!]$, we mean its interpretation in **Rel**, i.e, $[\![\pi]\!]_{\mathbf{Rel}}$. However, at the end of this section, we will see that indeed this $[\![\pi]\!]_{\mathbf{Rel}}$ is a total element, so, it is indeed in **Nuts** (but this needs to be proven).

As one can see in [21], given a closed formula $\nu\zeta\,.F$, we can define its interpretation in **Nuts** by a transfinite induction (using Knaster–Tarski theorem) considering sequences of totality candidate as follows: $U_0^A = \mathcal{P}([\![\nu\zeta\,.F]\!]_{\mathbf{Rel}})$ where $\mathcal{P}(X)$ is the power set of X, $U_{\alpha+1}^A = \mathcal{T}([\![F]\!])([\![\nu\zeta\,.F]\!]_{\mathbf{Rel}}, U_\alpha^A)$, $U_\delta^A = \bigcap_{\alpha<\delta} U_\alpha^A$, and finally, there is an ordinal λ such that $U_\lambda = U_{\lambda+1}$, and we use λ_A for the least such ordinal.

To have simpler notation, we use the notation U_α (and U_λ) freely without mentioning the formula. One can find what the corresponding formula is from the context.

The following definition is borrowed from [18].

▶ **Definition 16.** *The marked formulas of $\mu\mathsf{LL}_\infty$ are defined as follows where α is an ordinal:*

$$A, B, \cdots := 1 \mid 0 \mid \bot \mid \top \mid A \oplus B \mid A \otimes B \mid A \,\&\, B \mid A \parr B \mid ?A \mid !B \mid \zeta \mid \mu\zeta\,.F \mid \nu^\alpha\zeta.F \quad (1)$$

We denote by A° the label-stripped formula A.

[2] The idea of this orthogonality is that $\mathcal{T}(X)$ intuitively is a denotational account for the normalization of the programs. This is explained in Section 5.

The interpretation of $\nu^\alpha\zeta.F$ in **Nuts** is $[\![\nu^\alpha\zeta.F]\!] = ([\![\nu\zeta\,.F]\!]_{\mathbf{Rel}}, U_\alpha)$, and the other marked formulas are interpreted as usual.

▶ **Proposition 17.** *Let A be a $\mu\mathsf{LL}_\infty$ formula. Then we have $[\![\overline{A}]\!] = [\![A]\!]$ where $\overline{A}$ is the marked formula, obtained from A by marking every ν binder of A by the ordinal λ_A.*

The proof of this proposition is obvious.

See proof in Appendix F.4.

▶ **Lemma 18.** *If A is a $\mu\mathsf{LL}_\infty$ formula and $t \notin \mathcal{T}([\![\nu^\alpha\zeta.F]\!])$ $(t \subseteq [\![\nu^\alpha\zeta.F]\!])$, then there exists an ordinal $\gamma < \alpha$ such that $t \notin \mathcal{T}([\![F\,[\nu^\gamma\zeta.F/\zeta]]\!])$.*

See proof in Appendix F.5

▶ **Lemma 19.** $\mathcal{T}([\![F\,[\mu\zeta\,.F/\zeta]]\!]) = \mathcal{T}([\![\mu\zeta\,.F]\!])$.

▶ **Lemma 20.** *If π is a proof of $\vdash \Gamma$ and $[\![\pi]\!] \notin \mathcal{T}(([\![\Gamma]\!]))$, then*

1. *π has an infinite branch $\gamma = (\vdash \Gamma_i)_{i\in\omega}$ such that $[\![\pi_i]\!] \notin \mathcal{T}(([\![\Gamma_i]\!]))$ where π_i is the sub-proof of π rooted in $\vdash \Gamma_i$;*
2. *and there exists a sequence of functions $(f_i)_{i\in\omega}$ where f_i maps all formulas D of Γ_i to a marked formula $f_i(D)$ such that*
   - *$(f_i(D))^\circ = D$,*
   - *one can write $\Gamma_i = \Gamma'_i, C$,*
   - *and there exists $x \in \mathcal{T}([\![(f_i(\Gamma'_i))^\perp]\!])$ such that $[\![\pi_i]\!].x \notin \mathcal{T}([\![f_i(C)]\!])$ where $\Gamma'_i = A^i_1, \cdots, A^i_{n_i}$ and $[\![(f_i(\Gamma'_i))^\perp]\!] = ([\![f_i(A^i_1)]\!])^\perp \otimes \cdots \otimes ([\![f_i(A^i_{n_i})]\!])^\perp$.*

Proof. See proof in Appendix F.6. The idea is essentially to construct the infinite branch γ inductively using properties of totality. ◀

Now, we can state and prove our main result of this section.

▶ **Theorem 21.** *If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!] \in \mathcal{T}([\![\Gamma]\!])$.*

Proof. Let us assume $[\![\pi]\!] \notin \mathcal{T}([\![\Gamma]\!])$. We can then apply Lemma 20 to obtain an infinite branch $(\vdash \Gamma_i)_{i\in\omega}$ and a sequence $(f_i)_{i\in\omega}$ satisfying properties 1 and 2 of Lemma 20. By the definition of valid proof (Definition 8), there exists a valid thread $t = (F_i)_{i\in\omega}$ for the infinite branch $(\vdash \Gamma_i)_{i\in\omega}$. Let $\nu\zeta\,F$ be the minimal formula formula of t. So, there are infinitely many times in t that we use a ν rule to unfold $\nu\zeta\,F$. Let $(i_k)_{k\in\omega}$ be the sequence of indices where $\nu\zeta\,F$ gets unfolded. Then $\nu\zeta\,F$ in the sequent Γ_{i_k} is sub-occurrence of $\nu\zeta\,F$ in the sequent $\Gamma_{i_{k'}}$ for $k \geqslant k'$. By the property 2 of Lemma 20, $f_{i_k}(\nu\zeta\,F) = \nu^{\alpha_k}\zeta.f_{i_k}(F)$. Therefore, by the property 2 of Lemma 20 and by the construction of the f_i in the proof of Lemma 20, the sequence $(\alpha_k)_{k\in\omega}$ is strictly decreasing. As this contradicts the well-foundedness property of the ordinals we obtain the required contradiction and conclude that $[\![\pi]\!] \in \mathcal{T}([\![\Gamma]\!])$. ◀

5 What totality tells us

One might think of the following statement as the converse of Theorem 21. If π is a pre-proof of the sequent $\vdash \Gamma$ such that $[\![\pi]\!] \in \mathcal{T}([\![\Gamma]\!])$, then π is a valid proof. This statement is not necessarily true, and there are many counterexamples indeed. For instance, take $F = \mu\zeta\,.(\bot \,\&\, (\zeta \parr \zeta))$ and $G = \nu\xi\,.(1 \oplus (\xi \parr \xi))$ and the pre-proofs π defined in Figure 5, where $\pi_{\Gamma;G}$ is defined (corecursively) on the right of the figure.

This pre-proof is not valid, since there is no valid thread in the rightmost branch. The interpretation of π in **Rel** is $[\![\pi]\!]_{\mathbf{Rel}} = \{((1,*),(1,*))\}$. However, $[\![\pi]\!]_{\mathbf{Rel}} \in \mathcal{T}([\![F \parr G]\!])$.

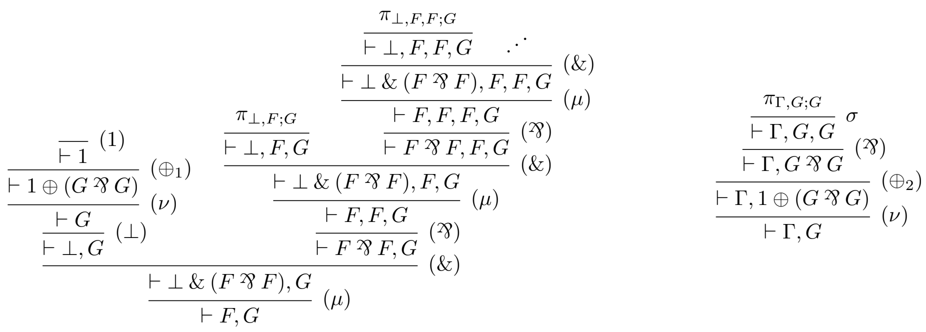


Figure 5 Proofs π and $\pi_{\Gamma;G}$

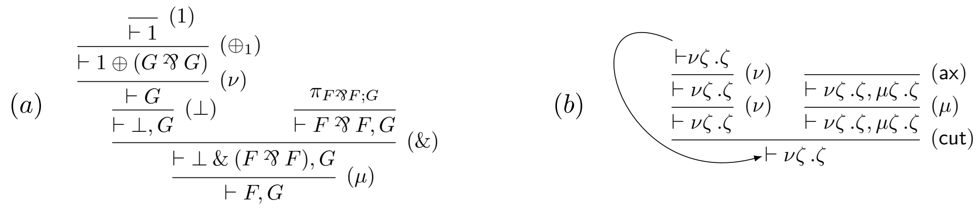


Figure 6 Examples of non-valid proof with total interpretations.

Notice that there are two ways to see that $[\![\pi]\!]_{\mathbf{Rel}} \in \mathcal{T}([\![F \parr G]\!])$. One can compute the interpretation of the formula $F \parr G$ in **Nuts**. And one can also provide a valid proof π' of $\vdash F, G$ such that $[\![\pi]\!]_{\mathbf{Rel}} = [\![\pi']\!]_{\mathbf{Rel}}$. Consider indeed the pre-proof π' of Figure 6 (a). This proof π' is a valid proof, since the thread $t = G_\alpha, (1 \oplus (G \parr G))_{\alpha i}, (G \parr G)_{\alpha ir}, G_{\alpha iri}, \cdots$ is a valid thread ($\mathsf{min}(\mathsf{Inf}(t)) = G$). We also have $[\![\pi']\!]_{\mathbf{Rel}} = \{((1,*),(1,*))\}$, and hence using Theorem 21, we know that $[\![\pi]\!] = [\![\pi']\!] \in \mathcal{T}([\![F \parr G]\!])$. The proof given in Figure 6 (b) is another example of non-valid proof whose interpretation is total. This examples differs however from π' (the proof given in Figure 6 (a)). It is true that this pre-proof does not respect the validity criterion, but it is valid with respect to the more recent criterion of [18, 6]. However this proof is considered as a valid proof in a more recent work [4]. That is why we hope that denotational semantic helps us to understand which validity conditions is more appropriate. However, this is just a hope till now. The only thing that we can say for the moment is that the notion of totality provides a sort of maximal notion for validity as valid proofs should be interpreted as total elements. To see this, let us assume that there is another notion of validity which will not be interpreted as total elements. In particular, take a proof π of $\vdash \mathsf{nat}^\perp, \mathsf{nat}$ such that $[\![\pi]\!] \notin \mathcal{T}([\![\mathsf{nat}]\!] \multimap [\![\mathsf{nat}]\!])$. As $\mathcal{T}([\![\mathsf{nat}]\!] \multimap [\![\mathsf{nat}]\!]) = \{f \subseteq \mathbb{N} \times \mathbb{N} \mid \forall n\, \exists m \text{ s.t } (n,m) \in f\}$, then this says that there is a finite, hence valid, proof σ of a natural number such that the cut-elimination procedure of π and σ will not terminate. Hence we can learn that that notion of validity is not a good one, as it does not enjoy cut-elimination.

6 On the semantics of circular proofs

The semantics developed in the previous section allows us to interpret both general non-wellfounded and circular proofs, but it presents two drawbacks:

- We only know how to interpret those proofs in concrete models and we do not provide a general categorical semantics for those proofs, even in the circular fragment, contrarily to

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash F,G,H,I,J}{\vdash F,G,\underline{H},I,J}\,(\nu)(\oplus_2)}{\vdash F,G,H,\underline{I}}\,(\mu)(\oplus_1)(\parr)}{\vdash F,G,H,I,\underline{J}}\,(\mu)(\oplus_2)(\bot)}{\vdash \underline{F \parr G},H,I,J}\,(\parr)}{\vdash F \parr G,\underline{G},H,I,J}\,(\nu)(\oplus_2)(\bot) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash F,G,H,I,J}{\vdash F,\underline{G},H,I,J}\,(\nu)(\oplus_1)}{\vdash F,G,H,\underline{K},J}\,(\nu)}{\vdash F,G,H,\underline{J}}\,(\mu)(\oplus_1)(\parr)}{\vdash F,G,H,\underline{I},J}\,(\mu)(\oplus_2)(\bot)}{\vdash F,\underline{H,G},I,J}\,(\mathsf{X})}{\vdash \underline{F \parr H},G,I,J}\,(\parr)}{\vdash F \parr H,G,\underline{H},I,J}\,(\nu)(\oplus_1)(\bot)}{\vdash \underline{(F \parr G) \,\&\, (F \parr H)},G,H,I,J}\,(\&)}{\vdash \underline{F},G,H,I,J}\,(\mu)$$

with
$F = \mu X.((X \parr G) \,\&\, (X \parr H))$
$G = \nu X.(X \oplus \bot)$
$H = \nu X.(\bot \oplus X)$
$I = \mu Z.((Z \parr J) \oplus \bot)$
$J = \mu X.((K \parr X) \oplus \bot)$
$K = \nu Y.\mu Z.((Z \parr \mu X(Y \parr X) \oplus \bot) \oplus \bot)$

Figure 7 Proof π_∞.

what Santocanale [32] achieved for the purely additive fragment of μLL.

- In the case of circular proofs, the approximation semantics does not benefit from the circularity of the proof objects in any way.

In the present section, we will discuss what are the challenges and how to proceed to achieve those goals.

One of the main difficulties to extend Santocanale's approach to $\mu\mathsf{LL}_\infty$ can be seen in the example of π_∞ presented in Figure 7. This is an example of a valid circular proof but having a quite involved validity structure, with three types of infinite branches (see Appendix H and [30] for more details). Santocanale's interpretation method strongly relies on the possibility to identify a thread by a formula, therefore π_∞ falls out of the scope of that method.

Two natural options are either (i) to disregard validity in interpreting circular proofs, as we did for non-well-founded proofs in previous sections, or (ii) to constrain the validity condition to make Santocanale's method usable. We discuss those two options below.

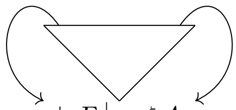
6.1 Interpreting circular pre-proofs

In this section, we will see how we can interpret circular pre-proofs in any model of μLL which is CPO-enriched. We first define $\mathsf{size}(\pi)$ of a pre-proof π as it is defined in [18]. Let $\mathsf{nax}(\pi)$ and $\mathsf{elc}(\pi)$ be the numbers of the non-axiom rules in π and the numbers of the elementary cycles in π respectively. Then we define $\mathsf{size}(\pi)$ as the pair $(\mathsf{elc}(\pi), \mathsf{nax}(\pi))$. We define the interpretation of a circular pre-proofs π by the lexicographic induction on $\mathsf{size}(\pi)$ providing a base case if $\mathsf{elc}(\pi) = 0$. In the base case, we simply interpret the proof as in μLL, since we have a finite proof (no cycle). If $\mathsf{elc}(\pi) > 0$, then we consider two cases. Either π is strongly connected as graph or not [3]. If π is not strongly connected, then there are two sequents $\vdash \Gamma$ and $\vdash \Delta$ such that there is no path from $\vdash \Gamma$ to $\vdash \Delta$. Let π_1 be the proof tree which is the reachable part of π from $\vdash \Gamma$, and let π_2 be the proof tree obtained from π by adding an auxiliary rule r on $\vdash \Gamma$ and taking the reachable part from the conclusion of π. Since π_1 does not have $\vdash \Gamma$, we have $\mathsf{nax}(\pi_1) < \mathsf{nax}(\pi)$, and then by induction hypothesis we have $[\![\pi_1]\!]$. We now take the interpretation of the rule r with the conclusion $\vdash \Gamma$ as $[\![\pi_1]\!]$. Then by removing $\vdash \Gamma$ from π_2, we have $\mathsf{nax}(\pi_2) < \mathsf{nax}(\pi)$. Hence we know how to compute the interpretation of π_2 by induction hypothesis, and finally we take $[\![\pi]\!]$ as $[\![\pi_2]\!]$ assuming $[\![\pi_1]\!]$ for $\vdash \Gamma$.

[3] We look at a proof as a directed graph where the nodes are the sequents and there is an edge from $\vdash \Gamma$ to $\vdash \Delta$ if there is an inference rule r such that $\vdash \Delta$ is one the premise of r and $\vdash \Gamma$ is its conclusion.

Now, we assume that π is strongly connected. Then, there is an infinite path p that visits all the sequents of π. Let t be a trace of p, and let $\vdash \Delta$ be a sequent where the minimal formula of t has been unfolded.

We suppose without loss of generality that $\vdash \Delta$ is $\vdash \Gamma^{\perp}, \nu\zeta\, A$ for some context Γ, and it is also the conclusion of π. Graphically, π is the proof depicted on the right.



$\vdash \Gamma^{\perp}, \nu\zeta\, A$

In this case, we have a morphism f in $\mathcal{L}(!(\llbracket\Gamma\rrbracket \multimap \llbracket\nu\zeta\, A\rrbracket), \llbracket\Gamma\rrbracket \multimap \llbracket\nu\zeta\, A\rrbracket)$. As the category $\mathcal{L}$ is a CPO-enriched category, we define $\llbracket\pi\rrbracket$ as the supremum of the chain $\{f^n \circ \bot_{!(\llbracket\Gamma\rrbracket \multimap \llbracket\nu\zeta\, A\rrbracket)} \mid n \in \mathbb{N}\}$ where $f^{n+1} \circ \bot = f \circ (f^n \circ \bot)$ and $\bot_{!\llbracket\Gamma\rrbracket \multimap \llbracket\nu\zeta\, A\rrbracket}$ is the least element of $\mathcal{L}(1, !(\llbracket\Gamma\rrbracket \multimap \llbracket\nu\zeta\, A\rrbracket))$.

6.2 About interpreting strongly valid proofs

In this direction, that we plan to pursue in future work, we plan to consider a proper fragment of $\mu\mathsf{LL}_\infty$, requiring that along every infinite branch, there exists a valid thread such that, each time it visits a sequent of the *circular representation*, the thread points to the same formula of the sequent. Such a fragment has indeed been considered in the literature of fixed-point logic, under the name of *strong validity*, in order to finitize circular proofs: first used to adapt Santocanale's argument to the interpretation of $\mu\mathsf{MALL}$ in Ludics [5], it was then used for the linear-time μ-calculus [19], and in a more general logical system in Doumane' PhD [18]. The class of strongly valid proofs contains all the unfoldings of finitary proofs. In Appendix G, we recall the definitions and the main result on strongly valid proofs.

Noticing that strongly valid proofs can be finitized, an alternative strategy is available to interpret strongly valid circular proofs: given a strongly valid proof π, one could interpret it using the interpretation of its finitization in $\mu\mathsf{LL}$. Of course, one needs to prove the interpretation via finitization coincides with the interpretation of the circular proof as a non-wellfounded proof, somehow requiring that interpretation is preserved via the finitization.

The preservation of the semantics by the finitization is somehow the converse of what we did in Section 3.2 (their composition is not identity) in any model of $\mu\mathsf{LL}$. We plan to check this coincidence in the case of the concrete models such as $\mathbf{Rel}$, as we already know how to interpret $\mu\mathsf{LL}_\infty$ proofs in $\mathbf{Rel}$. This is what we will consider as an immediate future work, and we just exemplify it in this paper. For instance, consider the following circular proof π:

$$\pi = \dfrac{\dfrac{\dfrac{\pi_0^{\mathsf{nat}}}{\vdash \bot, \mathsf{nat}} \qquad \vdash \mathsf{nat}^{\perp}, \mathsf{nat}}{\vdash \bot \,\&\, \mathsf{nat}^{\perp}, \mathsf{nat}}\,(\&)}{\vdash \mathsf{nat}^{\perp}, \mathsf{nat}}\,(\nu)$$

$$\pi' = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \bot \,\&\, I, 1 \oplus I^{\perp}}\,(\mathsf{ax}) \qquad \dfrac{\dfrac{\pi_0^{\mathsf{nat}}}{\vdash \bot, \mathsf{nat}} \qquad \dfrac{\dfrac{\dfrac{}{\vdash \mathsf{nat}^{\perp}, \mathsf{nat}}\,(\mathsf{ax})}{\vdash (\bot \,\&\, I) \oplus \mathsf{nat}^{\perp}, \mathsf{nat},}\,(\oplus_2)}{\vdash I, \mathsf{nat}}\,(\nu)}{\vdash \bot \,\&\, I, \mathsf{nat}}\,(\&)}{\vdash \bot \,\&\, I, (1 \oplus I^{\perp}) \,\&\, \mathsf{nat}}\,(\&)}{\vdash \bot \,\&\, I, I^{\perp}}\,(\mu)}{\vdash \mathsf{nat}^{\perp}, I^{\perp}}\,(\nu'_{\mathsf{rec}}) \qquad \dfrac{\dfrac{\dfrac{}{\vdash \mathsf{nat}^{\perp}, \mathsf{nat}}\,(\mathsf{ax})}{\vdash (\bot \,\&\, I) \oplus \mathsf{nat}^{\perp}, \mathsf{nat},}\,(\oplus_2)}{\vdash I, \mathsf{nat}}\,(\nu)}{\vdash \mathsf{nat}^{\perp}, \mathsf{nat}}\,(\mathsf{cut})$$

We can see that $\llbracket\pi\rrbracket_{\mathbf{Rel}} = \{(n, 0) \mid n \in \mathbb{N}\}$. If we apply the finitization given in [18] to this proof, we will get the finite proof π' above where $I = \nu\zeta\,.((\bot \,\&\, \zeta) \oplus \mathsf{nat}^{\perp})$. Then when computing the interpretation of π', we see that $\llbracket\pi'\rrbracket_{\mathbf{Rel}} = \llbracket\pi\rrbracket_{\mathbf{Rel}}$. Of course, it is just an example but we conjecture that it holds on all strongly valid proofs. The main difficulty is then to provide a more "structural" finitization procedure such as an (co)inductive definition.

7 Conclusion

In this paper, we studied the non-wellfounded proof system $\mu\mathsf{LL}_\infty$ from a Curry-Howard perspective, by providing a denotational semantics of $\mu\mathsf{LL}_\infty$. We first showed that the category **Rel** of sets and relations is a sound model of $\mu\mathsf{LL}_\infty$. More precisely, we interpret formula and pre-proofs of $\mu\mathsf{LL}_\infty$ in **Rel**, and prove that the semantic is preserved via a possibly infinite reduction sequence of cut-elimination. Then we investigated the relationship between the interpretation of inductive proofs and their image as circular proofs in a categorical model of $\mu\mathsf{LL}$ in the sense of [21]: it is shown that the translation from finitary proofs to circular ones is sound (*i.e.* it preserves the semantics), bringing evidence of the computational soundness of this translation. Finally, we provided another concrete model of $\mu\mathsf{LL}_\infty$ based on the category **Nuts** of non-uniform totality spaces and relations preserving totality, which is not the case, in general, for pre-proofs. Although the interpretation of proofs in both models of **Rel** and **Nuts** are the same, one can obtain more information by looking at the interpretation in **Nuts** as we showed any valid proofs will be interpreted as a total element.

Santocanale considered circular proofs in the framework of purely Additive linear logic, and he provided a categorical interpretation of circular proofs in μ-bicomplete categories [32, 23]. On the one hand, we treat both the multiplicatives and the exponentials, but on the other hand, we have concrete models and no categorical axiomatization yet. Moreover, on the one hand we can interpret arbitrary non-wellfounded proofs, but on the down side, we cannot benefit from the finitely presentable structure of circular proof, this is for future work.

We conclude by mentioning some directions for future work.

- We only provided concrete models of $\mu\mathsf{LL}_\infty$ in this paper, and not a categorical axiomatization. For instance, in [23, 32], there is categorical model for the circular proof in the additive fragment of LL. So, one can wonder if that can be extended to full $\mu\mathsf{LL}_\infty$. If we do not restrict ourselves to circular proofs and consider all valid $\mu\mathsf{LL}_\infty$ proofs, there is little hope to provide a categorical model for them, because it is not even clear how to interpret an arbitrary non-wellfounded proof. Of course, one can assume some structure on the category in order to interpret those proofs. For instance, one can work with CPO-enriched categories, and interpret proofs using the same idea as we did in **Rel**. Nevertheless, it is worth trying to find a class of categories as model of $\mu\mathsf{LL}_\infty$, rather than finding a free category for $\mu\mathsf{LL}_\infty$ logic in the sense of what we have for CCC categories and simple typed λ-calculus.
- Another question could be seeking for a complete denotational model of $\mu\mathsf{LL}_\infty$ in the sense of Girard and Streicher [26, 36]. This could be useful to tackle the Brotherston-Simpson's conjecture for $\mu\mathsf{LL}$ (this conjecture says that inductive proofs and circular proofs have the same provability) as well as a *proof-relevant/denotational* version of the conjecture which would read as follows (the converse of this conjecture is Theorem 11):

  ▶ **Conjecture 22.** *Let $\vdash \Gamma$ be a $\mu\mathsf{LL}$ sequent and π be a circular $\mu\mathsf{LL}^\infty$ proof of $\vdash \Gamma$. There exists a $\mu\mathsf{LL}$ (finite) proof π' of $\vdash \Gamma$ such that $[\![\pi]\!] = [\![\pi']\!]$.*
- Some non-valid $\mu\mathsf{LL}_\infty$ have a total interpretation. A natural question is to understand what sort of information can be obtained from a total interpretation, if not syntactic validity. We saw in the paper that, for functions from nat to nat, the totality of $\mathsf{nat} \multimap \mathsf{nat}$ is all total relations on natural numbers; as a consequence it is not possible for a non-terminating program of type $\mathsf{nat} \multimap \mathsf{nat}$ to have a total interpretation in **Nuts**. A natural (but difficult) question is whether this can be lifted to all $\mu\mathsf{LL}_\infty$ types. The same question is asked by Girard for second-order types, and it is still an open problem [24].

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A On the two notions of subformulas

In the following, we shall consider two notions of sub-formulas, the usual one and a notion of sub-formula which is specific to the μ-calculus, the Fischer-Ladner subformulas.

▶ **Definition 23.** *The sub-formula relation on $\mu\mathsf{LL}_\infty$ is defined as follows:*

- $A * B \to_{sub} A$ *and* $A * B \to_{sub} B$ *where $*$ is a binary* LL *connective.*
- $@A \to_{sub} A$ *where $@$ is either $!$ or $?$.*
- $\sigma\zeta\, F \to_{sub} F$ *where σ is either ν or μ.*

G is a subformula of F when $F \to^{\star}_{sub} G$.

Notice that the usual sub-formula relation is an ordering, so, we write $A \leqslant_{sub} B$ if A is sub-formula of B, i.e, we have $B \to^{\star}_{sub} A$.

▶ **Definition 24.** *We define the relation $\to_{FL}$ on formulas as follows:*

- $A * B \to_{FL} A$ *and* $A * B \to_{FL} B$ *where $*$ is a binary* LL *connective.*
- $@A \to_{FL} A$ *where $@$ is either $!$ or $?$.*
- $\sigma\zeta\, F \to_{FL} F\,[\sigma\zeta\, F/\zeta]$ *where σ is either ν or μ.*

A formula G is a Fischer-Ladner sub-formula of F when $F \to^{\star}_{FL} G$.

It is a well-known fact that the Fischer-Ladner closure of any formula (*ie.* the set of its Fischer-Ladner sub-formulas) is finite, see for instance Corollary 2.1 of [18].

▶ **Proposition 25** (Proposition 2.7 of [18])**.** *If a thread t is coming from a branch of an $\mu\mathsf{LL}_\infty$ pre-proof, then $\mathsf{Inf}(t)$ admits a minimum with respect to the usual sub-formula ordering $\leqslant_{sub}$ (see Definition 23), denoted $\mathsf{min}(\mathsf{Inf}(t))$.*

Proof. Proposition 2.7 of [18]. The idea of the proof is based on the observation that $\mathsf{Inf}(t)$ forms a cycle, and roughly speaking, the minimum of $\mathsf{Inf}(t)$ is the nearest to the root in that cycle. ◀

B Interpretation of LL rules

$$\left[\!\!\left[\frac{}{\vdash A, A^\perp}\ (\mathsf{ax})\right]\!\!\right] = \mathsf{Id}_A \qquad \left[\!\!\left[\frac{\begin{matrix}\vdots\,\pi_1 & & \vdots\,\pi_2\\ \vdash \Gamma, A & & \vdash A^\perp, \Delta\end{matrix}}{\vdash \Gamma, \Delta}\ (\mathsf{cut})\right]\!\!\right] = ([\![\Gamma]\!])^\perp \xrightarrow{[\![\pi_1]\!]} [\![A]\!] \xrightarrow{[\![\pi_2]\!]} [\![\Delta]\!]$$

$$\left[\!\!\left[\frac{}{\vdash 1}\ (1)\right]\!\!\right] = \mathsf{Id}_1 \qquad \left[\!\!\left[\frac{\begin{matrix}\vdots\,\pi_1 & & \vdots\,\pi_2\\ \vdash \Gamma, A & & \vdash \Delta, B\end{matrix}}{\vdash \Gamma, \Delta, A \otimes B}\ (\otimes)\right]\!\!\right] = ([\![\Gamma]\!])^\perp \otimes ([\![\Delta]\!])^\perp \xrightarrow{[\![\pi_1]\!] \otimes [\![\pi_2]\!]} [\![A]\!] \otimes [\![B]\!]$$

$$\left[\!\!\left[\frac{\begin{matrix}\vdots\,\pi\\ \vdash \Gamma\end{matrix}}{\vdash \Gamma, \bot}\ (\bot)\right]\!\!\right] = \mathsf{cur}([\![\pi]\!]) \qquad \left[\!\!\left[\frac{\begin{matrix}\vdots\,\pi\\ \vdash \Gamma, A, B\end{matrix}}{\vdash \Gamma, A \parr B}\ (\parr)\right]\!\!\right] = [\![\pi]\!]$$

$$\left[\!\!\left[\frac{}{\vdash \Gamma, \top}\ (\top)\right]\!\!\right] = \mathsf{t}_{([\![\Gamma]\!])^\perp} \qquad \left[\!\!\left[\frac{\begin{matrix}\vdots\,\pi\\ \vdash \Gamma, A_i\end{matrix}}{\vdash \Gamma, A_1 \oplus A_2}\ (\oplus_i)\right]\!\!\right] = ([\![\Gamma]\!])^\perp \xrightarrow{[\![\pi]\!]} [\![A_i]\!] \xrightarrow{\overline{\pi}_i} [\![A_1]\!] \oplus [\![A_2]\!]$$

$$\left[\!\!\left[\frac{\begin{matrix}\vdots\,\pi_1 & & \vdots\,\pi_2\\ \vdash \Gamma, A & & \vdash \Gamma, B\end{matrix}}{\vdash \Gamma, A \,\&\, B}\ (\&)\right]\!\!\right] = ([\![\Gamma]\!])^\perp \xrightarrow{\langle [\![\pi_1]\!], [\![\pi_2]\!] \rangle} [\![A]\!] \,\&\, [\![B]\!]$$

$$\left\llbracket \dfrac{\begin{array}{c}\vdots\,\pi\\ \vdash \Gamma\end{array}}{\vdash \Gamma, ?A}\,(\mathsf{w}) \right\rrbracket = \mathsf{cur}(f) \text{ where } f \text{ is } \begin{tikzcd} (\llbracket\Gamma\rrbracket)^{\perp} \otimes \llbracket !A^{\perp}\rrbracket \arrow[r, "\mathsf{Id}\,\otimes\,\mathsf{w}_{A^{\perp}}"] & (\llbracket\Gamma\rrbracket)^{\perp} \otimes 1 \arrow[r, "\simeq"] & (\llbracket\Gamma\rrbracket)^{\perp} \arrow[d, "\llbracket\pi\rrbracket"] \\ & & \perp \end{tikzcd}$$

$$\left\llbracket \dfrac{\begin{array}{c}\vdots\,\pi\\ \vdash \Gamma, ?A, ?A\end{array}}{\vdash \Gamma, ?A}\,(\mathsf{c}) \right\rrbracket = \mathsf{cur}(f) \text{ where } f \text{ is } (\llbracket\Gamma\rrbracket)^{\perp} \otimes (!A^{\perp} \otimes !A^{\perp}) \xrightarrow{\mathsf{Id}\,\otimes\,\mathsf{contr}_{A^{\perp}}} (\llbracket\Gamma\rrbracket)^{\perp} \otimes !A^{\perp} \xrightarrow{\llbracket\pi\rrbracket} \perp$$

$$\left\llbracket \dfrac{\begin{array}{c}\vdots\,\pi\\ \vdash \Gamma, A\end{array}}{\vdash \Gamma, ?A}\,(\mathsf{d}) \right\rrbracket = \mathsf{cur}(f) \text{ where } f = (\llbracket\Gamma\rrbracket)^{\perp} \otimes \llbracket !A^{\perp}\rrbracket \xrightarrow{\mathsf{Id}\,\otimes\mathsf{der}_{A^{\perp}}} (\llbracket\Gamma\rrbracket)^{\perp} \otimes A^{\perp} \xrightarrow{\llbracket\pi\rrbracket} \perp$$

$$\left\llbracket \dfrac{\begin{array}{c}\vdots\,\pi\\ \vdash ?\Gamma, A\end{array}}{\vdash ?\Gamma, !A}\,(\mathsf{p}) \right\rrbracket = \llbracket \textstyle\bigotimes_{B_i^{\perp}\in\Gamma} (!B_i)\rrbracket \xrightarrow{\bigotimes \mathsf{dig}_{B_i}} \llbracket \textstyle\bigotimes_{B_i^{\perp}\in\Gamma} !!B_i\rrbracket \xrightarrow{\mu^{n}} !(\llbracket \textstyle\bigotimes_{B_i^{\perp}\in\Gamma} !B_i\rrbracket) \xrightarrow{!\llbracket\pi\rrbracket} !\llbracket A\rrbracket$$

C Metric completion of finite proofs

The purpose of this section is to develop a precise characterization of non-wellfounded proofs as the completion of a space of finite proof with a notion of approximant, much in the same way a Böhm trees for the λ-calculus. As such, the material in this section should not surprise the reader in its technical development but it is nevertheless necessary for a precise definition of the semantics of non-wellfounded proofs that we consider in the latter sections.

We consider the proof system of $\mu\mathsf{LL}_{\infty}$ extended with the following rule: $\dfrac{}{\vdash \Gamma}\,(\Omega)$ for any sequent Γ. The reason why we consider this assumption will be clear later, for instance in Definition 28. Here, we can say that we are using this auxiliary rule in order to cut the infinite proofs at different levels and consider all its finite approximation.

▶ **Definition 26.** *Given a $\mu\mathsf{LL}_{\infty}$ pre-proof π, we associate a set $\mathsf{Pos}(\pi)$ of positions corresponding to each sequent of π as follows:*

- $\langle 0\rangle \in \mathsf{Pos}(\pi)$
- *Let r be an occurrence of an inference rule in π and that $\langle x\rangle$, which belongs to $\mathsf{Pos}(\pi)$, is the location of this occurrence in π*
  - *If $r \in \{(\otimes), (\&), (\mathsf{cut})\}$, then both $\langle x0\rangle$ and $\langle x1\rangle$ are in $\mathsf{Pos}(\pi)$;*
  - *Otherwise $\langle x0\rangle \in \mathsf{Pos}(\pi)$.*

The elements of $\mathsf{Pos}(\pi)$ are finite sequences of 0 and 1.

▶ **Definition 27.** *Given a pre-proof π and $p \in \mathsf{Pos}(\pi)$, we denote by $\mathsf{Proof}(\pi, p)$ the last sequent of the sub-pre-proof of π rooted at position p.*

As an example, consider the following proof π:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(\mathsf{ax})}{\vdash ?A^{\perp}, A}\,(\mathsf{d})}{\vdash ?A^{\perp}, ?B^{\perp}, A}\,(\mathsf{w}) \qquad \dfrac{\dfrac{\dfrac{}{\vdash B^{\perp}, B}\,(\mathsf{ax})}{\vdash ?B^{\perp}, B}\,(\mathsf{d})}{\vdash ?A^{\perp}, ?B^{\perp}, B}\,(\mathsf{w})}{\dfrac{\vdash ?A^{\perp}, ?B^{\perp}, A \,\&\, B}{\vdash ?A^{\perp}, ?B^{\perp}, !(A \,\&\, B)}\,(\mathsf{p})}\,(\&)$$

Then one can represent it by the $\mathsf{Pos}(\pi)$ as follows which is also annotated by the sequents. One can also label the edges by the inference rules.

$$
\begin{array}{ccc}
 & \vdash ?A^{\perp}, ?B^{\perp}, !(A \mathbin{\&} B)\ \langle 0 \rangle & \\
 & | & \\
 & \vdash ?A^{\perp}, ?B^{\perp}, A \mathbin{\&} B\ \langle 00 \rangle & \\
 \swarrow & & \searrow \\
\vdash ?A^{\perp}, ?B^{\perp}, A\ \langle 000 \rangle & & \vdash ?A^{\perp}, ?B^{\perp}, B\ \langle 001 \rangle \\
 | & & | \\
\vdash ?A^{\perp}, A\ \langle 0000 \rangle & & \vdash ?B^{\perp}, B\ \langle 0010 \rangle \\
 | & & | \\
\vdash A^{\perp}, A\ \langle 00000 \rangle & & \vdash B^{\perp}, B\ \langle 00100 \rangle
\end{array}
$$

▶ **Definition 28.** *Let π be a pre-proof and P be a prefix-closed subset of $\mathsf{Pos}(\pi)$. We denote by $\pi(P)$ the sub-pre-proof of π whose set of positions is P, i.e, $\mathsf{Pos}(\pi(P)) = P$.*

Notice that if we do not assume having the (Ω) rule, then $\pi(P)$ might not exist.

▶ **Definition 29.** *If π is a pre-proof we denote by $\mathsf{Pos}_i(\pi)$ the subset of $\mathsf{Pos}(\pi)$ that contains only all position of length i, i.e, $\mathsf{Pos}_i(\pi) = \pi(\mathsf{Pos}(\pi) \cap \{0,1\}^i)$.*

Let $\mathcal{X}$ be the set of all $\mu\mathsf{LL}_\infty$ finite proofs. One can define a distance $d : \mathcal{X} \times \mathcal{X} \to [0,1]$: $d(\pi, \pi') = 0$ if two proofs π and π' are identical, otherwise $d(\pi, \pi') = \frac{1}{2^k}$, where k is the length of the shortest position at which π and π' differ.

Denote by $C[\mathcal{X}]$ the collection of all Cauchy sequences in $\mathcal{X}$. Define a relation $\sim$ on $C[\mathcal{X}]$ by

$$(\pi_n) \sim (\pi'_n) \Leftrightarrow \lim_{n\to\infty} d(\pi_n, \pi'_n) = 0$$

It is easy to see that this is an equivalence relation on $C[\mathcal{X}]$. This definition does not depend on the choice of representatives in the two equivalence classes. Let $\mathcal{X}^*$ be the set of all equivalence classes for $\sim$. One can define the metric d^* on $\mathcal{X}^*$ as follows where $[(\pi_n)]$ is an equivalence class:

$$d^*([(\pi_n)], [(\pi'_n)]) = \lim_{n\to\infty} d(\pi_n, \pi'_n)$$

The metric space $(\mathcal{X}^*, d^*)$ is called *metric completion* of $\mathcal{X}$, and there is standard result showing that this is a complete space.

▶ **Proposition 30.** *Let $\mathcal{X}_\infty$ be set of all (potentially infinite) $\mu\mathsf{LL}_\infty$ proofs. Then the metric space $(\mathcal{X}^*, d^*)$ is isomorphic to $\mathcal{X}_\infty$.*

Proof. Since the completion of a metric space is unique up to isometry, it is enough to show that $(\mathcal{X}_\infty, d')$ is the completion of $\mathcal{X}$ for a metric d'. That is to show $\mathcal{X}$ is dense in $\mathcal{X}_\infty$ for taking d' same as d.

Take $\pi \in \mathcal{X}_\infty$. Consider the sequence (π_n) where $\pi_n = \pi(\bigcup_{i<n} \mathsf{Pos}_i(\pi))$. We have now $d(\pi, \pi_n) = \frac{1}{2^n}$, so, π is the limit of the sequence (π_n) of finite proofs. ◀

As the direct conclusion of 30, the metric space $(\mathcal{X}_\infty, d)$ is complete, that is to say every Cauchy sequence of proofs in $\mathcal{X}_\infty$ has a limit inside of $\mathcal{X}_\infty$.

▶ Remark 31. In the cut-elimination process of $\mu\mathsf{LL}_\infty$, for any natural number n, the number of steps of the sequence which reduces a (cut) rule at depth less that n is finite [18]. So, the cut-elimination reduction has countable length.

We saw that the metric space $(\mathcal{X}_\infty, d)$ is a complete space, but this was a result of the proposition 30. Here we show the completeness of this metric space directly.

▶ **Proposition 32.** *The metric space $(\mathcal{X}_\infty, d)$ is complete.*

Proof. Take a Cauchy sequence (π_n). First, we define the set P as $\bigcup_i \bigcap_{j>i} \mathsf{Pos}(\pi_i)$. And we also provide a function f that sends a $p \in P$ to a sequent as follows: Since $p \in P$, $\exists i \forall j > i (p \in \mathsf{Pos}(\pi_j) \wedge (\mathsf{Proof}(\pi_i, p) = \mathsf{Proof}(\pi_j, p))$. So, we define $f(p)$ as $\mathsf{Proof}(\pi_i, p)$ (this does not depend on the choice of i). Now since the sequence (π_n) is a Cauchy sequence, we have $\forall k, \exists N \forall i, j > N (d(\pi_i, \pi_j) < \frac{1}{2^k})$, and therefore $d(\Pi(P, f), \pi_i) < \frac{1}{2^k}$ where $\Pi(P, f)$ is the pre-proof tree that has P as set of its positions and it is labeled by element of $f(P)$ (one can deduce it by the contradiction). Hence the proof $\Pi(P, f)$ is the limit of the (π_n). ◀

We will use this direct proof later in proof of Theorem 12.

D Definition of $\mathsf{Trans}\,()$

- We have the following for $r \in \{(1), (\mathsf{ax}), (\bot), (⅋), (\top), (\oplus_1), (\oplus_2), (\mathsf{w}), (\mathsf{c}), (\mathsf{d}), (\mathsf{p}), (\mu)\}$:

$$\mathsf{Trans}\left(\cfrac{\begin{array}{c}\pi\\ \vdash \Delta\end{array}}{\vdash \Gamma}\,\mathsf{r}\right) = \cfrac{\begin{array}{c}\mathsf{Trans}\,(\pi)\\ \vdash \Delta\end{array}}{\vdash \Gamma}\,\mathsf{r}$$

- We have the following for $r \in \{(\mathsf{cut}), (\otimes), (\&)\}$:

$$\mathsf{Trans}\left(\cfrac{\begin{array}{c}\pi_1\\ \vdash \Delta_1\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash \Delta_2\end{array}}{\vdash \Gamma}\,\mathsf{r}\right) = \cfrac{\begin{array}{c}\mathsf{Trans}\,(\pi_1)\\ \vdash \Delta_1\end{array} \qquad \begin{array}{c}\mathsf{Trans}\,(\pi_2)\\ \vdash \Delta_2\end{array}}{\vdash \Gamma}\,\mathsf{r}$$

- And finally $\mathsf{Trans}\left(\cfrac{\begin{array}{c}\pi\\ \vdash ?\Gamma, A^\perp, F\,[A/\zeta]\end{array}}{\vdash ?\Gamma, A^\perp, \nu\zeta\,.F}\,(\nu'_{\mathsf{rec}})\right)$ is the following circular proof using the functoriality of formulas given in Section 2.3:

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdash ?\Gamma, A^\perp, \nu\zeta\,.F}{\vdash ?\Gamma, (F\,[A/\zeta])^\perp, F\,[\nu\zeta\,.F/\zeta]}\,(\mathfrak{F}_F)}{\vdash ?\Gamma, (F\,[A/\zeta])^\perp, \nu\zeta\,.F}\,(\nu) \qquad \begin{array}{c}\pi\\ \vdash A^\perp, F\,[A/\zeta]\end{array}}{\vdash ?\Gamma, ?\Gamma, A^\perp, \nu\zeta\,.F}\,(\mathsf{cut})}{\vdash ?\Gamma, A^\perp, \nu\zeta\,.F}\,(\mathsf{c})$$

E Proofs of Section 3

E.1 Proof of Lemma 10

▶ **Lemma 33.** *Let A be an onject of a category $\mathcal{A}$ and let $f_1, f_2 \in \mathcal{A}(A, \nu\mathcal{F})$. If there exists $l \in \mathcal{A}(A, \mathcal{F}(A))$ such that $\mathcal{F}(f_i)\, l = f_i$ for $i = 1, 2$, then $f_1 = f_2$.*

Proof. Since $\mathcal{F}(f_i)\, l = f_i$ for $i = 1, 2$, we have $f_i \in \mathbf{Coalg}_{\mathcal{A}}(\mathcal{F})((A, l), (\nu\mathcal{F}, \mathsf{Id}))$ for $i = 1, 2$. $(\nu\mathcal{F}, \mathsf{Id})$ is the final object in $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F})((A, l), (\nu\mathcal{F}, \mathsf{Id}))$, so there is a unique morphism from (A, l) to $(\nu\mathcal{F}, \mathsf{Id})$. Hence $f_1 = f_2$. ◀

▶ Remark 34. In the proof of Lemma 10, we refer to the identity for the coalgebra morphism of $\nu\mathcal{F}$ but never use any of its property and the proof would go through using any iso instead of Id: it is just a consequence of the universal property of a final coalgebra.

E.2 Proof of Theorem 11

The interpretation of a μLL formulas F that contains n free variable is an n-ary strong functor $\llbracket F \rrbracket$ [21]. We use the notations $\overline{\llbracket F \rrbracket}$ and $\widehat{\llbracket F \rrbracket}$ for the underlying functor and strength of the strong functor $\llbracket F \rrbracket$ respectively.

▶ **Theorem 35.** *Let π be a μLL proof. Then we have $\llbracket \pi \rrbracket = \llbracket \mathsf{Trans}\,(\pi) \rrbracket$ where the interpretation is given in a model $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ of μLL.*

Proof. The proof is by induction on π. Let us assume that the last inference rule is a (ν) rule so that π is the following proof:

$$\dfrac{\begin{array}{c}\pi' \\ \vdash ?\Gamma, A^{\perp}, F\,[A/\zeta]\end{array}}{\vdash ?\Gamma, A^{\perp}, \nu\zeta\, F}\ (\nu'_{\mathsf{rec}})$$

Let $f = \llbracket \mathsf{Trans}\,(\pi) \rrbracket$. By definition of $\mathsf{Trans}\,(\pi)$ given above, f should satisfy the following diagram:

$$\begin{tikzcd}
!\llbracket \Gamma^{\perp} \rrbracket \otimes A \arrow[rr, "f"] \arrow[dd, "\mathsf{c}_{!\llbracket \Gamma^{\perp} \rrbracket} \otimes \mathsf{Id}"] & & \llbracket \nu\zeta\, F \rrbracket \\
 & & \overline{\llbracket F \rrbracket}(\llbracket \nu\zeta\, F \rrbracket) \arrow[u, "\simeq"'] \\
!\llbracket \Gamma^{\perp} \rrbracket \otimes !\llbracket \Gamma^{\perp} \rrbracket \otimes A \arrow[d, "\mathsf{Id} \otimes \llbracket \pi' \rrbracket"] & & \\
!\llbracket \Gamma^{\perp} \rrbracket \otimes \overline{\llbracket F \rrbracket}(\llbracket A \rrbracket) \arrow[rr, "\widehat{\llbracket F \rrbracket}"] & & \overline{\llbracket F \rrbracket}(!\llbracket \Gamma^{\perp} \rrbracket \otimes \llbracket A \rrbracket) \arrow[uu, "\overline{\llbracket F \rrbracket}(f)"']
\end{tikzcd}$$

By the construction given in [21] to interpret formulas and proofs of μLL, the interpretation of π is the unique morphism $\llbracket \pi \rrbracket \in \mathcal{L}(!\llbracket \Gamma^{\perp} \rrbracket \otimes A, \llbracket \nu\zeta\, F \rrbracket)$ satisfying the following diagram:

$$\begin{tikzcd}
!\llbracket \Gamma^{\perp} \rrbracket \otimes A \arrow[rr, "\llbracket \pi \rrbracket"] \arrow[dd, "\mathsf{c}_{!\llbracket \Gamma^{\perp} \rrbracket} \otimes \mathsf{Id}"] & & \llbracket \nu\zeta\, F \rrbracket \\
 & & \overline{\llbracket F \rrbracket}(\llbracket \nu\zeta\, F \rrbracket) \arrow[u, "\simeq"'] \\
!\llbracket \Gamma^{\perp} \rrbracket \otimes !\llbracket \Gamma^{\perp} \rrbracket \otimes A \arrow[d, "\mathsf{Id} \otimes \llbracket \pi' \rrbracket"] & & \\
!\llbracket \Gamma^{\perp} \rrbracket \otimes \overline{\llbracket F \rrbracket}(\llbracket A \rrbracket) \arrow[rr, "\widehat{\llbracket F \rrbracket}"] & & \overline{\llbracket F \rrbracket}(!\llbracket \Gamma^{\perp} \rrbracket \otimes \llbracket A \rrbracket) \arrow[uu, "\overline{\llbracket F \rrbracket}(\llbracket \pi \rrbracket)"']
\end{tikzcd}$$

Hence, by Lemma 10, we have $\llbracket \pi \rrbracket = \llbracket \mathsf{Trans}\,(\pi) \rrbracket$. ◀

F Proofs of Section 4

F.1 Proof of Theorem 12

▶ **Theorem 36.** *Given two finite $\mu\mathsf{LL}_{\infty}$ proofs π and π' such that π' is obtained from π via an one-step cut-elimination rule, then $\llbracket \pi \rrbracket = \llbracket \pi' \rrbracket$.*

Proof. We only need to check the reduction of $(\mu)-(\nu)$ given in Section 2.3. And this is trivial, as both (μ) and (ν) rules have no effect on the interpretation as we saw in Section 3.1. ◀

F.2 Proof of Lemma 13

▶ **Lemma 37.** *Let (π_i) be a Cauchy sequence. Then* $[\![\lim_{n\to\infty}\pi_i]\!]_{\mathbf{Rel}} = \bigcup_i\bigcap_{j>i}[\![\pi_j]\!]_{\mathbf{Rel}}$.

Proof. By Proposition 32, $\lim_{n\to\infty}\pi_i = \Pi(P,f)$ (we are using a notation introduced in the proof of Proposition 32). By definition, $[\![\Pi(P,f)]\!]_{\mathbf{Rel}} = \bigcup_{\pi\in\mathsf{fin}(\Pi(P,f))}[\![\pi]\!]_{\mathbf{Rel}}$. Take a $\pi'\in\mathsf{fin}(\Pi(P,f))$. For each $p\in\mathsf{fin}(\Pi(P,f))$, we have $\exists i_p\forall j>i_p(p\in\mathsf{Pos}(\pi_j)\wedge(\mathsf{Proof}(\pi_j,p) = \mathsf{Proof}(\pi',p))$, by definition. Let i be the maximum among all i_p's (The set $\mathsf{Pos}(\pi')$ is finite). Then for all $j>i$ we have $\pi'\in\pi_j$. Hence we have the following:

$$\forall\pi'\in\mathsf{fin}(\Pi(P,f))\ \forall p\in\pi'\ \exists i\ \forall j>i\ (p\in\pi_j\wedge\ (\mathsf{Proof}(\pi_j,p)=\mathsf{Proof}(\pi',p)))$$

And that is to say for each $\pi'\in\mathsf{fin}(\Pi(P,f))$, there exists an i such that for all $j>i$, π' is a finite sub-pre-proof of all π_j. Hence $[\![\pi']\!]_{\mathbf{Rel}}$ is a subset of $[\![\pi_j]\!]_{\mathbf{Rel}}$ for all $j>i$, so, $[\![\pi']\!]_{\mathbf{Rel}}\subseteq\bigcap_{j>i}[\![\pi_j]\!]_{\mathbf{Rel}}$. ◀

F.3 Proof of Theorem 14

▶ **Theorem 38.** *Let $(\pi_i)_{i\in\omega}$ be a Cauchy sequence such that $\forall i,j\in\omega$ we have $[\![\pi_i]\!]_{\mathbf{Rel}} = [\![\pi_j]\!]_{\mathbf{Rel}}$. Then $[\![\lim_{n\to\infty}\pi_i]\!]_{\mathbf{Rel}} = [\![\pi_0]\!]$.*

Proof.

$$\begin{aligned}[\![\lim_{n\to\infty}\pi_i]\!]_{\mathbf{Rel}} &= \bigcup_i\bigcap_{j>i}[\![\pi_j]\!]_{\mathbf{Rel}} && \text{By Lemma 13}\\ &= \bigcup\bigcap[\![\pi_0]\!]_{\mathbf{Rel}} \qquad = [\![\pi_0]\!]_{\mathbf{Rel}}\end{aligned}$$

◀

F.4 Proof of Lemma 18

▶ **Lemma 39.** *If A is a $\mu\mathsf{LL}_\infty$ formula and $t\notin\mathcal{T}([\![\nu^\alpha\zeta.F]\!])$ ($t\subseteq[\![\nu^\alpha\zeta.F]\!]$), then there exists an ordinal $\gamma<\alpha$ such that $t\notin\mathcal{T}([\![F\,[\nu^\gamma\zeta.F/\zeta]]\!])$.*

Proof. If α is a successor ordinal $\delta+1$ then $U_\alpha = \mathcal{T}([\![F]\!])([\![\nu\zeta\,F]\!]_{\mathbf{Rel}},U_\delta)$ by definition, and obviously $t\notin\mathcal{T}([\![F]\!])(([\![\nu\zeta\,F]\!]_{\mathbf{Rel}},U_\delta))$. And so $t\notin\mathcal{T}([\![F\,[\nu^\gamma\zeta.F/\zeta]]\!])$ for $\gamma=\delta$.

If α is a limit ordinal, then: $U_\alpha = \bigcap_{\gamma<\alpha}U_\gamma$, and $t\notin\bigcap_{\gamma<\alpha}U_\gamma = \bigcap_{\delta+1<\alpha}U_{\delta+1}$. So, there exists an ordinal $\delta+1<\alpha$ such that $t\notin U_{\delta+1}$ and we continue as before.

◀

F.5 Poof of Lemma 19

▶ **Lemma 40.** $\mathcal{T}([\![F\,[\mu\zeta\,F/\zeta]]\!]) = \mathcal{T}([\![\mu\zeta\,F]\!])$.

Proof. The interpretation of $\mu\zeta\,F$ is the least fixed-point of θ_F where θ_F is $\mathcal{T}([\![F]\!])$. So, we have:

$$\begin{aligned}\mathcal{T}([\![\mu\zeta\,F]\!]) &= \theta_F(\mathcal{T}([\![\mu\zeta\,F]\!]))\\ &= \mathcal{T}([\![F]\!])(([\![\mu\zeta\,F]\!]_{\mathbf{Rel}},\mathcal{T}([\![\mu\zeta\,F]\!]))) && \text{by definition of } \theta_F\\ &= \mathcal{T}([\![F\,[\mu\zeta\,F/\zeta]]\!])\end{aligned}$$

◀

F.6 Proof of Lemma 20

▶ **Lemma 41.** *If π is a proof of $\vdash \Gamma$ and $[\![\pi]\!] \notin \mathcal{T}(([\![\Gamma]\!]))$, then*

1. *π has an infinite branch $\gamma = (\vdash \Gamma_i)_{i\in\omega}$ such that $[\![\pi_i]\!] \notin \mathcal{T}(([\![\Gamma_i]\!]))$ where π_i is the sub-proof of π rooted in $\vdash \Gamma_i$;*
2. *and there exists a sequence of functions $(f_i)_{i\in\omega}$ where f_i maps all formulas D of Γ_i to a marked formula $f_i(D)$ such that*
   - *$(f_i(D))^\circ = D$,*
   - *one can write $\Gamma_i = \Gamma'_i, C$,*
   - *and there exists $x \in \mathcal{T}([\![(f_i(\Gamma'_i))^\perp]\!])$ such that $[\![\pi_i]\!].x \notin \mathcal{T}([\![f_i(C)]\!])$ where $\Gamma'_i = A^i_1, \cdots, A^i_{n_i}$ and $[\![(f_i(\Gamma'_i))^\perp]\!] = ([\![f_i(A^i_1)]\!])^\perp \otimes \cdots \otimes ([\![f_i(A^i_{n_i})]\!])^\perp$.*

Proof. We set $\Gamma_0 = \Gamma$, and $f_0(D) = \overline{D}$ for all $D \in \Gamma_0$:

- Since $\pi_0 = \pi$, $[\![\pi_0]\!] \notin \mathcal{T}([\![\Gamma_0]\!])$.
- Let C be the principal formula in Γ_0. The sequent $\vdash f_0(\Gamma_0)$ is denotationally the same as $\vdash (f_0(\Gamma'_0))^\perp \multimap f_0(C)$. By the proposition 17, $[\![f_0(D)]\!] = [\![D]\!]$ for all $D \in \Gamma_0$. So, $[\![\pi_0]\!] \notin \mathcal{T}(f_0(\Gamma_0))$. That is to say $[\![\pi_0]\!] \notin \mathcal{T}([\![(f_0(\Gamma'_0))^\perp \multimap f_0(C)]\!])$. Therefore, by definition, there exists $x \in \mathcal{T}([\![(f_0(\Gamma'_0))^\perp]\!])$ such that $[\![\pi_0]\!].x \notin \mathcal{T}([\![f_0(C)]\!])$.

Suppose that we have provided Γ_i and f_i for $i \in \omega$. We then define Γ_{i+1} and f_{i+1} depending on the rule applied on $\vdash \Gamma_i$ in π. Let us assume that the formula C is the principal in Γ_i:

- If $C = C_1 \parr C_2$, then Γ_{i+1} is the unique premise of $\vdash \Gamma_i$. $f_i(C) = B^1_C \parr B^2_C$ where B^1_C and B^2_C are two marked formulas, so, we set $f_{i+1}(C_1) = B^1_C$, $f_{i+1}(C_2) = B^2_C$, and $f_{i+1}(F) = f_i(F)$ for the other $F \in \Gamma_{i+1}$:
  - Since Γ_i is obtained by applying the $\parr$ rule on Γ_{i+1}, we have $[\![\pi_{i+1}]\!] = [\![\pi_i]\!]$, and $[\![\Gamma_{i+1}]\!] = [\![\Gamma_i]\!]$. By induction hypothesis, $[\![\pi_{i+1}]\!] \notin \mathcal{T}([\![\Gamma_{i+1}]\!])$.
  - By induction hypothesis, there exists $x \in \mathcal{T}([\![(f_i(\Gamma'_i))^\perp]\!])$ such that $[\![\pi_i]\!].x \notin \mathcal{T}([\![f_i(C)]\!])$. So, $[\![\pi_{i+1}]\!].x = [\![\pi_i]\!].x \notin \mathcal{T}([\![B^1_C \parr B^2_C]\!]) = ((\mathcal{T}(([\![((B^1_C))^\perp]\!] \otimes [\![((B^2_C))^\perp]\!]))))^\perp$. So, there is a $y \in \mathcal{T}(([\![((B^1_C))^\perp]\!] \otimes [\![((B^2_C))^\perp]\!]))$ such that $[\![\pi_{i+1}]\!].x \cap y \neq \emptyset$. Since $y \in \mathcal{T}(([\![(B^1_C)^\perp]\!] \otimes [\![(B^2_C)^\perp]\!]))$, there is $u' \in \mathcal{T}([\![(B^1_C)^\perp]\!])$ and $v' \in \mathcal{T}([\![(B^1_C)^\perp]\!])$ such that $u' \times v' \subseteq y$. So, $[\![\pi_{i+1}]\!].x \cap (u' \times v') = \emptyset$. This statement is equivalent to $([\![\pi_{i+1}]\!].x).u' \cap v' \neq \emptyset$. $[\![\pi_{i+1}]\!].x \in$, and this is equivalent to $[\![\pi_{i+1}]\!].(x \times u') \cap v' \neq \emptyset$. We have shown till now that there exists $v' \in \mathcal{T}([\![(B^1_C)^\perp]\!])$ such that $[\![\pi_{i+1}]\!].x' \cap v' \neq \emptyset$ where $x' = x \times u'$. So, by definition, $[\![\pi_{i+1}]\!].x' \notin \mathcal{T}([\![B^1_C]\!])$.
- If $C = C_1 \oplus C_2$, then we proceed as above.
- If $C = C_1 \otimes C_2$. Let us call Γ^1_{i+1} and Γ^2_{i+1} for the two premises of $\vdash \Gamma_i$. $f_i(C) = B^1_C \otimes B^2_C$ where B^1_C and B^2_C are two marked formulas. Since $[\![\pi_i]\!] \notin \mathcal{T}([\![\Gamma_i]\!])$, we have $[\![\pi^j_{i+1}]\!] \notin \mathcal{T}([\![\Gamma^j_{i+1}]\!])$ for either $j = 1$ or $j = 2$ where π^1_{i+1} (respectively π^2_{i+1}) is the left (respectively the right) subproof of π_i. Let us assume that it is true for $j = 1$ (the proof of the case $j = 2$ is is identical to the case $j = 1$). So we set $\Gamma_{i+1} = \Gamma^1_{i+1}$, $f_{i+1}(C_1) = B^1_C$, and $f_{i+1}(D) = f_i(D)$ for the other $D \in \Gamma^1_{i+1}$.

  - By induction hypothesis, there exists $x' \in \mathcal{T}([\![(f_i(\Gamma^1_{i+1} \parr \Gamma^2_{i+1}))^\perp]\!])$ such that $[\![\pi_i]\!].x' \notin \mathcal{T}([\![B^1_C \otimes B^2_C]\!])$. Hence $[\![\pi_i]\!] \notin \mathcal{T}([\![f_i(\Gamma_i)]\!])$ by definition. So, we have $[\![\pi^j_{i+1}]\!] \notin \mathcal{T}([\![f_{i+1}(\Gamma_{i+1}j') \parr B^j_C]\!])$ for either $j = 1$ or $j = 2$. Let us assume that is true for $j = 1$ (the proof of the case $j = 2$ is identical to the case $j = 1$). So, $[\![\pi^1_{i+1}]\!] \notin \mathcal{T}([\![(f_{i+1}(\Gamma_{i+1}j'))^\perp \multimap B^1_C]\!])$. And therefore, by definition, there is a $y \in [\![(f_{i+1}(\Gamma_{i+1}j'))^\perp]\!]$ such that $[\![\pi^1_{i+1}]\!].y \notin \mathcal{T}([\![B^1_C]\!])$.

- If $C = C_1 \& C_2$, then we proceed as above.
- IF $C = \mu\zeta\, F$, then Γ_{i+1} is the unique premise of $\vdash \Gamma_i$. Wlog let us say $\Gamma_i = A^i_1, \cdots, A^i_{n_i}, \mu\zeta\, F$. $f_i(C) = \mu\zeta\, B_C$ where B_C is a marked formula. By induction hypothesis, there exists $x \in \mathcal{T}([\![(f_i(\Gamma'_i))^\perp]\!])$ such that $[\![\pi_i]\!].x \notin \mathcal{T}([\![\mu\zeta\, B_C]\!])$ where $\Gamma'_i = A^i_1, \cdots, A^i_{n_i}$. So, $[\![\pi_{i+1}]\!].x \notin \mathcal{T}([\![B_C\, [\mu\zeta\, B_C/\zeta]]\!])$, since $[\![\pi_{i+1}]\!] = [\![\pi_i]\!]$ and lemma 19. Then we set $f_{i+1}(F\,[C/\zeta]) = B_C\,[\mu\zeta\, B_C/\zeta]$ and $f_{i+1}(D) = f_i(D)$ for all the other formula $D \in \Gamma_{i+1}$ in order to have the second property of the lemma 20.
- If $C = \nu\zeta\, F$, then Γ_{i+1} is the unique premise of $\vdash \Gamma_i$. Wlog, let us say $\Gamma_i = A^i_1, \cdots, A^i_{n_i}, \nu\zeta\, F$. $f_i(C) = \nu^\theta\zeta.B_C$ where B_C is a marked formula. By induction hypothesis, there exists $x \in \mathcal{T}([\![(f_i(\Gamma'_i))^\perp]\!])$ such that $[\![\pi_i]\!].x \notin \mathcal{T}([\![\nu^\theta\zeta.B_C]\!])$ where $\Gamma'_i = A^i_1, \cdots, A^i_{n_i}$. By Lemma 18, there is an ordinal $\delta < \theta$ such that $[\![\pi_{i+1}]\!].x \notin \mathcal{T}([\![B_C\,[\nu^\delta\zeta.B_C/\zeta]]\!])$, since $[\![\pi_{i+1}]\!] = [\![\pi_i]\!]$. So, we set $f_{i+1}(F\,[C/\zeta]) = f_i(F)\,[\nu^\delta\zeta.B_C/\zeta]$ and $f_{i+1}(D) = f_i(D)$ for all the other formula $D \in \Gamma_{i+1}$ in order to have the second property of the lemma.
- If the rule applied to $\vdash \Gamma_i$ is a (cut) rule on the C. Let us say Γ_i is Γ^1_i, Γ^2_i. By induction hypothesis, $[\![\pi_i]\!] \notin \mathcal{T}([\![\Gamma_i]\!])$. So, we have either $[\![\pi_{i+1}]\!] \notin \mathcal{T}([\![\Gamma^1_i \parr C]\!])$ or $[\![\pi_{i+1}]\!] \notin \mathcal{T}([\![\Gamma^2_i \parr C^\perp]\!])$. Wlog let us say $[\![\pi_{i+1}]\!] \notin \mathcal{T}([\![\Gamma^1_i \parr C]\!])$. Then we take $\Gamma_{i+1} = \Gamma^1_i, C$. And for the f_{i+1}, we define $f_{i+1}(D) = f_i(D)$ for all $D \in \Gamma^1_i$, and $f_i(C) = \overline{C}$.

  - By induction hypothesis, $[\![\pi_i]\!] \notin \mathcal{T}([\![f_i(\Gamma_i)]\!])$. So, we have either $[\![\pi_{i+1}]\!] \notin \mathcal{T}([\![f_i(\Gamma^1_i) \parr \overline{C}]\!])$ or $[\![\pi_{i+1}]\!] \notin \mathcal{T}([\![f_i(\Gamma^1_i) \parr \overline{C^\perp}]\!])$. So, we can use definition of morphisms in the category **Nuts** to deduce the second property as we proceed as the case $C = C_1 \otimes C_2$.

- If the rule applied to $\vdash \Gamma_i$ is a (w) rule, then Γ_{i+1} is the unique premise of the (w) rule. And $f_{i+1}(D) = f_i(D)$ for all $D \in \Gamma_{i+1}$. We have $[\![\pi_{i+1}]\!] \notin \mathcal{T}([\![f_i(\Gamma_{i+1})]\!]) = \mathcal{T}([\![f_{i+1}(\Gamma_{i+1})]\!])$, since $[\![\pi_i]\!] \notin \mathcal{T}([\![f_i(\Gamma_i)]\!])$ (here we are also using Theorem **??** of $\mu\mathsf{LL}$).
- If the rule applied to $\vdash \Gamma_i$ is (c) rule on the formula $?C$, then we proceed as above.
- If the rule applied to $\vdash \Gamma_i$ is (d) rule on the formula $?C$. Let us say $\Gamma_i = \Gamma'_i, ?C$. Then $\Gamma_{i+1} = \Gamma'_i, C$. $f_{i+1}(D) = f_i(D)$ for all $D \in \Gamma'_i$. $f_i(?C) = ?B_C$ where B_C is a marked formula. Then we take $f_{i+1}(C) = B_C$. To show the second property, we can again use soundness theorem of of $\mu\mathsf{LL}$ [21].
- If the rule applied to $\vdash \Gamma_i$ is (p) rule on the formula $!C$, then we proceed as above.

◀

F.7 Proof of Theorem 21

▶ **Theorem 42.** *If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!] \in \mathcal{T}([\![\Gamma]\!])$.*

Proof. Let us assume $[\![\pi]\!] \notin \mathcal{T}([\![\Gamma]\!])$. We can then apply Lemma 20 to obtain an infinite branch $(\vdash \Gamma_i)_{i\in\omega}$ and a sequence $(f_i)_{i\in\omega}$ satisfying properties 1 and 2 of Lemma 20. By the definition of valid proof (Definition 8), there exists a valid thread $t = (F_i)_{i\in\omega}$ for the infinite branch $(\vdash \Gamma_i)_{i\in\omega}$. Let $\nu\zeta\, F$ be the minimal formula formula of t. So, there are infinitely many times in t that we use a ν rule to unfold $\nu\zeta\, F$. Let $(i_k)_{k\in\omega}$ be the sequence of indices where $\nu\zeta\, F$ gets unfolded. Then $(\nu\zeta\, F)_{\alpha_{i_k}}$ is sub-occurrence (Definition **??**) of $(\nu\zeta\, F)_{\alpha_{i'_k}}$ for $k \geqslant k'$ where α_{i_k} (respectively $\alpha_{i'_k}$) is the address of $\nu\zeta\, F$ in sequent i_k (respectively i'_k). By the property 2 of Lemma 20, $f_{i_k}(\nu\zeta\, F) = \nu^{\alpha_k}\zeta.f_{i_k}(F)$. Therefore, by the property 2 of Lemma 20 and by the construction of the f_i in the proof of Lemma 20, the sequence $(\alpha_k)_{k\in\omega}$ is strictly decreasing. As this contradicts the well-foundedness property of the ordinals we obtain the required contradiction and conclude that $[\![\pi]\!] \in \mathcal{T}([\![\Gamma]\!])$. ◀

G On strong validity and finitization of circular proofs

▶ **Definition 43.** *Let π be a circular pre-proof and β an infinite branch. A thread $t = (F_i)_{k \leq i \in \omega}$ is said to be strongly valid if t is valid and if there is $k \in \omega$ such that $\forall i, j > k$, if $\beta_i = \beta_j$, then $F_i = F_j$. We say a circular pre-proof π is strongly valid, if every infinite branch of π has a strongly valid thread.*

▶ **Proposition 44.** *Let $\nu\zeta\, A$ be a μLL formula. Then, for any context Γ there is a formula I such that the following rules are derivable in $\mu\mathsf{LL}_\infty$.*

$$\frac{\vdash \Delta\,[\nu\zeta\,.F/\zeta]}{\vdash \Delta\,[I/\zeta]} \qquad \frac{}{\vdash I, \Gamma}$$

The formula I in the proposition above is called the invariant formula and is defined as $\nu\zeta\,.(A \oplus (\parr I)^{\perp})$.

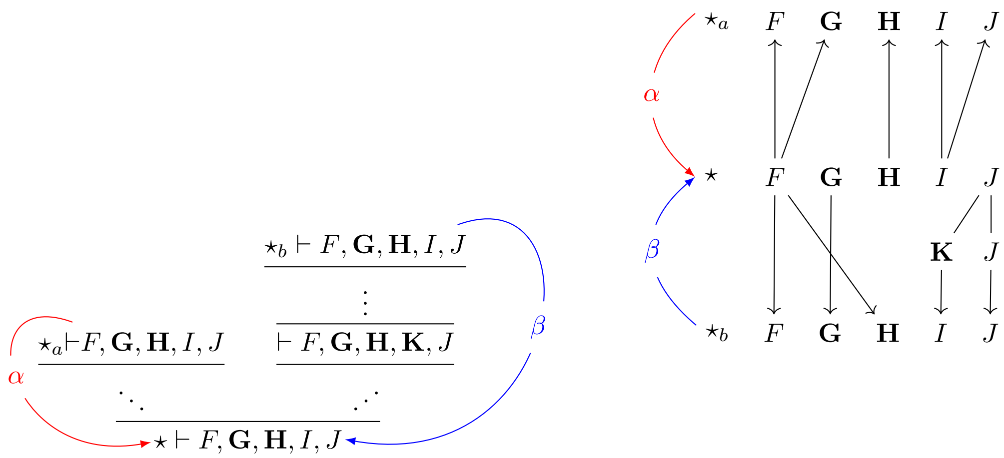
▶ **Proposition 45.** *Let π be a circular pre-proof of $\vdash \Gamma$. If π is strongly valid, then there is a finite proof $\tilde{\pi}$ of $\vdash \Gamma$ in μLL.*

H On the validity of π_∞

In this appendix, we provide some additional details on the derivation π_∞ which is considered in Section 6 (and defined in Figure 7) and we discuss in details the structure of its validating threads.

We present below an abstracted version of the pre-proof π_∞ to outline its threading structure and its "validation modes". In what follows, coinductive formulas (namely G, H, K) are depicted in **bold face**. Note that K is a ν-subformula of I and J.

On the left, we only show four sequents of the proof, with the two back-edges. On the right, we show the threading structure of the pre-proof, showing the recreationg of fixed-point formulas as well as the progress.



The circular proof has two back-edges which induce, in its infinite unfolding, three types of infinite branches (or three types of infinite paths in the circular representation, which is equivalent):

1. those branches which ultimately only visit the red back-edge, labeled α (visiting the blue back-edge only finitely many times);

2. those branches which ultimately only visit the blue back-edge, labeled β (visiting the red back-edge only finitely many times);
3. those branches which visit the red and blue back-edges, labeled α and β respectively, infinitely many times, that is such that in the "future", there will always be a change of direction.

Considering that validaty in non-wellfounded proofs is expressed in terms of recurring sequents, it is only a matter of its behaviour at the limit and one can neglect the transitory phase at the start and considering only the above three cases to classify all infinite branches.

1. the infinite branches containing only the red back-edge, α, validate via a thread on H only: indeed, K is never principal and formula G is erased and recreated at each iteration of the branch, making no progress.
2. the infinite branches containing only the blue back-edge, β, validate via a thread on G only: indeed, K is principal on the branch but unfolds into I which is erased in the following iteration, no coinductive progrees is made there, while formula H is erased and recreated at each iteration of the branch, making no progress either.
3. the infinite branches containing both infinitely many blue and red back-edges, α and β, validate via a thread on K only: indeed, G progresses on the left path but is erased next time the branches goes to the right while similarly, H progresses along the right path but is erased next time the branches goes to the left. On the other hand, K progresses infinitely: each time the branch **switches** from the right path to the left path, a coinductive progress is made on K, which is then stored in I and J until the next shift from a right path to a left path is made.

To sum up, one can then understand in the above example the complexity of the validation mode of π_∞: in each different class of branches, there is just one validating thread.

Moreover, π_∞ is not strongly valid since no unfolding of π_∞ to another circular proof can allow to synthesize the transition from a right path to a left path that is described above in order to ensure that one can specifically identify the occurrences of sequence on which K actually contributes to a coinductive progress.

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