A linear perspective on cut-elimination for non-wellfounded sequent calculi with least and greatest fixed-points

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1. Introduction and Background
   - Aim
   - Classical and Linear Logic
   - μLL^{∞}: Circular and Non-wellfounded Proofs for Linear Logic with Least and Greatest Fixed-points

2. μLL^{∞} Cut-elimination
   - Reviewing $\mu MALL^{\infty}$ cut-elimination
   - Encoding μLL^{∞} in $\mu MALL^{\infty}$
   - Simulation of μLL^{∞} cut-elimination steps
   - Cut-elimination for μLL^{∞}

3. Applications, Remarks and Conclusion
   - Cut-elimination for μLK^{∞}, μLJ^{∞}
   - Remarks on the encoding of the exponentials
   - Conclusion

Aim

A linear perspective on cut-elimination for
non-wellfounded sequent calculi with least and greatest fixed-points

- *Long-term goal:* contribute to improve the support for inductive and coinductive constructs in ATP, FPL and ITP.
- *Medium-term:* Design and study logical frameworks combining good properties wrt. (i) proof construction and (ii) proof normalization
  (that is, open-goal-elimination and cut-elimination)
  & admitting support for inductive and coinductive reasoning;
- Study circular & non-wellfounded proof systems for the μ-calculus:
  - pioneering works by Santocanale; Sprenger and Dam; Studer; Brotherston & Simpson; Dax, Hoffman & Lange;
  - recently, numerous developments of circular/cyclic proof systems (Afshari & Leigh, Baelde & Doumane & S., Berardi & Tatsuta, Cohen & Rowe, Das & Doumane & Pous, ... + a new generation of young researchers)
- This paper establishes a cut-elimination theorem for non-wellfounded proofs for μLL^{∞}, μLK^{∞} and μLJ^{∞} by relying on (a tiny bit of) *what we have learnt from linear logic* in the past 35 years.

Relating additive & multiplicative inferences

- In LK, additive and multiplicative inferences for $\wedge$ and $\vee$ are interderivable *thanks to availability of structural rules*:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash A}\,(\mathsf{Ax})}{A, B \vdash A}\,(\mathsf{W_l}) \qquad \cfrac{\cfrac{}{B \vdash B}\,(\mathsf{Ax})}{A, B \vdash B}\,(\mathsf{W_l})}{A, B \vdash A \wedge^a B}\,(\wedge^{\mathsf{a}}{}_{\mathsf{r}})}{A \wedge^m B \vdash A \wedge^a B}\,(\wedge^{\mathsf{m}}{}_{\mathsf{l}})
\qquad
\cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash A}\,(\mathsf{Ax})}{A \wedge^a B \vdash A}\,(\wedge^{\mathsf{a}2}_{\mathsf{l}}) \qquad \cfrac{\cfrac{}{B \vdash B}\,(\mathsf{Ax})}{A \wedge^a B \vdash B}\,(\wedge^{\mathsf{a}1}_{\mathsf{l}})}{A \wedge^a B, A \wedge^a B \vdash A \wedge^m B}\,(\wedge^{\mathsf{m}}{}_{\mathsf{r}})}{A \wedge^a B \vdash A \wedge^m B}\,(\mathsf{C_l})$$

Relating additive & multiplicative inferences

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A, B are weakened on the left, $A \wedge^a B$ is contracted on the left.

Exponentials: Relating additive & multiplicative inferences

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$$
\dfrac{
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\qquad
\dfrac{\dfrac{}{B \vdash B}\,(\mathrm{Ax})}{A, B \vdash B}\,(\mathrm{W_l})
}{
\dfrac{A, B \vdash A \wedge^a B}{A \wedge^m B \vdash A \wedge^a B}\,(\wedge^{\mathrm{m}}{}_{\mathrm{l}})
}\,(\wedge^{\mathrm{a}}{}_{\mathrm{r}})
\qquad\qquad
\dfrac{
\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{Ax})}{A \wedge^a B \vdash A}\,(\wedge^{\mathrm{a2}}_{\mathrm{l}})
\qquad
\dfrac{\dfrac{}{B \vdash B}\,(\mathrm{Ax})}{A \wedge^a B \vdash B}\,(\wedge^{\mathrm{a1}}_{\mathrm{l}})
}{
\dfrac{A \wedge^a B, A \wedge^a B \vdash A \wedge^m B}{A \wedge^a B \vdash A \wedge^m B}\,(\mathrm{C_l})
}\,(\wedge^{\mathrm{m}}{}_{\mathrm{r}})
$$

A, B are weakened on the left, $A \wedge^a B$ is contracted on the left.

- In LL, no free structural rules: need to tag formulas with exponentials where structural rules are needed, leading to: $!A \otimes !B \dashv\vdash !(A \& B)$.

  (with one-sided sequents: $\vdash ?A^\perp ⅋ ?B^\perp, !(A \& B)$ *and* $\vdash !A \otimes !B, ?(A^\perp \oplus B^\perp)$*)*

Exponentials: Relating additive & multiplicative inferences

- In LK, additive and multiplicative inferences for $\wedge$ and $\vee$ are interderivable *thanks to availability of structural rules*:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A\vdash A}(\mathrm{Ax})}{A,{\color{red}B}\vdash A}(\mathrm{W_l}) \quad \dfrac{\dfrac{}{B\vdash B}(\mathrm{Ax})}{{\color{red}A},B\vdash B}(\mathrm{W_l})}{{\color{red}A},{\color{red}B}\vdash A\wedge^a B}(\wedge^{a}{}_{r})}{{\color{red}A}\wedge^m {\color{red}B}\vdash A\wedge^a B}(\wedge^{m}{}_{l})
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{A\vdash A}(\mathrm{Ax})}{A\wedge^a B\vdash A}(\wedge^{a2}_{l}) \quad \dfrac{\dfrac{}{B\vdash B}(\mathrm{Ax})}{A\wedge^a B\vdash B}(\wedge^{a1}_{l})}{{\color{blue}A\wedge^a B},{\color{blue}A\wedge^a B}\vdash A\wedge^m B}(\wedge^{m}{}_{r})}{{\color{blue}A\wedge^a B}\vdash A\wedge^m B}(\mathrm{C_l})
$$

${\color{red}A},{\color{red}B}$ are weakened on the left, ${\color{blue}A\wedge^a B}$ is contracted on the left.

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  (with one-sided sequents: $\vdash ?A^\perp \parr ?B^\perp, !(A\,\&\,B)$ and $\vdash !A\otimes !B, ?(A^\perp\oplus B^\perp)$)

$$
\pi_{\otimes\vdash\&} = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}(\mathrm{Ax})}{\vdash ?A^\perp, A}(?\mathrm{d})}{\vdash ?A^\perp, ?B^\perp, A}(?\mathrm{w}) \quad \dfrac{\dfrac{\dfrac{}{\vdash B^\perp, B}(\mathrm{Ax})}{\vdash ?B^\perp, B}(?\mathrm{d})}{\vdash ?A^\perp, ?B^\perp, B}(?\mathrm{w})}{\vdash ?A^\perp, ?B^\perp, A\,\&\,B}(\&)}{\vdash ?A^\perp, ?B^\perp, !(A\,\&\,B)}(!\mathrm{p})}{\vdash ?A^\perp \parr ?B^\perp, !(A\,\&\,B)}(\parr)
$$

$$
\pi_{\&\vdash\otimes} = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}(\mathrm{Ax})}{\vdash A^\perp\oplus B^\perp, A}(\oplus_1)}{\vdash ?(A^\perp\oplus B^\perp), A}(?\mathrm{d})}{\vdash ?(A^\perp\oplus B^\perp), !A}(!\mathrm{p}) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B^\perp, B}(\mathrm{Ax})}{\vdash A^\perp\oplus B^\perp, B}(\oplus_2)}{\vdash ?(A^\perp\oplus B^\perp), B}(?\mathrm{d})}{\vdash ?(A^\perp\oplus B^\perp), !B}(!\mathrm{p})}{\vdash ?(A^\perp\oplus B^\perp), ?(A^\perp\oplus B^\perp), !A\otimes !B}(\otimes)}{\vdash ?(A^\perp\oplus B^\perp), !A\otimes !B}(?\mathrm{c})
$$

Non-Wellfounded Sequent Calculus

Consider your favourite logic $\mathscr{L}$ & add fixed points as in the μ-calculus

Pre-proofs are the trees **coinductively** generated by:

- $\mathscr{L}$ inference rules
- inference for μ, ν:

$$\frac{\Gamma, F[\mu X.F/X] \vdash \Delta}{\Gamma, \mu X.F \vdash \Delta}\ (\mu_l) \qquad \frac{\Gamma, F[\nu X.F/X] \vdash \Delta}{\Gamma, \nu X.F \vdash \Delta}\ (\nu_l)$$

$$\frac{\Gamma \vdash F[\mu X.F/X], \Delta}{\Gamma \vdash \mu X.F, \Delta}\ (\mu_r) \qquad \frac{\Gamma \vdash F[\nu X.F/X], \Delta}{\Gamma \vdash \nu X.F, \Delta}\ (\nu_r)$$

Fischer-Ladner Subformulas: induced by fixed-point unrolling:

$$F[\sigma X.F/X] \in \mathsf{FL}(\sigma X.F) \text{ with } \sigma \in \{\mu, \nu\}.$$

Circular (pre-)proofs: the regular fragment of infinite (pre-)proofs, ie finitely many sub-(pre)proofs.

Pre-proofs are unsound!!
Need for a global validity condition

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X}\ (\mu)}{\vdash \mu X.X}\ (\mu) \qquad \dfrac{\dfrac{\vdots}{\vdash \nu X.X, F}\ (\nu)}{\vdash \nu X.X, F}\ (\nu)}{\vdash F}\ (\mathrm{Cut})$$

μLL$^\infty$ Non-Wellfounded Sequent Calculus

Consider your favourite logic LL & add fixed points as in the μ-calculus

μLL$^\infty$ Pre-proofs are the trees **coinductively** generated by:

- LL inference rules
- inference for μ, ν: $\dfrac{\vdash F[\mu X.F/X], \Delta}{\vdash \mu X.F, \Delta}\ (\mu_r) \qquad \dfrac{\vdash F[\nu X.F/X], \Delta}{\vdash \nu X.F, \Delta}\ (\nu_r)$

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$F[\sigma X.F/X] \in \mathrm{FL}(\sigma X.F)$ with $\sigma \in \{\mu, \nu\}$.

Circular (pre-)proofs: the regular fragment of infinite (pre-)proofs, ie finitely many sub-(pre)proofs. μLL$^\omega$

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$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X}\ (\mu)}{\vdash \mu X.X}\ (\mu) \qquad \dfrac{\dfrac{\vdots}{\vdash \nu X.X, F}\ (\nu)}{\vdash \nu X.X, F}\ (\nu)}{\vdash F}\ (\mathrm{Cut})$$

Involutive negation, $(\ )^\perp$: operator on formula, not a connective.

One-sided sequents as <u>lists</u>: $\vdash A_1, \ldots, A_n$. *($\Gamma \vdash \Delta$ is a short for $\vdash \Gamma^\perp, \Delta$)*

μ and ν are dual binders. <u>Ex:</u> $(\nu X.X \otimes X)^\perp = \mu X.X \parr X$.

μLK$^\infty$ Inferences

Inference Rules

$$\frac{}{\vdash F, F^\perp}\ (\mathrm{Ax}) \qquad \frac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta}\ (\mathrm{Cut}) \qquad \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ (\mathrm{ex})$$

$$\frac{\vdash \ F, \ F, \Gamma}{\vdash \ F, \Gamma}\ (?\mathrm{c}) \qquad \frac{\vdash \Gamma}{\vdash \ F, \Gamma}\ (?\mathrm{w})$$

$$\frac{}{\vdash \top, \Gamma}\ (\top) \qquad \frac{\vdash F, \Gamma \qquad \vdash G, \Gamma}{\vdash F \wedge G, \Gamma}\ (\wedge) \qquad \frac{\vdash A_i, \Gamma}{\vdash A_1 \vee A_2, \Gamma}\ (\vee_i)$$

$$\frac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma}\ (\nu) \qquad \frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\ (\mu)$$

μLL$^\infty$ Inferences

μLL formulas

$$
\begin{array}{lllllllll}
F & ::= & a & \mid F \mathbin{\&} F & \mid F \parr F & \mid \top & \mid \bot & \mid ?F & \textbf{negative } \textit{LL} \textbf{ formulas} \\
 & & \mid a^\perp & \mid F \otimes F & \mid F \oplus F & \mid 1 & \mid 0 & \mid !F & \textbf{positive } \textit{LL} \textbf{ formulas} \\
 & & \mid X & \mid \mu X.F & \mid \nu X.F & & & & \textbf{lfp \& gfp}
\end{array}
$$

μLL$^\infty$ Inference Rules

$$
\frac{}{\vdash F, F^\perp}\ (\mathrm{Ax}) \qquad \frac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta}\ (\mathrm{Cut}) \qquad \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ (\mathrm{ex})
$$

$$
\frac{}{\vdash \top, \Gamma}\ (\top) \qquad \frac{\vdash F, \Gamma \qquad \vdash G, \Gamma}{\vdash F \mathbin{\&} G, \Gamma}\ (\&) \qquad \frac{\vdash A_i, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ (\oplus_i) \qquad \text{(no rule for 0)}
$$

$$
\frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ (\parr) \qquad \frac{\vdash F, \Gamma \qquad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ (\otimes) \qquad \frac{}{\vdash \mathbf{1}}\ (\mathbf{1})
$$

$$
\frac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma}\ (\nu) \qquad \frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\ (\mu)
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$$\frac{}{\vdash \top, \Gamma}\ (\top) \qquad \frac{\vdash F, \Gamma \quad \vdash G, \Gamma}{\vdash F \& G, \Gamma}\ (\&) \qquad \frac{\vdash A_i, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ (\oplus_i) \qquad \text{(no rule for 0)}$$

$$\frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{\vdash F, G, \Gamma}{\vdash F ⅋ G, \Gamma}\ (⅋) \qquad \frac{\vdash F, \Gamma \quad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ (\otimes) \qquad \frac{}{\vdash \mathbf{1}}\ (\mathbf{1})$$

$$\frac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma}\ (\nu) \qquad \frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\ (\mu)$$

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μLL$^\infty$ Inference Rules (with ancestor relation)

$$\dfrac{}{\vdash F, F^\perp}\ (\mathrm{Ax}) \qquad \dfrac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta}\ (\mathrm{Cut}) \qquad \dfrac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ (\mathrm{ex})$$

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$$\dfrac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \dfrac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ (\parr) \qquad \dfrac{\vdash F, \Gamma \qquad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ (\otimes) \qquad \dfrac{}{\vdash \mathbf{1}}\ (\mathbf{1})$$

$$\dfrac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma}\ (\nu) \qquad \dfrac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\ (\mu)$$

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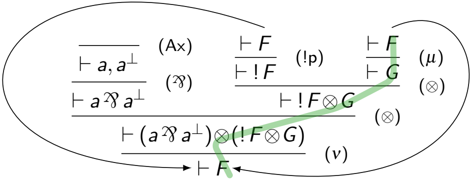
$$\frac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma}\ (\nu) \qquad \frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\ (\mu)$$

How to distinguish valid nwf proofs from invalid ones?

Infinite traces, validity

$F = \nu X.((a \parr a^{\perp}) \otimes (!X \otimes \mu Y.X))$.
$G = \mu Y.F$



A ***trace (or thread)*** on an infinite branch $(\Gamma_i)_{i \in \omega}$ is an infinite sequence of formula occurrences $(F_i)_{i \geq k}$ such that $\forall i \geq k$, $F_i \in \Gamma_i$ and F_{i+1} is an immediate ancestor of F_i.

Infinite traces, validity

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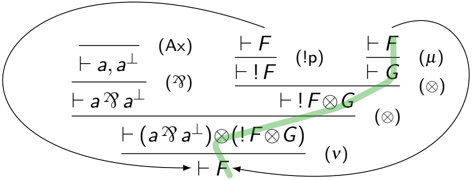

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A trace (or thread) is valid if the minimal ***recurring*** principal formula of the trace is a ν-formula ($\approx$ if it unfolds infinitely many ν).

A proof is valid if every infinite branch contains a valid trace.

Infinite traces, validity

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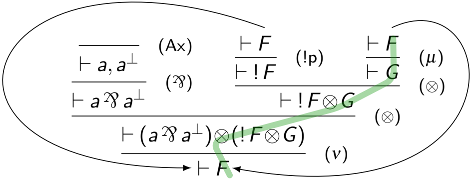

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Validity criteria ensure (productive) cut-elimination:

μALL$^{\omega}$: Santocanale in 2002, with Fortier in 2013;
μMALL$^{\infty}$: Baelde, Doumane and S. in 2016;
Bouncing μMALL$^{\infty}$: Baelde, Doumane, Kuperberg and S. in 2022.

μMALL$^\infty$ Cut Elimination Theorem

Theorem (Baelde, Doumane & S, 2016)

Fair μMALL$^\infty$ cut-reduction sequences converge to cut-free μMALL$^\infty$ proofs.

Strategy: "push" the cuts away from the root.

- **Cut-Cut Case**

$$\dfrac{\dfrac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta, G}{\vdash \Gamma, \Delta, G}\,(\mathrm{Cut}) \qquad \vdash G^\perp, \Sigma}{\vdash \Gamma, \Delta, \Sigma}\,(\mathrm{Cut}) \longleftrightarrow \dfrac{\vdash \Gamma, F \qquad \dfrac{\vdash F^\perp, \Delta, G \qquad \vdash G^\perp, \Sigma}{\vdash F^\perp, \Delta, \Sigma}\,(\mathrm{Cut})}{\vdash \Gamma, \Delta, \Sigma}\,(\mathrm{Cut})$$

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- **Cut-commutation steps** are productive:

$$\dfrac{\dfrac{\vdash \Delta, F \qquad \vdash \Delta, G}{\vdash \Delta, F \,\&\, G}\ (\&) \qquad \dots}{\vdash \Sigma, F \,\&\, G}\ (\mathrm{mcut}) \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F \qquad \dots}{\vdash \Sigma, F}\ (\mathrm{mcut}) \qquad \dfrac{\vdash \Delta, G \qquad \dots}{\vdash \Sigma, G}\ (\mathrm{mcut})}{\vdash \Sigma, F \,\&\, G}\ (\&)$$

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Strategy: "push" the cuts away from the root.

- **Cut-Cut Case** merge in a multicut:

$$\dfrac{\dfrac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta, G}{\vdash \Gamma, \Delta, G}\,(\mathrm{Cut}) \qquad \vdash G^\perp, \Sigma}{\vdash \Gamma, \Delta, \Sigma}\,(\mathrm{Cut}) \quad \Rightarrow \quad \dfrac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta, G \qquad \vdash G^\perp, \Sigma}{\vdash \Gamma, \Delta, \Sigma}\,(\mathrm{mcut})$$

- **Cut-commutation steps** are productive:

$$\dfrac{\dfrac{\vdash \Delta, F \qquad \vdash \Delta, G}{\vdash \Delta, F \,\&\, G}\,(\&) \qquad \dots}{\vdash \Sigma, F \,\&\, G}\,(\mathrm{mcut}) \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F \qquad \dots}{\vdash \Sigma, F}\,(\mathrm{mcut}) \qquad \dfrac{\vdash \Delta, G \qquad \dots}{\vdash \Sigma, G}\,(\mathrm{mcut})}{\vdash \Sigma, F \,\&\, G}\,(\&)$$

- **Key cases** are not productive:

$$\dfrac{\dots \qquad \dfrac{\vdash \Delta, F[\mu X.F]}{\vdash \Delta, \mu X.F}\,(\mu) \qquad \dfrac{\vdash \Gamma, F^\perp[\nu X.F^\perp]}{\vdash \Gamma, \nu X.F^\perp}\,(\nu)}{\vdash \Sigma}\,(\mathrm{mcut}) \quad \Rightarrow \quad \dfrac{\dots \qquad \vdash \Delta, F[\mu X.F] \qquad \vdash \Gamma, F^\perp[\nu X.F^\perp]}{\vdash \Sigma}\,(\mathrm{mcut})$$

Cut-elimination for μLL$^\infty$

Theorem

Fair μLL$^\infty$ mcut-reduction sequences converge to cut-free μLL$^\infty$ proofs.

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Why is it difficult?

μMALL$^\infty$ cut-elimination proof uses a locative sequent calculus, that is difficult to extend to a non-linear calculus.

Cut-elimination for μLL$^\infty$

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Fair μLL$^\infty$ mcut-reduction sequences converge to cut-free μLL$^\infty$ proofs.

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Idea. The proof goes by:

1. considering the following encoding of LL exponential modalities:
   $\underset{\mu}{?}F = \mu X.F \oplus (\bot \oplus (X \parr X)) \qquad \underset{\nu}{!}F = \nu X.F \,\&\, (\mathbf{1} \,\&\, (X \otimes X))$
   and translating μLL$^\infty$ sequents & proofs in μMALL$^\infty$: $\pi^\bullet \vdash \Gamma^\bullet$,
2. simulating μLL$^\infty$ cut-reduction sequences in μMALL$^\infty$ and
3. applying μMALL$^\infty$ cut-elimination theorem.

1) Encoding μLL$^\infty$ in μMALL$^\infty$: $\pi^\bullet \vdash \Gamma^\bullet$.

$$?_\mu F = \mu X.F \oplus (\bot \oplus (X \parr X)) \qquad !_\nu F = \nu X.F \,\&\, (\mathbf{1} \,\&\, (X \otimes X))$$

μMALL$^\infty$ derivability of the exponential rules ($?d^\bullet$, $?c^\bullet$, $?w^\bullet$, $!p^\bullet$):

Dereliction :

$$\dfrac{\dfrac{\vdash F, \Delta}{\vdash F \oplus (\bot \oplus (?_\mu F \parr ?_\mu F)), \Delta}\,(\oplus_1)}{\vdash ?_\mu F, \Delta}\,(\mu)$$

Contraction :

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash ?_\mu F, ?_\mu F, \Delta}{\vdash ?_\mu F \parr ?_\mu F, \Delta}\,(\parr)}{\vdash \bot \oplus (?_\mu F \parr ?_\mu F), \Delta}\,(\oplus_2)}{\vdash F \oplus (\bot \oplus (?_\mu F \parr ?_\mu F)), \Delta}\,(\oplus_2)}{\vdash ?_\mu F, \Delta}\,(\mu)$$

Weakening :

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Delta}{\vdash \bot, \Delta}\,(\bot)}{\vdash \bot \oplus (?_\mu F \parr ?_\mu F), \Delta}\,(\oplus_1)}{\vdash F \oplus (\bot \oplus (?_\mu F \parr ?_\mu F)), \Delta}\,(\oplus_2)}{\vdash ?_\mu F, \Delta}\,(\mu)$$

Promotion:

$$\dfrac{\vdash F, ?_\mu \Delta \qquad \dfrac{\dfrac{}{\vdash \mathbf{1}}\,(\mathbf{1})}{\vdash \mathbf{1}, ?_\mu \Delta}\,(?w^\bullet) \qquad \dfrac{\dfrac{\vdash !_\nu F, ?_\mu \Delta \qquad \vdash !_\nu F, ?_\mu \Delta}{\vdash !_\nu F \otimes !_\nu F, ?_\mu \Delta, ?_\mu \Delta}\,(\otimes)}{\vdash !_\nu F \otimes !_\nu F, ?_\mu \Delta}\,(?c^\bullet)}{\vdash !_\nu F, ?_\mu \Delta}\,(\nu)\ ,\ (\&)\ ,\ (\&)$$

Preservation of validity

π is a valid μLL$^\infty$ pre-proof of $\vdash \Gamma$ iff
$\pi^\bullet$ is a valid μMALL$^\infty$ pre-proof of $\vdash \Gamma^\bullet$.

2) Simulation of μLL$^\infty$ cut-elimination steps

μLL$^\infty$ cut-elimination steps can be simulated by the previous encoding. For instance, commutation of (Cut) with (?d) can be simulated as:

$$\dfrac{\dfrac{\vdash F, G, \Gamma}{\vdash ?_\mu F, G, \Gamma}\,(?d^\bullet) \qquad \vdash G^\perp, \Delta}{\vdash ?_\mu F, \Gamma, \Delta}\,(\mathrm{Cut}) \quad \longrightarrow^2 \quad \dfrac{\dfrac{\vdash F, G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash F, \Gamma, \Delta}\,(\mathrm{Cut})}{\vdash ?_\mu F, \Gamma, \Delta}\,(?d^\bullet)$$

by applying the commutations $(\mu)/(\mathrm{Cut})$ followed by $(\oplus)/(\mathrm{Cut})$.

Challenge: to show that the simulation of derivation also holds

(i) for the reductions involving (!p) as well as
(ii) for reductions occurring ***above*** a promotion rule (aka. in a box) since the encoding of (!p) uses an infinite, circular derivation.

2) Simulation of μLL$^\infty$ cut-elimination steps

Cut-commutation rules

$$\dfrac{\dfrac{\vdash F, G, \Gamma}{\vdash ?_{\mu} F, G, \Gamma}\,(?\mathrm{d}^\bullet) \qquad \vdash G^\perp, \Delta}{\vdash ?_{\mu} F, \Gamma, \Delta}\,(\mathrm{Cut}) \longrightarrow^2 \dfrac{\dfrac{\vdash F, G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash F, \Gamma, \Delta}\,(\mathrm{Cut})}{\vdash ?_{\mu} F, \Gamma, \Delta}\,(?\mathrm{d}^\bullet)$$

$$\dfrac{\dfrac{\vdash ?_{\mu} F, ?_{\mu} F, G, \Gamma}{\vdash ?_{\mu} F, G, \Gamma}\,(?\mathrm{c}^\bullet) \qquad \vdash G^\perp, \Delta}{\vdash ?_{\mu} F, \Gamma, \Delta}\,(\mathrm{Cut}) \longrightarrow^3 \dfrac{\dfrac{\vdash ?_{\mu} F, ?_{\mu} F, G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash ?_{\mu} F, ?_{\mu} F, \Gamma, \Delta}\,(\mathrm{Cut})}{\vdash ?_{\mu} F, \Gamma, \Delta}\,(?\mathrm{c}^\bullet)$$

$$\dfrac{\dfrac{\vdash G, \Gamma}{\vdash ?_{\mu} F, G, \Gamma}\,(?\mathrm{w}^\bullet) \qquad \vdash G^\perp, \Delta}{\vdash ?_{\mu} F, \Gamma, \Delta}\,(\mathrm{Cut}) \longrightarrow^3 \dfrac{\dfrac{\vdash G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash \Gamma, \Delta}\,(\mathrm{Cut})}{\vdash ?_{\mu} F, \Gamma, \Delta}\,(?\mathrm{w}^\bullet)$$

$$\dfrac{\dfrac{\vdash F, ?_{\mu} G, ?_{\mu} \Gamma}{\vdash !_{\nu} F, ?_{\mu} G, ?_{\mu} \Gamma}\,(!\mathrm{p}^\bullet) \qquad \dfrac{\vdash G, ?_{\mu} \Delta}{\vdash !_{\nu} G^\perp, ?_{\mu} \Delta}\,(!\mathrm{p}^\bullet)}{\vdash !_{\nu} F, ?_{\mu} \Gamma, ?_{\mu} \Delta}\,(\mathrm{Cut}) \longrightarrow^\omega \dfrac{\dfrac{\vdash F, ?_{\mu} G, ?_{\mu} \Gamma \qquad \dfrac{\vdash G, ?_{\mu} \Delta}{\vdash !_{\nu} G^\perp, ?_{\mu} \Delta}\,(!\mathrm{p}^\bullet)}{\vdash F, ?_{\mu} \Gamma, ?_{\mu} \Delta}\,(\mathrm{Cut})}{\vdash !_{\nu} F, ?_{\mu} \Gamma, ?_{\mu} \Delta}\,(!\mathrm{p}^\bullet)$$

2) Simulation of μLL$^\infty$ cut-elimination steps

Key-cut rules

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash F,\Gamma}}{\vdash ?_\mu F,\Gamma}\,(?\mathrm{d}^\bullet) \qquad \dfrac{\dfrac{\pi'}{\vdash F^\perp, ?_\mu\Delta}}{\vdash !_\nu F^\perp, ?_\mu \Delta}\,(!\mathrm{p}^\bullet)}{\vdash \Gamma, ?_\mu\Delta}\,(\mathrm{Cut}) \longrightarrow^{2} \dfrac{\dfrac{\pi}{\vdash F,\Gamma} \qquad \dfrac{\pi'}{\vdash F^\perp,?_\mu\Delta}}{\vdash \Gamma,?_\mu\Delta}\,(\mathrm{Cut})$$

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash ?_\mu F, ?_\mu F,\Gamma}}{\vdash ?_\mu F,\Gamma}\,(?\mathrm{c}^\bullet) \qquad \dfrac{\dfrac{\pi'}{\vdash F^\perp, ?_\mu\Delta}}{\vdash !_\nu F^\perp, ?_\mu \Delta}\,(!\mathrm{p}^\bullet)}{\vdash \Gamma, ?_\mu\Delta}\,(\mathrm{Cut}) \longrightarrow^{4\times(\#\Delta=1)} \dfrac{\dfrac{\dfrac{\pi}{\vdash ?_\mu F, ?_\mu F,\Gamma} \qquad \dfrac{\pi'}{\vdash !_\nu F^\perp,?_\mu\Delta} \qquad \dfrac{\pi'}{\vdash !_\nu F^\perp,?_\mu\Delta}}{\vdash \Gamma,?_\mu\Delta,?_\mu\Delta}\,(\mathrm{mcut})}{\vdash \Gamma,?_\mu\Delta}\,(?\mathrm{c}^\bullet)^\star$$

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash \Gamma}}{\vdash ?_\mu F,\Gamma}\,(?\mathrm{w}^\bullet) \qquad \dfrac{\dfrac{\pi'}{\vdash F^\perp, ?_\mu\Delta}}{\vdash !_\nu F^\perp, ?_\mu \Delta}\,(!\mathrm{p}^\bullet)}{\vdash \Gamma, ?_\mu\Delta}\,(\mathrm{Cut}) \longrightarrow^{3int,3\times\#\Delta ext} \dfrac{\dfrac{\pi}{\vdash \Gamma}}{\vdash \Gamma,?_\mu\Delta}\,(?\mathrm{w}^\bullet)^\star$$

3) Cut-elimination for μLL$^\infty$

1. Let $\sigma = (\pi_i)_{i\in\omega}$ be a fair μLL$^\infty$ cut-reduction seq. from π.

3) Cut-elimination for $\mu\mathsf{LL}^\infty$

1. Let $\sigma = (\pi_i)_{i\in\omega}$ be a fair $\mu\mathsf{LL}^\infty$ cut-reduction seq. from π.
2. σ converges to a cut-free $\mu\mathsf{LL}^\infty$ pre-proof π'. *By contradiction:* Otherwise, some suffix τ of σ contains only key-cut steps and $\tau^\bullet$ would be a fair $\mu\mathsf{MALL}^\infty$ mcut-reduction, contradicting $\mu\mathsf{MALL}^\infty$ cut-elim thm.

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   Therefore, by μMALL$^\infty$ cut-elimination theorem:
   ρ' has a limit, $\pi'^\bullet$, which is a valid cut-free μMALL$^\infty$ proof.

3) Cut-elimination for μLL^∞

1. Let $\sigma = (\pi_i)_{i\in\omega}$ be a fair μLL^∞ cut-reduction seq. from π.
2. σ converges to a cut-free μLL^∞ pre-proof π'. *By contradiction:* Otherwise, some suffix τ of σ contains only key-cut steps and $\tau^\bullet$ would be a fair $\mu MALL^\infty$ mcut-reduction, contradicting $\mu MALL^\infty$ cut-elim thm.
3. As σ is productive, it strongly converges to π'.
   $\sigma^\bullet$ is a transfinite sequence from $\pi^\bullet$ strongly converging to $\pi'^\bullet$.
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5. Fairness of σ transfers (almost) to ρ: ρ can be turned into a fair $\mu MALL^\infty$ cut-reduction sequence ρ'.
   Therefore, by $\mu MALL^\infty$ cut-elimination theorem:
   ρ' has a limit, $\pi'^\bullet$, which is a valid cut-free $\mu MALL^\infty$ proof.
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3) Cut-elimination for μLL^∞

1. Let $\sigma = (\pi_i)_{i\in\omega}$ be a fair μLL^∞ cut-reduction seq. from π.
2. σ converges to a cut-free μLL^∞ pre-proof π'. *By contradiction:* Otherwise, some suffix τ of σ contains only key-cut steps and $\tau^\bullet$ would be a fair $\mu MALL^\infty$ mcut-reduction, contradicting $\mu MALL^\infty$ cut-elim thm.
3. As σ is productive, it strongly converges to π'. $\sigma^\bullet$ is a transfinite sequence from $\pi^\bullet$ strongly converging to $\pi'^\bullet$.
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5. Fairness of σ transfers (almost) to ρ: ρ can be turned into a fair $\mu MALL^\infty$ cut-reduction sequence ρ'. Therefore, by $\mu MALL^\infty$ cut-elimination theorem: ρ' has a limit, $\pi'^\bullet$, which is a valid cut-free $\mu MALL^\infty$ proof.
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Corollary

Cut-elimination also holds for two-sided μLL^∞ and μILL^∞.

Cut-elimination for μLK$^\infty$, μLJ$^\infty$

The usual linear embeddings of LJ and LK into ILL (intuitionnistic LL) and LL can be adapted to μLJ$^\infty$ and μLK$^\infty$ and μLK$^\infty$ by adding exponentials in the translation of fixed-points in the natural way.

Theorem

If π is an μLK$^\infty$ (resp. μLJ$^\infty$) proof of $\vdash \Gamma$ (resp. $\Gamma \vdash F$), there exists a μLL$^\infty$ (resp. μILL$^\infty$) proof of the translated sequents.

Moreover, by erasing the exponentials (connectives and inferences) one obtains the skeleton of a μLL$^\infty$/ μILL$^\infty$ proof which is a μLK$^\infty$/ μLJ$^\infty$ proof, preserving validity. The skeleton of a μLL$^\infty$/ μILL$^\infty$ cut-reduction sequence is a μLK$^\infty$/ μLJ$^\infty$ cut-reduction sequence.
As a result, one has:

Theorem (Productive cut-elimination for μLK$^\infty$ and μLJ$^\infty$)

For any μLK$^\infty$ (resp. μLJ$^\infty$) proof, there are productive cut-reduction seq. producing cut-free μLK$^\infty$ (resp. μLJ$^\infty$) proofs of the same sequent.

About Seely isomorphisms $\quad !A \otimes !B \dashv\vdash !(A \& B)$

$$\pi_{\otimes\vdash\&} = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\ (\mathrm{Ax})}{\vdash ?A^\perp, A}\ (?\mathrm{d})}{\vdash ?A^\perp, ?B^\perp, A}\ (?\mathrm{w}) \qquad \dfrac{\dfrac{\dfrac{}{\vdash B^\perp, B}\ (\mathrm{Ax})}{\vdash ?B^\perp, B}\ (?\mathrm{d})}{\vdash ?A^\perp, ?B^\perp, B}\ (?\mathrm{w})}{\vdash ?A^\perp, ?B^\perp, A \& B}\ (\&)}{\vdash ?A^\perp, ?B^\perp, !(A \& B)}\ (!\mathrm{p})}{\vdash ?A^\perp \parr ?B^\perp, !(A \& B)}\ (\parr)$$

$$\pi_{\&\vdash\otimes} = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\ (\mathrm{Ax})}{\vdash A^\perp \oplus B^\perp, A}\ (\oplus_1)}{\vdash ?(A^\perp \oplus B^\perp), A}\ (?\mathrm{d})}{\vdash ?(A^\perp \oplus B^\perp), !A}\ (!\mathrm{p}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B^\perp, B}\ (\mathrm{Ax})}{\vdash A^\perp \oplus B^\perp, B}\ (\oplus_2)}{\vdash ?(A^\perp \oplus B^\perp), B}\ (?\mathrm{d})}{\vdash ?(A^\perp \oplus B^\perp), !B}\ (!\mathrm{p})}{\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !A \otimes !B}\ (\otimes)}{\vdash ?(A^\perp \oplus B^\perp), !A \otimes !B}\ (?\mathrm{c})$$

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$$\dfrac{(\pi_{\otimes\vdash\&})^\bullet \qquad (\pi_{\&\vdash\otimes})^\bullet}{\vdash ?_\mu A^\perp \parr ?_\mu B^\perp, !_\nu A \otimes !_\nu B}\,(\mathrm{Cut}) \quad \to^{\omega}_{cut} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(\mathrm{Ax})}{\vdash ?_\mu A^\perp, A}\,(?\mathrm{d}^\bullet)}{\vdash ?_\mu A^\perp, ?_\mu B^\perp, A}\,(?\mathrm{w}^\bullet)}{\vdash ?_\mu A^\perp, ?_\mu B^\perp, !_\nu A}\,(!\mathrm{p}^\bullet) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B^\perp, B}\,(\mathrm{Ax})}{\vdash ?_\mu B^\perp, B}\,(?\mathrm{d}^\bullet)}{\vdash ?_\mu A^\perp, ?_\mu B^\perp, B}\,(?\mathrm{w}^\bullet)}{\vdash ?_\mu A^\perp, ?_\mu B^\perp, !_\nu B}\,(!\mathrm{p}^\bullet)}{\vdash ?_\mu A^\perp, ?_\mu B^\perp, ?_\mu A^\perp, ?_\mu B^\perp, !_\nu A \otimes !_\nu B}\,(\otimes)}{\vdash ?_\mu A^\perp, ?_\mu B^\perp, !_\nu A \otimes !_\nu B}\,(?\mathrm{c}^\bullet)^2}{\vdash ?_\mu A^\perp \parr ?_\mu B^\perp, !_\nu A \otimes !_\nu B}\,(\parr)$$

The left occurrences of A, B require two unfoldings of the fixed-point, while the right occurrences of A, B require only one unfolding of the fixed-point.

The fixed-point unfolding structure tracks the history of the structural rules.

This witnesses the existence of a non-uniform exponential in the encoding of !.

Conclusion

To sum up:

- Fixed-point logics with circular or non-wellfounded proofs equipped with a parity condition to discriminate valid from invalid proofs;
- Syntactic cut-elimination for various non well-founded calculi: $\mu\mathsf{MALL}^\infty$, $\mu\mathsf{LL}^\infty$, $\mu\mathsf{LJ}^\infty$, $\mu\mathsf{LK}^\infty$;
- Using an fixed-point encoding of LL exponentials to deduce cut-elimination for $\mu\mathsf{LL}^\infty$ from that of $\mu\mathsf{MALL}^\infty$;
- Application to $\mu\mathsf{LK}^\infty$ & $\mu\mathsf{LJ}^\infty$ using standard tools from LL proof-theory.

Ongoing and future work:

- $\mu\mathsf{MALL}^\infty$ cut-elimination proof is used as a black-box: potential to apply the same method for other validity conditions;
- currently working at relaxing bouncing validity; (jww Bauer)
- the encoded exponential have very odd properties (loss of Seely iso, non-uniformity): explore the potentiality of this non-uniformity.

Announcements:

- We are looking for post-docs, to be funded by ANR RECIPROG...
- FICS workshop (Fixed-points in CS) to be held as a satellite of CSL 2024 next february in Naples: call for contributions published soon !

Questions?