TLLA 2023 Tutorial:
A tour of the non-wellfounded, linear proof-theory of the μ-calculus: on the virtues of circular reasoning

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Infinite descent and Circular proofs

An old mathematical story, in Euclid's *Elements* (Book VII)

PROPOSITION 31

Any composite number is measured by some prime number.

Let A be a composite number;

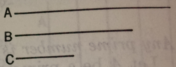
I say that A is measured by some prime number.

For, since A is composite,

some number will measure it.

Let a number measure it, and let it be B.

Now, if B is prime, what was enjoined will have been done.



But if it is composite, some number will measure it.

Let a number measure it, and let it be C.

Then, since C measures B,

and B measures A,

therefore C also measures A.

And, if C is prime, what was enjoined will have been done.

But if it is composite, some number will measure it.

Thus, if the investigation be continued in this way, some prime number will be found which will measure the number before it, which will also measure A.

For, if it is not found, an infinite series of numbers will measure the number A, each of which is less than the other:

which is impossible in numbers.

Therefore some prime number will be found which will measure the one before it, which will also measure A.

Therefore any composite number is measured by some prime number.

Q. E. D.

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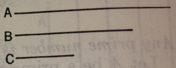

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Root of Fermat's **infinite descent** proof method.

Infinite descent and Circular proofs

An old mathematical story: Fermat identifies a powerful heuristics

Pierre de Fermat studied in depth infinite descent and used it extensively. See letter of August 1659 to Carcavaci where Fermat listed 10 theorems he "proved" using infinite descent:

1. *Aucun nombre de la forme, moindre de l'unité qu'un multiple de 3, ne peut être composé d'un carré et du triple d'un autre carré.*
2. *Aucun triangle rectangle en nombres n'a une aire carrée.*
3. *Tout nombre premier qui surpasse de l'unité un multiple de 4 est somme de deux carrés.*

(...)

9. *Toutes les puissances carrées de 2, augmentées de 1, sont des nombres premiers.*
10. *Il n'y a que 1 et 7 qui sont moindres de 1 qu'un double carré et aient un carré de même nature.*

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... even for proving wrong statements! Property 9 asserts that every Fermat number is prime, which was later disproved by Euler who factorized $F_5 = 2^{2^5} + 1$ as $641 \times 6,700,417$.

Infinite descent and Circular proofs

For any integer m, $\sqrt{m}$ is either an integer, or irrational.

Proof

Let $m \in \mathbb{N}$ and for the sake of contradiction, assume $\sqrt{m} \in \mathbb{Q} \setminus \mathbb{N}$.

1. Choose $a_0, b_0, q \in \mathbb{N}$ st. $\sqrt{m} = a_0 / b_0$ and $q < \sqrt{m} < q + 1$.
2. One has $b_0 \sqrt{m} = a_0 \in \mathbb{N}$ and $a_0 \sqrt{m} = m b_0 \in \mathbb{N}$.
3. Therefore by setting $a_1 \triangleq b_0 m - a_0 q = a_0(\sqrt{m} - q)$ and $b_1 \triangleq a_0 - b_0 q = b_0(\sqrt{m} - q)$, we have:
   - a_1, b_1 are integers,
   - $0 < a_1 < a_0$, $\quad 0 < b_1 < b_0$ and
   - $\sqrt{m} = a_1 / b_1$.
4. In a similar way, one can build $(a_i)_{i \in \mathbb{N}}$ and $(b_i)_{i \in \mathbb{N}}$ infinite sequences of integers, *"each of which is less than the other"*. This is impossible.
5. Therefore $\sqrt{m}$ is either integer or irrational. □

Infinite descent and Circular proofs

Towards sequent calculus: Irrationality of $\sqrt{2}$

$$
\dfrac{
\dfrac{}{\begin{array}{c} 0 < x_0, x_0^2 = 2x_1^2 \vdash \\ 0 < x_1 < x_0 \wedge \\ \exists x_2 . x_0 = 2x_2 \end{array}}
\qquad
\dfrac{
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\qquad
\dfrac{
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}{\underline{x_3 < x_2}, 0 < x_3, 4x_4^2 = 2x_3^2 \vdash}
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}{0 < x_0, x_0^2 = 2x_1^2 \vdash}
$$

Infinite descent and Circular proofs

Inductive and coinductive cases

Inductive case:

$$\dfrac{\dfrac{}{\vdash nat\ 0} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdots}}{even\ y \vdash nat\ y}}{even\ y \vdash nat\ (s\ y)}}{even\ y \vdash nat\ (s\ (s\ y))}}{even\ x \vdash nat\ x}$$

The infinite branch unfolds the inductive predicate *even* infinitely often on the left: valid!

Infinite descent and Circular proofs

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Coinductive case:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{step\ p\ \alpha\ q \vdash step\ p\ \alpha\ q} \qquad \dfrac{\iddots}{\vdash sim\ q\ q}}{step\ p\ \alpha\ q \vdash step\ p\ \alpha\ q \wedge sim\ q\ q}}{\vdash \forall\alpha\forall q.\ step\ p\ \alpha\ q \supset \exists q'.\ step\ p\ \alpha\ q' \wedge sim\ q\ q'}}{\vdash sim\ p\ p}$$

The infinite branch unfolds the coinductive predicate *sim* infinitely often on the right: valid!

Mixing inductive and coinductive definitions

A matter of priority

$$p \triangleq_{ind} q \qquad\qquad q \triangleq_{coind} p \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash p}}{\vdash q}}{\vdash p} \qquad \dfrac{\dfrac{\dfrac{\vdots}{p \vdash}}{q \vdash}}{p \vdash}}{\vdash}$$

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$$p \triangleq_{ind} q \qquad\qquad q \triangleq_{coind} p$$

Choose *which matters most* between p and q:

	$p < q$	$q < p$
p	$\mu X.\nu Y.X$	$\mu X.q$
q	$\nu Y.p$	$\nu Y.\mu X.Y$

Circular LL ?

In which case is it safe to allow infinite branches in a LL proof?

1. Applying infinitely many MALL rules?
2. Applying infinitely many cut rules?
3. Applying infinitely many structural rules?

Or...

Circular LL ?

In which case is it safe to allow infinite branches in a LL proof?

1. Applying infinitely many MALL rules?
   No deductive progress
2. Applying infinitely many cut rules?
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   Or...

Impossible! They strictly reduce the size of the sequent.
The length of a (cut-free) branch is bounded by the size of the conclusion sequent.

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What the Tortoise said to Achilles (1895, L. Carroll), revisited by J-YG:

Achilles' goal: proving $A \multimap B, A \vdash B$
The Tortoise rejects ($\multimap_L$) but accepts all the $T_i, i \geq 2$:

$$
\begin{array}{rcl}
T_0 & \triangleq & A \\
T_{k+1} & \triangleq & (\bigotimes_{i=0}^{k} T_i) \multimap B
\end{array}
$$

$$
\dfrac{\vdash T_2 \qquad \dfrac{\vdash T_3 \qquad \dfrac{\vdash T_4 \qquad \dfrac{\vdash T_5 \qquad \dfrac{\vdots}{T_0, \ldots, T_5 \vdash B}}{T_0, \ldots, T_4 \vdash B}\,(\mathrm{Cut})}{T_0, \ldots, T_3 \vdash B}\,(\mathrm{Cut})}{T_0, T_1, T_2 \vdash B}\,(\mathrm{Cut})}{T_0, T_1 \vdash B}\,(\mathrm{Cut})
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Infinite structural trees:

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash ?F, ?F, ?F}}{\vdash ?F, ?F}\ (?c)}{\vdash ?F}\ (?c)$$

$$\dfrac{\dfrac{\vdash ?F}{\vdash ?F, ?F}\ (?w)}{\vdash ?F}\ (?c)$$

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A promotion must react to any (finite) structural tree.

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$$\longrightarrow_{cut} \quad \frac{\dfrac{\dfrac{\dfrac{\pi}{\vdash \Gamma, ?F^{\perp}, ?F^{\perp}} \quad \dfrac{\pi_2}{\vdash !F, ?\Delta}}{\vdash \Gamma, ?\Delta, ?F^{\perp}}\;(\mathrm{Cut}) \qquad \dfrac{\pi_3}{\vdash !F, ?\Delta}}{\vdash \Gamma, ?\Delta, ?\Delta}\;(\mathrm{Cut})}{\vdash \Gamma, ?\Delta}\;(?\mathrm{c})^{\star}$$

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Usual promotion is derivable as:

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Condition to admit a nwf branch:
a !-*formula occurrence* must be principal infinitely often along the branch.

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5. μLL^{∞}

μLL^{∞}:

In the following, we will extend the language of LL formulas to admit sort of "infinite" formulas, defined by fixed-point constructions:

$$\mu X.F, \nu X.F.$$

In some cases (use of an inductive hypothesis, production of a coinductive conclusion), one can allow non-wellfounded branches.

Circular & non-wellfounded proofs in the litterature

- **As verification device**: Complete deduction sytem giving algorithms for checking validity (Tableaux, sequent calculi),

$$\mu\text{-calculus formula} \quad \rightarrow \quad \text{Proof search} \begin{array}{l} \nearrow \ \text{Success} \quad \rightarrow \quad \text{Validity} \\ \searrow \ \text{Failure} \quad \rightarrow \quad \text{Invalidity} \end{array}$$

- **Completeness arguments**: Intermediate objects between syntax and semantics for modal μ-calculus (Kozen, Kaivola, Walukiewicz)

$$\mu\text{-calulus formula} \quad \rightarrow \quad \text{Circular proof} \quad \rightarrow \quad \text{Finite axiomatization}$$

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μ-calulus formula $\to$ Circular proof $\to$ Finite axiomatization

- **But rarely as proof–program objects in themselves**:
  - pioneering works by Santocanale; Studer; Brotherston & Simpson; Dax, Hoffman & Lange.
  - develop such a proof-theoretical study, from a Curry-Howard perspective: study the dynamics of cut-elimination.

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  - develop such a proof-theoretical study, from a Curry-Howard perspective: study the dynamics of cut-elimination.
- Recently, development of numerous circular/cyclic proof systems (Afshari & Leigh, Das, Doumane & Pous, Cohen & Rowe, Tatsuta et al. etc.)

Proof theory of fixed-point logics

- Various deductive frameworks for (co)inductive reasoning (Martin-Löf's inductive definitions, μ-calculi, ...), suitable to represent and reason about (co)inductive data structures.
- Structural proof-theory, Curry-Howard-oriented: not only to express statements and their provability relation, but stressing the proof objects themselves, in particular in the substructural setting.
- LL *with fixed points*, considered with proofs as finite trees (μLL) or proofs as infinite, non-wellfounded trees ($\mu\mathrm{LL}^{\infty}$) with a special fragment of interest, circular proofs.

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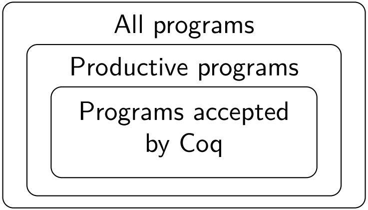
  - Can we extend the *proof–program correspondence* to circular proofs?
  - E.g., in Coq proof assistant, syntactic productivity conditions are required: assert progress after every step. Many productive programs are rejected by Coq type-checker.



Some valid and invalid definitions

```
CoInductive stream := Cons : (nat * stream) → stream.
CoInductive bstream := BCons : (bool * bstream) → bstream.

Definition hdinc (s: stream) : stream := match s with
  | Cons (a, s') ⇒ Cons (S a, s') end.
CoFixpoint enum (n:nat) : stream := Cons (n, (enum (S n))).
CoFixpoint drop (s : stream) : stream := match s with
  | Cons (a, Cons (b, s')) ⇒ Cons (b, (drop s')) end.
CoFixpoint incdrop (s : stream) : stream := match s with
  | Cons (a, Cons (b, s')) ⇒ hdinc (Cons (b, incdrop s')) end.

Definition hdneg (s: bstream) : bstream := match s with
  | BCons (a, s') ⇒ BCons (negb a, s') end.
CoFixpoint filter1everyk (m : nat) (s : bstream) :
  bstream := match (m,s) with
  | (0, BCons (a, s')) ⇒ BCons (a, filter1everyk k s')
  | (S m', BCons (a, s')) ⇒ hdneg (filter1everyk m' s') end.
```

Aim of this talk

- Our goal: investigate productivity conditions which are proof-theoretically grounded, by considering circular and non-wellfounded linear proofs in μ-calculi.
- Ideally, accept such proof objects:

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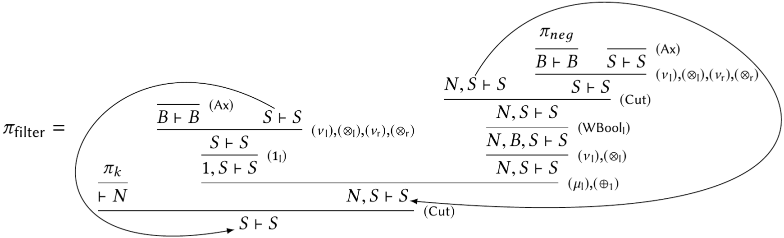

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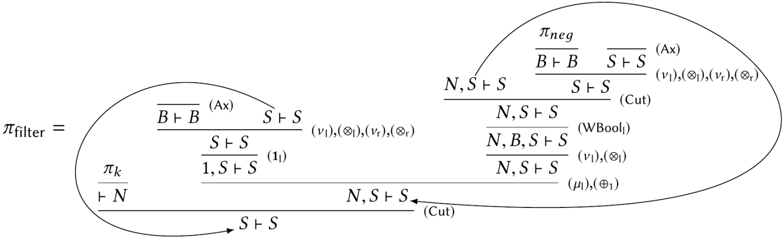

- Report on some progress in designing more flexible validity conditions for circular and non-wellfounded proofs in linear logic with fixed-points as well as some proof invariants.
- **Based on joint works with Baelde, Bauer, Chardonnet, Das, De, Doumane, Ehrhard, Jaber, Jafarrahmani, Kuperberg, Nollet, Pellissier and Tasson.**

μLL formulas and sequent calculus

(Baelde & Miller 2007, Baelde 2012)

μLL formulas

LL formula grammar extended with fixed-point constructs:

$$F ::= \ \dots \ \mid X \mid \mu X.F \mid \nu X.F$$

- μ and ν are binders, consider closed formulas only.
- μ and ν are dual. <u>Ex:</u> $(\nu X.X \otimes X)^\perp = \mu X.X \parr X$.
- One-sided sequents: $\vdash A_1, \dots, A_n$. *($\Gamma \vdash \Delta$ is a short for $\vdash \Gamma^\perp, \Delta$)*

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- Data types encodings:

$$\begin{array}{rcl|rcl}
\mathsf{Nat} & \triangleq & \mu X.1 \oplus X & \mathsf{Nat}^{\perp} & = & \nu X.\perp \& X \\
\mathsf{List}(A) & \triangleq & \mu X.1 \oplus (A \otimes X) & \mathsf{List}(A)^{\perp} & = & \nu X.\perp \& (A^{\perp} \parr X) \\
\mathsf{Stream}(A) & \triangleq & \nu X.1 \& (A \otimes X) & \mathsf{Stream}(A)^{\perp} & = & \mu X.\perp \oplus (A^{\perp} \parr X)
\end{array}$$

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μLL Sequent Calculus

- LL inference rules together with
- Inference rules for μ and ν $\Rightarrow$ ***See following slides***

Knaster-Tarski fixed-point theorem

Let C be a complete lattice and F a monotonic operator on C.

Theorem

F has a **least** F.P. μF.

μF: **least prefixed**-point:
– $F(\mu F) \sqsubseteq \mu F$ and
– $\forall S, F(S) \sqsubseteq S \Rightarrow \mu F \sqsubseteq S$.

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F has a **greatest** F.P. νF.

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Proof by induction:
To prove that $\mu F \subseteq P$, it is sufficient to find some $S \subseteq P$ and to prove that $\forall x \in F(S),\ x \in S$.

Theorem

F has a **greatest** F.P. νF.

νF **greatest postfixed**-point:
– $\nu F \sqsubseteq F(\nu F)$ and
– $\forall S, S \sqsubseteq F(S) \Rightarrow S \sqsubseteq \nu F$.

Proof by coinduction:
To prove that $P \subseteq \nu F$, it is sufficient to find some $S \supseteq P$ and to prove that $\forall x \in S,\ x \in F(S)$.

Knaster-Tarski fixed-point theorem

Let C be a complete lattice and F a monotonic operator on C.

Theorem

F has a **least** F.P. μF.

μF: **least prefixed**-point:
– $F(\mu F) \sqsubseteq \mu F$ and
– $\forall S, F(S) \sqsubseteq S \Rightarrow \mu F \sqsubseteq S$.

Proof by induction:
To prove that $\mu F \subseteq P$, it is sufficient to find some $S \subseteq P$ and to prove that $\forall x \in F(S)$, $x \in S$.

$$\frac{H \vdash F[\mu X.F/X]}{H \vdash \mu X.F}\ (\mu_r) \qquad \frac{F[S/X] \vdash S}{\mu X.F \vdash S}\ (\mu_l)$$

Theorem

F has a **greatest** F.P. νF.

νF **greatest postfixed**-point:
– $\nu F \sqsubseteq F(\nu F)$ and
– $\forall S, S \sqsubseteq F(S) \Rightarrow S \sqsubseteq \nu F$.

Proof by coinduction:
To prove that $P \subseteq \nu F$, it is sufficient to find some $S \supseteq P$ and to prove that $\forall x \in S$, $x \in F(S)$.

$$\frac{F[\nu X.F/X] \vdash H}{\nu X.F \vdash H}\ (\nu_l) \qquad \frac{S \vdash F[S/X]}{S \vdash \nu X.F}\ (\nu_r)$$

Inferences for fixed-points

- One-sided version
- The inferences of the previous slides do not have cut-elimination:

$$
\dfrac{\dfrac{}{\vdash 0, 0, \top}\,(\top) \qquad \dfrac{\dfrac{}{\vdash 0, \top}\,(\top)}{\vdash 0, \nu X.X}}{\vdash 0, 0, \nu X.X}\,(\mathrm{Cut})
$$

- Consider branching ν-rule:

$$
\dfrac{\vdash \Gamma, S \qquad \vdash S^{\perp}, F[S/X]}{\vdash \Gamma, \nu X.F}\,(\nu)
$$

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$$\dfrac{\vdash \Gamma, S \qquad \vdash S^{\perp}, F[S/X], ?\Delta}{\vdash \Gamma, \nu X.F, ?\Delta}\ (\nu)$$

Inferences for fixed-points

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- Consider branching ν-rule:

$$\dfrac{\vdash \Gamma, S \qquad \vdash S^{\perp}, F[S/X]}{\vdash \Gamma, \nu X.F}\ (\nu)$$

- Cut-elimination holds in μMALL (Baelde, 2012).

The key step $(\mu) - (\nu)$ in μLL cut-elimination

Lemma: Functoriality

For any μLL pre-formula F with one free fixed-point variable,

$$\frac{\vdash ?\Delta, A, B}{\vdash ?\Delta, F^{\perp}[A/X], F[B/X]} \ (\mathfrak{F}_{\mathsf{F}}) \qquad \text{is cut-free derivable in } \mu\text{LL}.$$

(By induction on the maximal depth of free occurrences of X in F.)

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Key $(\mu)/(\nu)$ ***cut-reduction case (slightly simplified):***

$$\dfrac{\dfrac{\dfrac{\Pi}{\vdash \Gamma, N[\mu X.N/X]}}{\vdash \Gamma, \mu X.N}\,(\mu) \qquad \dfrac{\dfrac{\Theta}{\vdash ?\Delta, S, N^{\perp}[S^{\perp}/X]}}{\vdash ?\Delta, S, \nu X.(N^{\perp})}\,(\nu)}{\vdash \Gamma, ?\Delta, S}\,(\mathsf{Cut}) \qquad \longrightarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi}{\vdash \Gamma, N[\mu X.N/X]} \qquad \dfrac{\dfrac{\dfrac{\Theta}{\vdash ?\Delta, S, N^{\perp}[S^{\perp}/X]}}{\vdash ?\Delta, S, \nu X.(N^{\perp})}\,(\nu)}{\vdash ?\Delta, N[S/X], N^{\perp}[(\nu X.(N^{\perp}))/X]}\,(\mathfrak{F}_{\mathsf{N}})}{\vdash \Gamma, ?\Delta, N[S/X]}\,(\mathsf{Cut}) \qquad \dfrac{\Theta}{\vdash ?\Delta, S, N^{\perp}[S^{\perp}/X]}}{\vdash \Gamma, ?\Delta, ?\Delta, S}\,(\mathsf{Cut})}{\vdash \Gamma, ?\Delta, S}\,(?\mathsf{c})^{\star}$$

Denotational semantics of μLL

(Ehrhard & Jafarrahmani 2021, Jafarrahmani 2023)

Ehrhard and Jafarrahmani provide:

- A notion of categorical models of μLL as pairs $(\mathscr{L}, \overrightarrow{\mathscr{L}})$ such that $\mathscr{L}$ is a Seely category and $\overrightarrow{\mathscr{L}} = (\mathscr{L}_n)_{n\in\mathbb{N}}$ with $\mathscr{L}_n$ a class of strong functors $\mathscr{L}^n \to \mathscr{L}$, and $\mathscr{L}_0 = \mathrm{Obj}(\mathscr{L})$ which is closed by composition and s.t.:
  - (i) $\otimes, \& \in \mathscr{L}_2$, (ii) $!\_ \in \mathscr{L}_1$ and (iii) if $\mathbb{X} \in \mathscr{L}_n$ then $\mathbb{X}^\perp \in \mathscr{L}_n$
  - all (unary) strong functors of $\mathscr{L}$ induce final coalgebras and $\overrightarrow{\mathscr{L}}$ is closed by taking those final coalgebras (for any $\mathbb{X} \in \mathscr{L}_{k+1}$, $\nu\mathbb{X} \in \mathscr{L}_k$).
- Concrete models based on REL or COH and totality predicates.

Totality candidates on a set E

Given $\mathscr{T} \subseteq \mathscr{P}(E)$ we set

$$\mathscr{T}^{\perp} = \left\{u' \subseteq E \mid \forall u \in \mathscr{T} \;\; u \cap u' \neq \varnothing\right\}$$

Definition (Totality candidates)

$\mathscr{T}$ is a *totality candidate* for E if $\mathscr{T} = \mathscr{T}^{\perp\perp}$.

Fact

- *$\mathscr{T}$ is a totality candidate on E iff $\mathscr{T} \subseteq \mathscr{P}(E)$ and $\mathscr{T} = \uparrow\mathscr{T}$.*
- $\mathrm{Tot}(X)$ *(The set of all totality candidates on E), ordered with $\subseteq$, is a complete lattice (it is closed under arbitrary intersections).*

Non-uniform totality spaces (NUTS)

A NUTS is a pair $X = (|X|, \mathscr{T}X)$ where

- $|X|$ is a set
- $\mathscr{T}X$ is a totality candidate on $|X|$, that is, a $\uparrow$-closed subset of $\mathscr{P}(|X|)$.

$t \in \mathbf{NUTS}(X, Y)$ if:

- $t \in \mathbf{REL}(|X|, |Y|)$ and
- $\forall u \in \mathscr{T}X \quad t \cdot u \in \mathscr{T}Y$.

Fact

NUTS *is a model of LL where the proofs are interpreted exactly as in* **REL**.

Interpretation of $\mu X.F$ in **NUTS**

$$\begin{tikzcd}
\mathbf{NUTS} \arrow[r, "\overline{F}"] \arrow[d] & \mathbf{NUTS} \arrow[d] \\
\mathbf{REL} \arrow[r, "F"] & \mathbf{REL}
\end{tikzcd}$$

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$\overline{F} : (X, U) \mapsto (FX, \Phi U)$ where $\Phi U \in \mathscr{T}(FX)$.

Interpretation of $\mu X.F$ in **NUTS**

$$\begin{array}{ccc} \mathbf{NUTS} & \xrightarrow{\overline{F}} & \mathbf{NUTS} \\ \downarrow & & \downarrow \\ \mathbf{REL} & \xrightarrow{F} & \mathbf{REL} \end{array}$$

$\overline{F} : (X, U) \mapsto (FX, \Phi U)$ where $\Phi U \in \mathscr{T}(FX)$. Assume μF exists.

$$\varphi : \mathrm{Tot}(\mu F) \to \mathrm{Tot}(\mu F)$$

$$R \mapsto \Phi R$$

Interpretation of $\mu X.F$ in **NUTS**

$$\begin{tikzcd}
\mathbf{NUTS} \arrow[r, "\overline{F}"] \arrow[d] & \mathbf{NUTS} \arrow[d] \\
\mathbf{REL} \arrow[r, "F"] & \mathbf{REL}
\end{tikzcd}$$

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$$\varphi : \mathrm{Tot}(\mu F) \to \mathrm{Tot}(\mu F)$$

$$R \mapsto \Phi R$$

By Knaster-Tarski theorem, φ has a least fixed-point: $\mu\varphi$ exists.

$$\mu\overline{F} = (\mu F, \mu\varphi).$$

Proposition

NUTS *is a model of* μLL.

Non-Wellfounded Sequent Calculus

Consider your favourite logic $\mathscr{L}$ & add fixed-points as in the μ-calculus

Pre-proofs are the trees **coinductively** generated by:

- $\mathscr{L}$ inference rules
- inference for μ, ν:

$$\frac{\Gamma, F[\mu X.F/X] \vdash \Delta}{\Gamma, \mu X.F \vdash \Delta}\ (\mu_l) \qquad \frac{\Gamma, F[\nu X.F/X] \vdash \Delta}{\Gamma, \nu X.F \vdash \Delta}\ (\nu_l)$$

$$\frac{\Gamma \vdash F[\mu X.F/X], \Delta}{\Gamma \vdash \mu X.F, \Delta}\ (\mu_r) \qquad \frac{\Gamma \vdash F[\nu X.F/X], \Delta}{\Gamma \vdash \nu X.F, \Delta}\ (\nu_r)$$

Circular (pre-)proofs: the regular fragment of infinite (pre-)proofs, ie finitely many sub-(pre)proofs.

Pre-proofs are unsound!!
Need for a correctness criterion!

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X}\ (\mu)}{\vdash \mu X.X}\ (\mu) \qquad \dfrac{\dfrac{\vdots}{\vdash \nu X.X, F}\ (\nu)}{\vdash \nu X.X, F}\ (\nu)}{\vdash F}\ (cut)$$

μLL$^\infty$ Non-Wellfounded Sequent Calculus

Consider your favourite logic LL & add fixed-points as in the μ-calculus

μLL$^\infty$ **Pre-proofs** are the trees **coinductively** generated by:

- LL inference rules
- inference for μ, ν: $\dfrac{\vdash F[\mu X.F/X], \Delta}{\vdash \mu X.F, \Delta}\ (\mu_r) \qquad \dfrac{\vdash F[\nu X.F/X], \Delta}{\vdash \nu X.F, \Delta}\ (\nu_r)$

Circular (pre-)proofs: the regular fragment of infinite (pre-)proofs, ie finitely many sub-(pre)proofs. μLL$^\omega$

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One-sided sequents as lists: $\vdash A_1, \ldots, A_n$. *($\Gamma \vdash \Delta$ is a short for $\vdash \Gamma^\perp, \Delta$)*

μ **and** ν **are dual binders**. Ex: $(\nu X.X \otimes X)^\perp = \mu X.X ⅋ X$.

μLL$^\infty$ Inferences

μLL$^\infty$ Inference Rules

$$\frac{}{\vdash F, F^\perp}\ (ax) \quad \frac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta}\ (cut) \quad \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ (ex)$$

$$\frac{\vdash F, \Gamma}{\vdash ?F, \Gamma}\ (?) \quad \frac{\vdash \Gamma}{\vdash ?F, \Gamma}\ (w) \quad \frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma}\ (c) \quad \frac{\vdash F, ?\Gamma}{\vdash !F, ?\Gamma}\ (!)$$

$$\frac{\vdash \Gamma}{\vdash \perp, \Gamma}\ (\perp) \quad \frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ (\parr) \quad \frac{\vdash F, \Gamma \qquad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ (\otimes) \quad \frac{}{\vdash \mathbf{1}}\ (\mathbf{1})$$

$$\frac{}{\vdash \top, \Gamma}\ (\top) \quad \frac{\vdash F, \Gamma \qquad \vdash G, \Gamma}{\vdash F \,\&\, G, \Gamma}\ (\&) \quad \frac{\vdash A_i, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ (\oplus_i) \quad \text{(no rule for 0)}$$

$$\frac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma}\ (\nu) \quad \frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\ (\mu)$$

μLL$^\infty$ Inferences

μLL$^\infty$ Inference Rules (with ancestor relation)

$$\frac{}{\vdash F, F^\perp}\ (ax) \qquad \frac{\vdash \Gamma, F \quad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta}\ (cut) \qquad \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ (ex)$$

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$$\frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ (\parr) \qquad \frac{\vdash F, \Gamma \quad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ (\otimes) \qquad \frac{}{\vdash \mathbf{1}}\ (\mathbf{1})$$

$$\frac{}{\vdash \top, \Gamma}\ (\top) \qquad \frac{\vdash F, \Gamma \quad \vdash G, \Gamma}{\vdash F \,\&\, G, \Gamma}\ (\&) \qquad \frac{\vdash A_i, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ (\oplus_i) \qquad \text{(no rule for 0)}$$

$$\frac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma}\ (\nu) \qquad \frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\ (\mu)$$

How to distinguish valid nwf proofs from invalid ones?

Validity condition: Reachable formulas

Example: $F = \nu X.((a \parr a^{\perp}) \otimes (!X \otimes \mu Y.X))$

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Validity condition: Reachable formulas

Example: $F = \nu X.((a \parr a^{\perp}) \otimes (!X \otimes \mu Y.X))$

$$FL(F) = \left\{ F\ ,\ (a \parr a^{\perp}) \otimes (!F \otimes \mu Y.F),\ \begin{matrix} a \parr a^{\perp} & , & \begin{matrix} a \\ a^{\perp} \end{matrix} \\ !F \otimes \mu Y.F, & & \mu Y.F \\ !F & & \end{matrix} \right\}$$

Validity condition: Reachable formulas

Example: $F = \nu X.((a \parr a^\perp) \otimes (!X \otimes \mu Y.X))$

$$FL(F) = \begin{tikzcd} & & a \parr a^\perp \arrow[r] \arrow[dr] & a \\ F \arrow[r] & (a \parr a^\perp) \otimes (!F \otimes \mu Y.F) \arrow[ur] \arrow[dr] & & a^\perp \\ & & !F \otimes \mu Y.F \arrow[r] \arrow[d] & \mu Y.F \arrow[ulll, bend left] \\ & & !F \arrow[uull, bend left] & \end{tikzcd}$$

Validity condition: Reachable formulas

Example: $F = \nu X.((a \parr a^{\perp}) \otimes (!X \otimes \mu Y.X))$

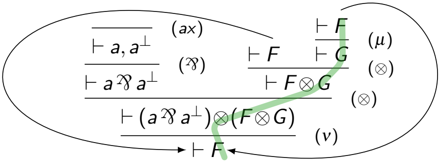
$$
\begin{tikzcd}
 & & a \parr a^{\perp} \arrow[r] \arrow[dr] & a \\
FL(F) = F \arrow[r] & (a \parr a^{\perp}) \otimes (!F \otimes \mu Y.F) \arrow[ur] \arrow[dr] & & a^{\perp} \\
 & & !F \otimes \mu Y.F \arrow[r] \arrow[dr] & \mu Y.F \arrow[ulll, bend left] \\
 & & & !F \arrow[uulll, bend left]
\end{tikzcd}
$$

$\mathrm{FL}(F)$ is the least set of formulas such that:

- $F \in \mathrm{FL}(F)$;
- $G_1 \star G_2 \in \mathrm{FL}(F) \Rightarrow G_1, G_2 \in \mathrm{FL}(F)$ for $\star \in \{\oplus, \&, \parr, \otimes\}$;
- $\sigma X.G \in \mathrm{FL}(F) \Rightarrow G[\sigma X.G/X] \in \mathrm{FL}(F)$ for $\sigma \in \{\mu, \nu\}$;
- $mG \in \mathrm{FL}(F) \Rightarrow G \in \mathrm{FL}(F)$ for $m \in \{!, ?\}$.

$\mathrm{FL}(F)$ is a finite set for any formula F.

$F = \nu X.((a \parr a^{\perp}) \otimes (X \otimes \mu Y.X))$.
$G = \mu Y.F$



A ***thread*** along an infinite branch $(\Gamma_i)_{i \in \omega}$ is an infinite sequence of formula occurrences $(F_i)_{i \geq k}$ such that for any $i \geq k$, $F_i \in \Gamma_i$ and F_{i+1} is an immediate ancestor of F_i.

Infinite threads, validity

$F = \nu X.((a \parr a^{\perp}) \otimes (X \otimes \mu Y.X))$.
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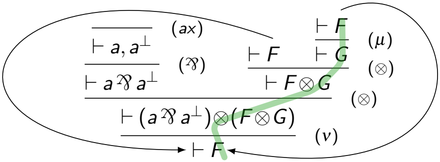

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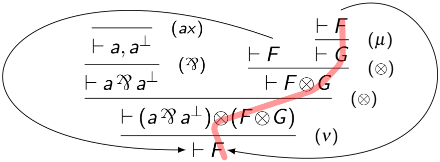
A thread is valid if it unfolds infinitely many ν. More precisely, if the ***minimal recurring*** principal formula of the thread is a ν-formula.

A proof is valid if every infinite branch contains a valid thread.

Infinite threads, validity

$F = \nu X.((a \parr a^{\perp}) \otimes (X \otimes G))$
$G = \mu Y.\nu X.((a \parr a^{\perp}) \otimes (X \otimes Y))$



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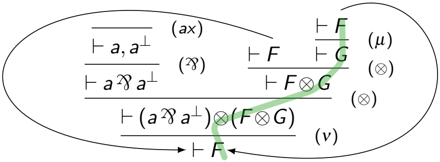

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Theorem (Baelde, Doumane & S, 2016)

μMALL$^{\infty}$ is sound, and admits cut-elimination.

Theorem (Doumane 2017 + Nollet, Tasson & S, 2019)

Validity of μLL$^{\omega}$ (circular) pre-proofs is decidable and PSPACE-complete.

Parity automata

Skip details

Definition

A *parity automaton* is a finite state word automaton,
whose states are ordered and given a parity bit ν/μ,
which accepts runs $(q_i)_{i\in\omega}$ such that $\min(\inf((q_i)_i))$ has parity ν.

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Remarks

- States are usually given a color in $\mathbb{N}$, equivalently.
- Only co-accessible states need to be ordered.

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Remarks

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- Only co-accessible states need to be ordered.

Properties

- PA can be determinized,
- PA are closed by complementation and intersection,
- The emptiness problem is decidable,
- (Thus) inclusion of parity automata is decidable.

Theorem: The validity of circular pre-proofs is decidable.

Proof.

Consider a pre-proof Π i.e. a graph with nodes $s_i = (F_i^j)_{j\in[1;n_i]}$.

The proof goes as follows:

- One builds a parity automaton recognizing the language $\mathscr{L}_B$ of infinite branches of Π;
- One builds a parity automaton recognizing the language $\mathscr{L}_T$ the valid branches of Π.
- Validity amounts to the inclusion of $\mathscr{L}_B$ in $\mathscr{L}_T$, that is showing that $\mathscr{L}_B \setminus \mathscr{L}_T = \varnothing$ which is decidable.

Branch automaton: Let $\mathscr{A}_B$ be the **branch automaton** with states s_i, transitions $s_i \to^k s_j$ when s_j is the k-th premise of s_i, and which accepts all runs.

(...)

Theorem: The validity of circular pre-proofs is decidable.

Proof.

Consider a pre-proof Π i.e. a graph with nodes $s_i = (F_i^j)_{j\in[1;n_i]}$.
(...)
Thread automaton: Let $\mathscr{A}_T$ be the **thread automaton** with states F_i^{j+}, F_i^{j-} or s_i, with transitions:

- $s_i \to^k s_p$ and $s_i \to^k F_p^{q-}$ when s_p is the k-th premise of s_i
- $F_i^{j+} \to^k F_p^{q\varepsilon} \quad (\varepsilon \in \{+,-\})$ when $s_i \to^k s_p$ and F_i^j is active in the rule of conclusion s_i and has ancestor F_p^q
- $F_i^{j-} \to^k F_p^{q\varepsilon} \quad (\varepsilon \in \{+,-\})$ when $s_i \to^k s_p$ and F_i^j is passive in the rule of conclusion s_i and has ancestor F_p^q

acceptance based on subformula ordering with the active/passive distinction: only active ν-formulas have coinductive parity.

Validity of Π equivalent to $\mathscr{L}(\mathscr{A}_B) \setminus \mathscr{L}(\mathscr{A}_T) = \varnothing$, thus decidable.

$\square$

Examples of circular proofs

$$\mathsf{N} = \mu X.1 \oplus X$$

$$\begin{array}{lcl} double & : & N \to N \\ double(0) & = & 0 \\ double(succ(m)) & = & succ(succ(double(m))) \end{array}$$

$$\pi_0 = \dfrac{\dfrac{\dfrac{}{\vdash 1}\,(\mathbf{1})}{\vdash 1 \oplus N}\,(\oplus_1)}{\vdash N}\,(\mu) \qquad \pi_{k+1} = \dfrac{\dfrac{\pi_k}{\vdash 1 \oplus N}\,(\oplus_2)}{\vdash N}\,(\mu) \qquad \pi_{succ} = \dfrac{\dfrac{\dfrac{}{N \vdash N}\,(ax)}{N \vdash 1 \oplus N}\,(\oplus_2)}{N \vdash N}\,(\mu)$$

$$\pi_{double} = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(\mathbf{1})}{\vdash 1 \oplus N}\,(\oplus_1)}{\vdash N}\,(\mu)}{1 \vdash N}\,(\bot) \qquad \dfrac{\dfrac{\dfrac{\dfrac{N \vdash N}{N \vdash 1 \oplus N}\,(\oplus_2)}{N \vdash N}\,(\mu)}{N \vdash 1 \oplus N}\,(\oplus_2)}{N \vdash N}\,(\mu)}{\dfrac{1 \oplus N \vdash N}{N \vdash N}\,(\nu)}\,(\&)$$

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$$\begin{array}{lcl} double & : & N \to N \\ double(0) & = & 0 \\ double(succ(m)) & = & succ(succ(double(m))) \end{array}$$

$$\pi_0 = \dfrac{\dfrac{\dfrac{}{\vdash 1}\ (\mathbf{1})}{\vdash 1 \oplus N}\ (\oplus_1)}{\vdash N}\ (\mu) \qquad \pi_{k+1} = \dfrac{\dfrac{\pi_k}{\vdash 1 \oplus N}\ (\oplus_2)}{\vdash N}\ (\mu) \qquad \pi_{succ} = \dfrac{\dfrac{\dfrac{}{N \vdash N}\ (ax)}{N \vdash 1 \oplus N}\ (\oplus_2)}{N \vdash N}\ (\mu)$$

$$\pi_{double} = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\ (\mathbf{1})}{\vdash 1 \oplus N}\ (\oplus_1)}{\vdash N}\ (\mu)}{1 \vdash N}\ (\perp) \qquad \dfrac{\dfrac{\dfrac{\dfrac{N \vdash N}{N \vdash 1 \oplus N}\ (\oplus_2)}{N \vdash N}\ (\mu)}{N \vdash 1 \oplus N}\ (\oplus_2)}{N \vdash N}\ (\mu)}{\dfrac{1 \oplus N \vdash N}{N \vdash N}\ (\nu)}\ (\&)$$

$$\dfrac{\pi_k \qquad \pi_{succ}}{\vdash N}\ (cut) \longrightarrow^\star \pi_{k+1} \qquad\qquad \dfrac{\pi_k \qquad \pi_{double}}{\vdash N}\ (cut) \longrightarrow^\star \pi_{2k}$$

Examples of circular proofs

$\mathsf{N} = \mu X.1 \oplus X$

$$\pi_{\text{dup}} \quad = \quad \dfrac{\dfrac{\dfrac{}{\vdash N_1}\,{\scriptstyle(\mu),(\oplus_1),(\mathbf{1})} \qquad \dfrac{}{\vdash N_2}\,{\scriptstyle(\mu),(\oplus_1),(\mathbf{1})}}{\mathbf{1} \vdash N_1 \otimes N_2}\,{\scriptstyle(\bot),(\otimes)} \qquad \dfrac{N' \vdash N_1' \otimes N_2' \qquad \dfrac{\pi_{\text{succ}} \qquad \pi_{\text{succ}}}{N_1' \otimes N_2' \vdash N_1 \otimes N_2}\,{\scriptstyle(\parr),(\otimes)}}{N' \vdash N_1 \otimes N_2}\,{\scriptstyle(\text{Cut})}}{N \vdash N_1 \otimes N_2}\,{\scriptstyle(\nu),(\&)}$$

$$\mathrm{WNat}(\pi) = \dfrac{\dfrac{\dfrac{\dfrac{\pi}{\vdash \Gamma}}{1 \vdash \Gamma}\,{\scriptstyle(\bot)} \qquad N \vdash \Gamma}{1 \oplus N \vdash \Gamma}\,(\&)}{N \vdash \Gamma}\,(\nu)$$

Examples of circular proofs

$\mathsf{N} = \mu X.1 \oplus X$

$$\pi_{\mathrm{dup}} \quad = \quad \dfrac{\dfrac{\dfrac{}{\vdash N_1}\,(\mu),(\oplus_1),(\mathbf{1}) \qquad \dfrac{}{\vdash N_2}\,(\mu),(\oplus_1),(\mathbf{1})}{\mathbf{1} \vdash N_1 \otimes N_2}\,(\bot),(\otimes) \qquad \dfrac{N' \vdash N_1' \otimes N_2' \qquad \dfrac{\pi_{\mathrm{succ}} \qquad \pi_{\mathrm{succ}}}{N_1' \otimes N_2' \vdash N_1 \otimes N_2}\,(\parr),(\otimes)}{N' \vdash N_1 \otimes N_2}\,(\mathrm{Cut})}{N \vdash N_1 \otimes N_2}\,(\nu),(\&)$$

$$\mathrm{WNat}(\pi) = \dfrac{\dfrac{\dfrac{\dfrac{\pi}{\vdash \Gamma}}{1 \vdash \Gamma}\,(\bot) \qquad N \vdash \Gamma}{1 \oplus N \vdash \Gamma}\,(\&)}{N \vdash \Gamma}\,(\nu)$$

$$\dfrac{\pi_k \qquad \pi_{\mathrm{dup}}}{\vdash N \otimes N}\,(cut) \quad \longrightarrow^{\star} \quad \dfrac{\pi_k \qquad \pi_k}{\vdash N \otimes N}\,(\otimes) \qquad\qquad \dfrac{\pi_k \qquad \mathrm{WNat}(\pi)}{\vdash \Gamma}\,(cut) \quad \longrightarrow^{\star} \quad \dfrac{\pi}{\vdash \Gamma}$$

Examples of circular proofs

$$\textcolor{red}{\mathsf{S}} = \nu X.(1 \,\&\, (N \otimes X))$$

$$\begin{array}{lcl} enum & : & Nat \rightarrow Stream \\ enum(n) & = & n :: enum(succ(n)) \end{array}$$

$$\pi_{enum} = \begin{prooftree}
\AxiomC{}
\RightLabel{$(\mathbf{1})$}
\UnaryInfC{$\vdash 1$}
\RightLabel{(w)}
\UnaryInfC{$!N \vdash 1$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$N \vdash N$}
\RightLabel{$(?)$}
\UnaryInfC{$!N \vdash N$}
\AxiomC{π_{succ}}
\UnaryInfC{$N \vdash N$}
\RightLabel{$(?)$}
\UnaryInfC{$!N \vdash N$}
\RightLabel{$(!)$}
\UnaryInfC{$!N \vdash !N$}
\AxiomC{$!N \vdash S$}
\RightLabel{(cut)}
\BinaryInfC{$!N \vdash S$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$!N, !N \vdash N \otimes S$}
\RightLabel{(c)}
\UnaryInfC{$!N \vdash N \otimes S$}
\RightLabel{$(\&)$}
\BinaryInfC{$!N \vdash 1 \,\&\, (N \otimes S)$}
\RightLabel{(ν)}
\UnaryInfC{$!N \vdash S$}
\end{prooftree}$$

Examples of circular proofs

$$\color{red}{S} = \nu X.(1 \,\&\, (N \otimes X))$$

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\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$N \vdash N$}
\RightLabel{$(?)$}
\UnaryInfC{$!N \vdash N$}
\AxiomC{π_{succ}}
\UnaryInfC{$N \vdash N$}
\RightLabel{$(?)$}
\UnaryInfC{$!N \vdash N$}
\RightLabel{$(!)$}
\UnaryInfC{$!N \vdash !N$}
\AxiomC{$!N \vdash S$}
\RightLabel{(cut)}
\BinaryInfC{$!N \vdash S$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$!N, !N \vdash N \otimes S$}
\RightLabel{(c)}
\UnaryInfC{$!N \vdash N \otimes S$}
\RightLabel{$(\&)$}
\BinaryInfC{$!N \vdash 1 \,\&\, (N \otimes S)$}
\RightLabel{(ν)}
\UnaryInfC{$!N \vdash S$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{π_k}
\UnaryInfC{$\vdash ?N$}
\RightLabel{$(?)$}
\AxiomC{π_{enum}}
\RightLabel{(cut)}
\BinaryInfC{$\vdash S$}
\end{prooftree}
\quad \longrightarrow^{\omega} \quad
\begin{prooftree}
\AxiomC{}
\RightLabel{$(\mathbf{1})$}
\UnaryInfC{$\vdash 1$}
\AxiomC{π_k}
\AxiomC{}
\RightLabel{$(\mathbf{1})$}
\UnaryInfC{$\vdash 1$}
\AxiomC{π_{k+1}}
\AxiomC{}
\RightLabel{$(\mathbf{1})$}
\UnaryInfC{$\vdash 1$}
\AxiomC{π_{k+2}}
\AxiomC{$\vdots$}
\UnaryInfC{$\vdash S$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$\vdash N \otimes S$}
\RightLabel{$(\nu)\ ,\ (\&)$}
\BinaryInfC{$\vdash S$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$\vdash N \otimes S$}
\RightLabel{$(\nu)\ ,\ (\&)$}
\BinaryInfC{$\vdash S$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$\vdash N \otimes S$}
\RightLabel{$(\&)$}
\BinaryInfC{$\vdash 1 \,\&\, (N \otimes S)$}
\RightLabel{(ν)}
\UnaryInfC{$\vdash S$}
\end{prooftree}$$

Fixed-point encoding the exponentials

Consider the following encoding of LL exponentials:

$$
\begin{array}{rcl}
?^{\bullet} F & \triangleq & \mu X. F \oplus (\bot \oplus (X \parr X)) \\
!^{\bullet} F & \triangleq & \nu X. F \,\&\, (\mathbf{1} \,\&\, (X \otimes X))
\end{array}
$$

Fixed-point encoding the exponentials

Consider the following encoding of LL exponentials:

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The exponential inferences can be derived, circularly:

Dereliction $(?\mathrm{d}^\bullet)$:

$$\dfrac{\dfrac{\vdash F, \Delta}{\vdash F \oplus (\bot \oplus (?^\bullet F \parr ?^\bullet F)), \Delta}\,(\oplus_1)}{\vdash ?^\bullet F, \Delta}\,(\mu)$$

Contraction $(?\mathrm{c}^\bullet)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash ?^\bullet F, ?^\bullet F, \Delta}{\vdash ?^\bullet F \parr ?^\bullet F, \Delta}\,(\parr)}{\vdash \bot \oplus (?^\bullet F \parr ?^\bullet F), \Delta}\,(\oplus_2)}{\vdash F \oplus (\bot \oplus (?^\bullet F \parr ?^\bullet F)), \Delta}\,(\oplus_2)}{\vdash ?^\bullet F, \Delta}\,(\mu)$$

Weakening $(?\mathrm{w}^\bullet)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Delta}{\vdash \bot, \Delta}\,(\bot)}{\vdash \bot \oplus (?^\bullet F \parr ?^\bullet F), \Delta}\,(\oplus_1)}{\vdash F \oplus (\bot \oplus (?^\bullet F \parr ?^\bullet F)), \Delta}\,(\oplus_2)}{\vdash ?^\bullet F, \Delta}\,(\mu)$$

Promotion $(!\mathrm{p}^\bullet)$:

$$\dfrac{\vdash F, ?^\bullet \Delta \qquad \dfrac{\dfrac{}{\vdash \mathbf{1}}\,(\mathbf{1})}{\vdash \mathbf{1}, ?^\bullet \Delta}\,(?\mathrm{w}^\bullet) \qquad \dfrac{\dfrac{\vdash !^\bullet F, ?^\bullet \Delta \qquad \vdash !^\bullet F, ?^\bullet \Delta}{\vdash !^\bullet F \otimes !^\bullet F, ?^\bullet \Delta, ?^\bullet \Delta}\,(\otimes)}{\vdash !^\bullet F \otimes !^\bullet F, ?^\bullet \Delta}\,(?\mathrm{c}^\bullet)}{\vdash !^\bullet F, ?^\bullet \Delta}\,(\nu)\ ,\ (\&)\ ,\ (\&)$$

Fixed-point encoding the exponentials

Consider the following encoding of LL exponentials:

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\end{array}
$$

Preservation of validity

π is a valid $\mu\mathrm{LL}^\infty$ pre-proof of $\vdash \Gamma$ iff
$\pi^\bullet$ is a valid $\mu\mathrm{MALL}^\infty$ pre-proof of $\vdash \Gamma^\bullet$.

Preservation of provability

If $\vdash \Gamma$ is provable in $\mu\mathrm{LL}^\infty$ (resp. $\mu\mathrm{LL}^\omega$),
then $\vdash \Gamma^\bullet$ is provable in $\mu\mathrm{MALL}^\infty$ (resp. $\mu\mathrm{MALL}^\omega$).

Fixed-point encoding the exponentials

Consider the following encoding of LL exponentials:

$$\begin{array}{rcl} ?^\bullet F & \triangleq & \mu X. F \oplus (\bot \oplus (X \parr X)) \\ !^\bullet F & \triangleq & \nu X. F \,\&\, (\mathbf{1} \,\&\, (X \otimes X)) \end{array}$$

Preservation of validity

π is a valid μLL$^\infty$ pre-proof of $\vdash \Gamma$ iff
$\pi^\bullet$ is a valid μMALL$^\infty$ pre-proof of $\vdash \Gamma^\bullet$.

Preservation of provability

If $\vdash \Gamma$ is provable in μLL$^\infty$ (resp. μLL$^\omega$),
then $\vdash \Gamma^\bullet$ is provable in μMALL$^\infty$ (resp. μMALL$^\omega$).

Shortcomings of this encoding

No soundness result for the encoding: converse result for the preservation of provability. Loss of Seely isomorphisms, etc.

Circular & finitary proofs

From finitary to circular proofs

Theorem

Finitary proofs can be transformed to (valid) circular proofs.

The key translation step is the following:

$$
\dfrac{\dfrac{\pi_1}{\vdash \Gamma, S} \qquad \dfrac{\pi_2}{\vdash S^\perp, F[S]}}{\vdash \Gamma, \nu X.F}\,(\nu)
\quad\longmapsto\quad
\dfrac{\dfrac{[\pi_1]}{\vdash \Gamma, S} \qquad \dfrac{\dfrac{\dfrac{[\pi_2]}{\vdash S^\perp, F[S]} \qquad \dfrac{\vdash S^\perp, \nu X.F}{\vdash F[S]^\perp, F[\nu X.F]}\,(\mathfrak{F}_F)}{\vdash S^\perp, F[\nu X.F]}\,(cut)}{\vdash S^\perp, \nu X.F}\,(\nu)}{\vdash \Gamma, \nu X.F}\,(cut)
$$

From circular to finitary proofs

Open problem for μLL^ω in general. Solved positively for ***strongly valid*** circular proofs.

Intuitive Idea of Focusing

Skip details

Idea of focusing: Reduce the proof search space by

- *Reversibility* of negatives: no choice to make, provability of conclusion entails provability of premisses.
- *Focusing* the positives: involves choice, but proofs can proceed in a stubborn way by committing hereditarily to a positive *focus* and its subformulas.

Γ contains a negative formula	Γ contains no negative formula
choose the leftmost negative formula and apply the unique negative rule available.	choose some positive formula and decompose it hereditarily until atoms or negative subformulas are reached.

Various proof methods:
by cut-elimination, inference permutations, etc.
Here application of a proof method designed jointly with Miller.

$\mu MALL^{\infty}$ focusing

By adapting the proof of focusing using Focalization graphs by Miller and S., 2007:

1. Reversibility of the negatives;
2. Focusing for positive sequents:
   1. Weak commutation properties among the positives;
   2. Positive Trunks;
   3. Focalization graph;
   4. Existence of a potential focus in a positive sequent.
3. Productivity of the focusing process;
4. Validity of the produced proof.

Polarity of fixed points:

ν must be negative and μ must be positive.

$\mu MALL^{\infty}$ focusing

Reversibility of negatives & focusing of positives

1. Reversibility of negative sequents:
   Similar to MALL except that it cannot be treated with local rule permutations as shown by the following example:

$$\dfrac{\dfrac{\dfrac{\vdash F, P \parr Q \qquad \dfrac{\dfrac{\pi}{\vdash F, P, Q}}{\vdash F, P \parr Q}\,(\parr)}{\vdash F \mathbin{\&} F, P \parr Q}\,(\&)}{\vdash (F \mathbin{\&} F) \oplus \mathbf{0}, P \parr Q}\,(\oplus_1)}{\vdash F, P \parr Q}\,(\nu)$$

(reminiscent to what happens with $\top$ in LL focusing)

2. Focusing of positive sequents:
   - Positive trunks are finite trees (due to the polarization of fixed points formulas);
   - The rest of the proof goes as for MALL.

$\mu MALL^{\infty}$ focusing

Productivity and validity of the focusing process

- Productivity of the focusing process is essentially direct from MALL case:
  - Reversibility is productive by construction;
  - The positive focusing takes place in a finite subtree (finite positive trunks): just as in MALL.
- Preservation of validity relies on an analysis of the kind of permutations involved in focusing. Since a positive never permutes below a positive, valid thread cannot be infinitely postponed.

The extension to μLL ***is achieved exactly as for*** LL.

μMALL$^\infty$ Cut Elimination Theorem

Theorem (Baelde, Doumane & S, 2016)

Fair μMALL$^\infty$ cut-reduction sequences converge to cut-free μMALL$^\infty$ proofs.

Previous result by Santocanale and Fortier
for the purely additive fragment of μMALL$^\infty$.
Proof uses a locative treatment of occurrences.

- **Strategy:** "push" the cuts away from the root.
- **Cut-Cut:**

$$\dfrac{\dfrac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta, G}{\vdash \Gamma, \Delta, G}\,(cut) \qquad \vdash G^\perp, \Sigma}{\vdash \Gamma, \Delta, \Sigma}\,(cut) \longleftrightarrow \dfrac{\vdash \Gamma, F \qquad \dfrac{\vdash F^\perp, \Delta, G \qquad \vdash G^\perp, \Sigma}{\vdash F^\perp, \Delta, \Sigma}\,(cut)}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$$

μMALL$^\infty$ Cut Elimination Theorem

Theorem (Baelde, Doumane & S, 2016)

Fair μMALL$^\infty$ mcut-reduction sequences converge to cut-free μMALL$^\infty$ proofs.

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- **Strategy:** "push" the cuts away from the root.
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$$\dfrac{\dfrac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta, G}{\vdash \Gamma, \Delta, G}\ (cut) \qquad \vdash G^\perp, \Sigma}{\vdash \Gamma, \Delta, \Sigma}\ (cut) \quad \longrightarrow \quad \dfrac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta, G \qquad \vdash G^\perp, \Sigma}{\vdash \Gamma, \Delta, \Sigma}\ (mcut)$$

Cut elimination procedure

External phase: Cut-commutation cases

$$\dfrac{\dfrac{\vdash \Delta, F, G}{\vdash \Delta, F \parr G}\ (\parr) \qquad \dots}{\vdash \Sigma, F \parr G}\ (mcut) \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F, G \qquad \dots}{\vdash \Sigma, F, G}\ (mcut)}{\vdash \Sigma, F \parr G}\ (\parr)$$

$$\dfrac{\dfrac{\vdash \Delta, F \qquad \vdash \Delta, G}{\vdash \Delta, F \,\&\, G}\ (\&) \qquad \dots}{\vdash \Sigma, F \,\&\, G}\ (mcut) \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F \qquad \dots}{\vdash \Sigma, F}\ (mcut) \qquad \dfrac{\vdash \Delta, G \qquad \dots}{\vdash \Sigma, G}\ (mcut)}{\vdash \Sigma, F \,\&\, G}\ (\&)$$

$$\dfrac{\dfrac{\vdash \Delta, F[\mu X.F/X]}{\vdash \Delta, \mu X.F}\ (\mu) \qquad \dots}{\vdash \Sigma, \mu X.F}\ (mcut) \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F[\mu X.F/X] \qquad \dots}{\vdash \Sigma, F[\mu X.F/X]}\ (mcut)}{\vdash \Sigma, \mu X.F}\ (\mu)$$

+ additional cases

Cut-commutation steps are productive

Cut elimination procedure

Internal Phase: Key cases

$$
\dfrac{\dots \quad \dfrac{\vdash \Delta, F_2 \quad \vdash \Delta, F_1}{\vdash \Delta, F_2 \,\&\, F_1}\,(\&) \quad \dfrac{\vdash \Gamma, F_i^\perp}{\vdash \Gamma, F_1^\perp \oplus F_2^\perp}\,(\oplus_i)}{\vdash \Sigma}\,(mcut)
\Rightarrow
\dfrac{\dots \quad \vdash \Delta, F_i \quad \vdash \Gamma, F_i^\perp}{\vdash \Sigma}\,(mcut)
$$

$$
\dfrac{\dots \quad \dfrac{\vdash \Delta, F[\mu X.F/X]}{\vdash \Delta, \mu X.F}\,(\mu) \quad \dfrac{\vdash \Gamma, F^\perp[\nu X.F^\perp/X]}{\vdash \Gamma, \nu X.F^\perp}\,(\nu)}{\vdash \Sigma}\,(mcut)
$$

$$
\Rightarrow \dfrac{\dots \quad \vdash \Delta, F[\mu X.F/X] \quad \vdash \Gamma, F^\perp[\nu X.F^\perp/X]}{\vdash \Sigma}\,(mcut)
$$

+ additional cases

Key cases are not productive

Cut elimination algorithm

- Internal phase: Perform key case reductions as long as you cannot do anything else.
- External phase: Build a part of the output tree by applying cut-commutation steps as soon possible, being fair.
- Repeat.

Cut elimination algorithm

- Internal phase: Perform key case reductions as long as you cannot do anything else.
- External phase: Build a part of the output tree by applying cut-commutation steps as soon possible, being fair.
- Repeat.

Remark: We consider a **fair** strategy ie. every reduction which is available at some point will be performed eventually.

Theorem

Internal phases always halt. Cut-elimination produces a pre-proof.

Theorem

The pre-proof obtained by the cut elimination algorithm is valid.

μLL^{ω} is not stable by cut-elimination

Eliminating cuts from a μLL^{ω} proof (circular) may result in a μLL^{∞}, non circular, proof.

Validity sensitive to cut-introduction in cycles

Circular derivations corresponding to:

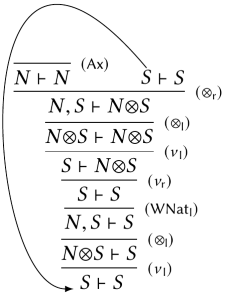
`drop`

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{N \vdash N}\,(\mathsf{Ax}) \qquad S \vdash S}{N, S \vdash N \otimes S}\,(\otimes_{\mathsf{r}})}{N \otimes S \vdash N \otimes S}\,(\otimes_{\mathsf{l}})}{S \vdash N \otimes S}\,(\nu_{\mathsf{l}})}{S \vdash S}\,(\nu_{\mathsf{r}})}{N, S \vdash S}\,(\mathsf{WNat}_{\mathsf{l}})}{N \otimes S \vdash S}\,(\otimes_{\mathsf{l}})}{S \vdash S}\,(\nu_{\mathsf{l}})
$$

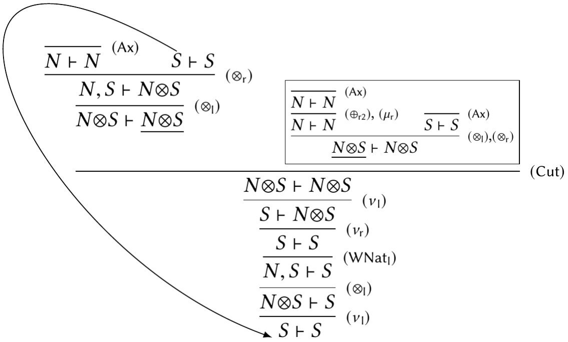
Validity sensitive to cut-introduction in cycles

Circular derivations corresponding to:

`drop`



`incdrop`



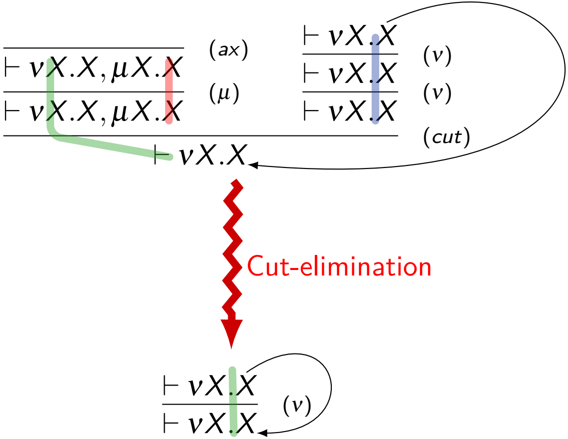
A invalid, though productive, proof with cut

Problem: Cuts are not well-managed by the validity condition.

$$\dfrac{\dfrac{\dfrac{}{\vdash \nu X.X, \mu X.X}\,(ax)}{\vdash \nu X.X, \mu X.X}\,(\mu) \qquad \dfrac{\dfrac{\vdash \nu X.X}{\vdash \nu X.X}\,(\nu)}{\vdash \nu X.X}\,(\nu)}{\vdash \nu X.X}\,(cut)$$

A invalid, though productive, proof with cut

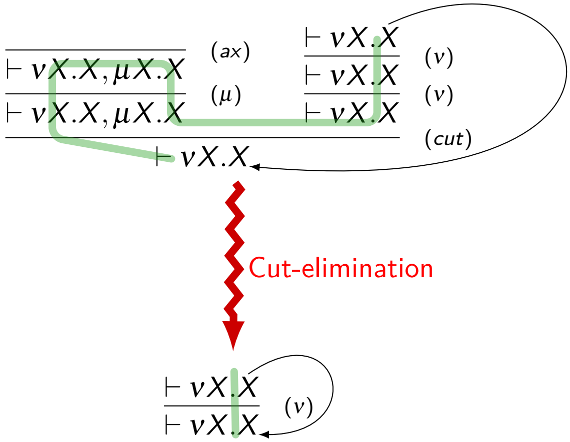
Problem: Cuts are not well-managed by the validity condition.



From now, we will refer to s-valid pre-proof for the previous validity condition and will consider alternative validity conditions.

A invalid, though productive, proof with cut

Problem: Cuts are not well-managed by the validity condition.



From now, we will refer to s-valid pre-proof for the previous validity condition and will consider alternative validity conditions.

What's in a bouncing thread?

To be persistent or not to be:

$$\dfrac{\dfrac{\dfrac{}{\vdash F, F'^{\perp}}\,(\mathrm{Ax}) \qquad \dfrac{}{\vdash G, G'^{\perp}}\,(\mathrm{Ax})}{\vdash F \parr G, F'^{\perp} \otimes G'^{\perp}}\,(\parr, \otimes) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\vdash F'', G'}}{\vdash F', G'}\,(\nu)}{\vdash F' \parr G'}\,(\parr)}{\vdash F \parr G}\,(\mathrm{Cut})$$

What's in a bouncing thread?

To be persistent or not to be:

$$
\dfrac{\dfrac{\dfrac{}{\vdash F, F'^{\perp}}\;(\mathrm{Ax}) \qquad \dfrac{}{\vdash G, G'^{\perp}}\;(\mathrm{Ax})}{\vdash F \parr G, F'^{\perp} \otimes G'^{\perp}}\;(\parr, \otimes) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\vdash F'', G'}}{\vdash F', G'}\;(\nu)}{\vdash F' \parr G'}\;(\parr)}{\vdash F \parr G}\;(\mathrm{Cut})
$$

To be visible or not to be:

$$
\dfrac{\dfrac{\dfrac{}{\vdash F, F'^{\perp}}\;(\mathrm{Ax}) \qquad \dfrac{}{\vdash G, G'^{\perp}}\;(\mathrm{Ax})}{\vdash F \parr G, F'^{\perp} \otimes G'^{\perp}}\;(\parr, \otimes) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\vdash F'', G'}}{\vdash F', G'}\;(\nu)}{\vdash F' \parr G'}\;(\parr)}{\vdash F \parr G}\;(\mathrm{Cut})
$$

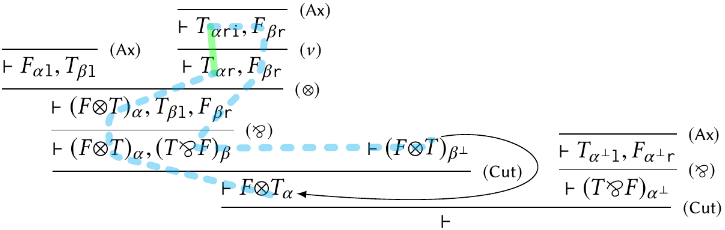
Valid and invalid bouncing proofs

Unsound proof with infinite visible part:

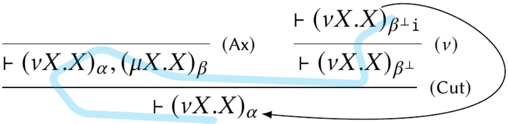
$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\vdash F_{\alpha 1}, T_{\beta 1}$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\vdash T_{\alpha \mathtt{ri}}, F_{\beta \mathtt{r}}$}
\RightLabel{(ν)}
\UnaryInfC{$\vdash T_{\alpha \mathtt{r}}, F_{\beta \mathtt{r}}$}
\RightLabel{($\otimes$)}
\BinaryInfC{$\vdash (F \otimes T)_{\alpha}, T_{\beta 1}, F_{\beta \mathtt{r}}$}
\RightLabel{($\parr$)}
\UnaryInfC{$\vdash (F \otimes T)_{\alpha}, (T \parr F)_{\beta}$}
\AxiomC{$\vdash (F \otimes T)_{\beta^\perp}$}
\RightLabel{(Cut)}
\BinaryInfC{$\vdash F \otimes T_{\alpha}$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\vdash T_{\alpha^\perp 1}, F_{\alpha^\perp \mathtt{r}}$}
\RightLabel{($\parr$)}
\UnaryInfC{$\vdash (T \parr F)_{\alpha^\perp}$}
\RightLabel{(Cut)}
\BinaryInfC{$\vdash$}
\end{prooftree}
$$

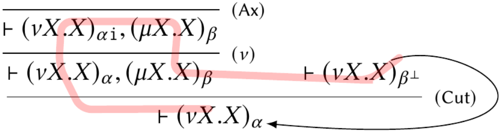
Valid and invalid bouncing proofs

Unsound proof with infinite visible part:



Restriction to the validity condition:





Bouncing threads: visible part and bouncing validity

Visible part: survives the cut-elimination.
Hidden part: Must satisfy matching constaints.

Summing up...

$$\dfrac{\dfrac{\dfrac{}{\vdash F, F^{\perp}}\,(ax) \qquad \dfrac{}{\vdash G, G^{\perp}}\,(ax)}{\vdash F \parr G, F^{\perp} \otimes G^{\perp}}\,(\parr,\otimes) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\vdash F', G}}{\vdash F, G}\,(\nu)}{\vdash F \parr G}\,(\parr)}{\vdash F \parr G}\,(cut)$$

Valid bouncing thread: ∞ ν-unfoldings in visible part.

Bouncing threads: visible part and bouncing validity

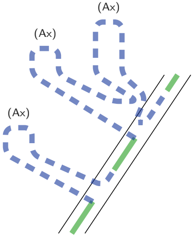
Visible part: survives the cut-elimination.
Hidden part: Must satisfy matching constaints.

Summing up...

$$\dfrac{\dfrac{\dfrac{}{\vdash F, F^\perp}(ax) \qquad \dfrac{}{\vdash G, G^\perp}(ax)}{\vdash F \parr G, F^\perp \otimes G^\perp}(\parr,\otimes) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\vdash F', G}}{\vdash F, G}(\nu)}{\vdash F \parr G}(\parr)}{\vdash F \parr G}(cut)$$

Valid bouncing thread: ∞ ν-unfoldings in visible part.

Valid branch B: there exists a valid bouncing thread with visible part included in B.
B-valid proof: all infinite branches are valid. (+ condition for the additives.)



Theorem (Baelde, Doumane, Kuperberg, S)

Soundness and cut-elimination hold for μMALL$^\infty$ b-valid proofs.

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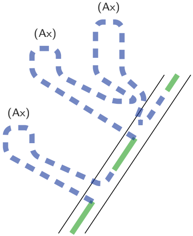


Theorem (Baelde, Doumane, Kuperberg, S)

Soundness and cut-elimination hold for μMALL$^{\infty}$ b-valid proofs.

Working conjecture (joint work with Bauer):

It is sufficient that the visible part intersects infinitely often the validated branch for cut-elimination.

Wait! What about the additives?

Wait! What about the additives?

Consider $S = \mu Y.((\bot \,\&\, \bot) \parr Y), T = \nu X.X$.

$$
\pi_k = \cfrac{\cfrac{\cfrac{\cfrac{}{\vdash T, T^\perp}\,(ax)\,\dagger}{\vdash \bot, T, T^\perp}\,(\bot) \qquad \cfrac{\cfrac{\cfrac{}{\vdash T, T^\perp}\,(ax)\,\ddagger}{\vdash \bot, T, T^\perp}\,(\bot)}{\vdash \bot, T, T^\perp}\,{\color{red}(\mu)^k}}{\vdash \bot \,\&\, \bot, T, T^\perp}\,(\&) \qquad \vdash S, T}{\cfrac{\cfrac{\vdash \bot \,\&\, \bot, S, T}{\vdash S, T}\,(\mu)\ ,\ (\parr)}{\vdash S, T}\,(\nu)}\,(cut)
\qquad k \geq 0
$$

For $k \geq 1$, cut elimination does not produce. Why?

Wait! What about the additives?

Consider $S = \mu Y.((\perp \& \perp) \parr Y), T = \nu X.X$.

$$\pi_k = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash T, T^\perp}\,(ax)\,\dagger}{\vdash \perp, T, T^\perp}\,(\perp) \qquad \dfrac{\dfrac{\dfrac{}{\vdash T, T^\perp}\,(ax)\,\ddagger}{\vdash \perp, T, T^\perp}\,(\perp)}{\vdash \perp, T, T^\perp}\,(\mu)^k}{\vdash \perp \& \perp, T, T^\perp}\,(\&) \qquad \vdash S, T}{\vdash \perp \& \perp, S, T}\,(cut)}{\vdash S, T}\,(\mu)\ ,\ (\parr)}{\vdash S, T}\,(\nu) \qquad\qquad k \geq 0$$

For $k \geq 1$, cut elimination does not produce. Why?

Consider validity condition slice-wise:

The sliced proof system is obtained from μMALL$^\infty$ by replacing the & rule with:

$$\dfrac{\vdash A, \Gamma}{\vdash A \& B, \Gamma}\,(\&_1) \qquad \dfrac{\vdash B, \Gamma}{\vdash A \& B, \Gamma}\,(\&_2)$$

Every persistent slices shall be bouncing valid.

Cut-elimination proof for bounding validity

skip details

Theorem

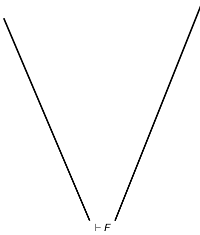
The internal phase always halts.

Theorem

The pre-proof obtained by the cut elimination algorithm is valid.

Definition: Trace of a cut-reduction sequence

Given a cut reduction sequence σ from π, the trace of σ is the collection of those sequents of π which becomes premises of a multicut in σ.



Cut-elimination proof for bounding validity

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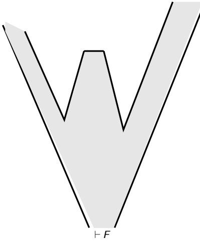
Definition: Trace of a cut-reduction sequence

Given a cut reduction sequence σ from π, the trace of σ is the collection of those sequents of π which becomes premises of a multicut in σ.

Lemma

Let π be a b-valid proof and σ a fair reduction sequence from π. Every infinite branch of π that is contained in the trace of σ has its (bouncing) validating thread also contained in the trace.



Cut elimination is productive

skip details

Theorem

The internal phase always halts.

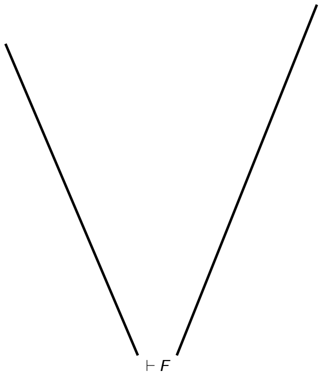
Cut elimination is productive

skip details

Theorem

The internal phase always halts.

Proof by contradiction: Suppose that there is a proof of F for which the internal phase does not halt.



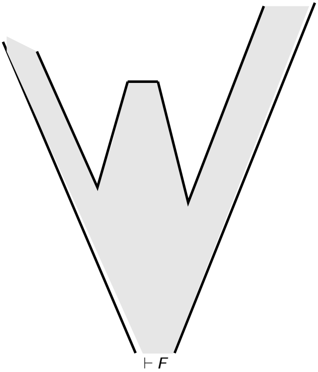
Cut elimination is productive

skip details

Theorem

The internal phase always halts.

Proof by contradiction: Consider the trace of this divergent reduction.



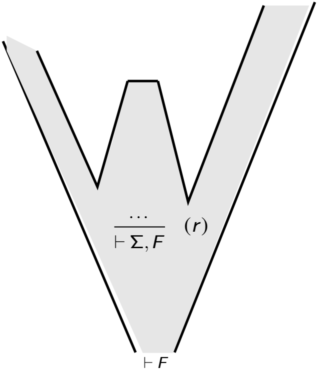
Cut elimination is productive

skip details

Theorem

The internal phase always halts.

Proof by contradiction: No rule on F is applied in the trace, otherwise the internal phase would halt.



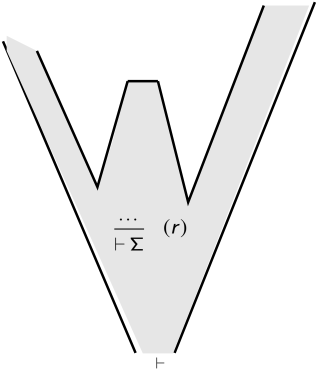
Cut elimination is productive

skip details

Theorem

The internal phase always halts.

Proof by contradiction: We can eliminate the occurrences of F from the trace. This yields a "proof" of $\vdash$.



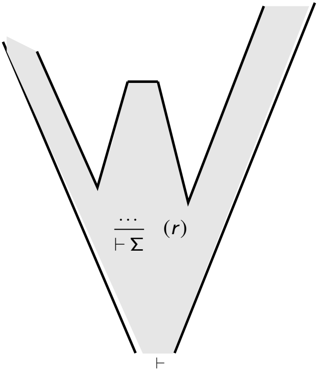
Cut elimination is productive

skip details

Theorem

The internal phase always halts.

Proof by contradiction: We show that the proof system is sound. Contradiction.



Cut elimination produces a proof

Theorem

The pre-proof obtained by the cut elimination algorithm is valid.

Cut elimination produces a proof

Theorem

The pre-proof obtained by the cut elimination algorithm is valid.

Proof: Let $\pi^\star$ be the pre-proof obtained from $\pi \vdash \Delta$ by cut elimination. Suppose that a branch b of $\pi^\star$ is not valid.

- Let θ be the sub-derivation of π explored by the reduction that produces b.
- **Fact:** Threads of θ are the threads of b, together with threads starting from cut formulas.
- The validity of θ cannot rely on the threads of b.
- θ^μ is θ where we replace in Δ any ν by a μ and any $1, \top$ by $\bot, 0$.
- Show that formulas containing only $\mu, \bot, 0$ and $MALL$ connectives are false.
- θ^μ proves a false sequent which contradicts soundness.

Decidability of the bouncing validity condition ?

Given a **circular** proof, can we decide b-validity ?

Decidability of the bouncing validity condition ?

Given a **circular** proof, can we decide b-validity ? **NO!**

$\Longrightarrow$ Reduce termination of Minsky machines to bouncing validity.

Decidability of the bouncing validity condition ?

Given a **circular** proof, can we decide b-validity ? **NO!**

$\Longrightarrow$ Reduce termination of Minsky machines to bouncing validity.

A hierarchy of decidable conditions: Height of a b-thread: parameter binding the height of bounces.

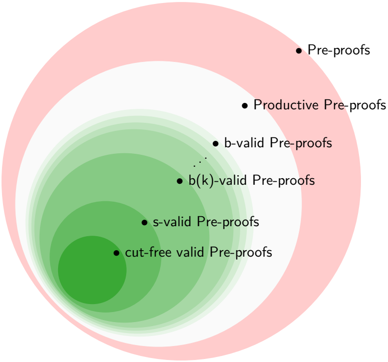
$b(k)$-valid proof: b-valid proof using only threads of height $\leq k$.

Theorem

Every b-valid circular proof is a $b(k)$-valid for some $k \in \mathbb{N}$.

Theorem

For all $k \in \mathbb{N}$, it is decidable whether a circular proof is a $b(k)$-proof.



Cut-elimination for μLL^∞

Theorem

Fair μLL^∞ mcut-reduction sequences converge to cut-free μLL^∞ proofs.

Cut-elimination for μLL$^\infty$

Theorem

Fair μLL$^\infty$ mcut-reduction sequences converge to cut-free μLL$^\infty$ proofs.

Idea. The proof goes by:

- considering the following encoding of LL exponential modalities:

$$
\begin{array}{rcl}
?^\bullet F & = & \mu X.F \oplus (\bot \oplus (X \parr X)) \\
!^\bullet F & = & \nu X.F \,\&\, (\mathbf{1} \,\&\, (X \otimes X))
\end{array}
$$

- translating μLL$^\infty$ sequents and proofs in μMALL$^\infty$,
- simulating μLL$^\infty$ cut-reduction sequences in μMALL$^\infty$ and
- applying μMALL$^\infty$ cut-elimination theorem.

Extends to the circular version of LL with $(!p^{\mathrm{nu}})$ (even with fixed-points):

Theorem (Cut-elimination for circular LL)

Circular LL (with $(!p^{\mathrm{nu}})$ for promotion) eliminates cuts (even with fixed-points).

Encoding μLL$^\infty$ in μMALL$^\infty$

$$?^\bullet F = \mu X. F \oplus (\bot \oplus (X \parr X)) \qquad !^\bullet F = \nu X. F \mathbin{\&} (\mathbf{1} \mathbin{\&} (X \otimes X))$$

μMALL$^\infty$ derivability of the exponential rules ($?d^\bullet$, $?c^\bullet$, $?w^\bullet$, $!p^\bullet$):

Dereliction :

$$\dfrac{\dfrac{\vdash F, \Delta}{\vdash F \oplus (\bot \oplus (?^\bullet F \parr ?^\bullet F)), \Delta}\,(\oplus_1)}{\vdash ?^\bullet F, \Delta}\,(\mu)$$

Contraction :

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash ?^\bullet F, ?^\bullet F, \Delta}{\vdash ?^\bullet F \parr ?^\bullet F, \Delta}\,(\parr)}{\vdash \bot \oplus (?^\bullet F \parr ?^\bullet F), \Delta}\,(\oplus_2)}{\vdash F \oplus (\bot \oplus (?^\bullet F \parr ?^\bullet F)), \Delta}\,(\oplus_2)}{\vdash ?^\bullet F, \Delta}\,(\mu)$$

Weakening :

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Delta}{\vdash \bot, \Delta}\,(\bot)}{\vdash \bot \oplus (?^\bullet F \parr ?^\bullet F), \Delta}\,(\oplus_1)}{\vdash F \oplus (\bot \oplus (?^\bullet F \parr ?^\bullet F)), \Delta}\,(\oplus_2)}{\vdash ?^\bullet F, \Delta}\,(\mu)$$

Promotion:

$$\dfrac{\vdash F, ?^\bullet \Delta \qquad \dfrac{\dfrac{}{\vdash \mathbf{1}}\,(\mathbf{1})}{\vdash \mathbf{1}, ?^\bullet \Delta}\,(?w^\bullet) \qquad \dfrac{\dfrac{\vdash !^\bullet F, ?^\bullet \Delta \qquad \vdash !^\bullet F, ?^\bullet \Delta}{\vdash !^\bullet F \otimes !^\bullet F, ?^\bullet \Delta, ?^\bullet \Delta}\,(\otimes)}{\vdash !^\bullet F \otimes !^\bullet F, ?^\bullet \Delta}\,(?c^\bullet)}{\vdash !^\bullet F, ?^\bullet \Delta}\,(\nu)\ ,\ (\&)\ ,\ (\&)$$

Preservation of validity

π is a valid μLL$^\infty$ pre-proof of $\vdash \Gamma$ iff
$\pi^\bullet$ is a valid μMALL$^\infty$ pre-proof of $\vdash \Gamma^\bullet$.

Simulation of μLL$^\infty$ cut-elimination steps

μLL$^\infty$ cut-elimination steps can be simulated by the previous encoding.

For instance, the following reduction can be simulated by applying the external reduction rule $(\mu)/(cut)$ followed by the external reduction rule $(\oplus)/(cut)$.

$$\dfrac{\dfrac{\vdash F, G, \Gamma}{\vdash ?^\bullet F, G, \Gamma}\,(?\mathrm{d}^\bullet) \qquad \vdash G^\perp, \Delta}{\vdash ?^\bullet F, \Gamma, \Delta}\,(cut) \quad \longrightarrow^2 \quad \dfrac{\dfrac{\vdash F, G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash F, \Gamma, \Delta}\,(cut)}{\vdash ?^\bullet F, \Gamma, \Delta}\,(?\mathrm{d}^\bullet)$$

Challenge: to show that the simulation of derivation also holds

(i) for the reductions involving $[!p]$ as well as
(ii) for reductions occurring ***above*** a promotion rule (aka. in a box) since the encoding of $[!p]$ uses an infinite, circular derivation.

Simulation of μLL$^\infty$ cut-elimination steps

Cut-commutation rules

$$\dfrac{\dfrac{\vdash F, G, \Gamma}{\vdash ?^\bullet F, G, \Gamma}\ (?\mathrm{d}^\bullet) \qquad \vdash G^\perp, \Delta}{\vdash ?^\bullet F, \Gamma, \Delta}\ (cut) \quad \longrightarrow^2 \quad \dfrac{\dfrac{\vdash F, G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash F, \Gamma, \Delta}\ (cut)}{\vdash ?^\bullet F, \Gamma, \Delta}\ (?\mathrm{d}^\bullet)$$

$$\dfrac{\dfrac{\vdash ?^\bullet F, ?^\bullet F, G, \Gamma}{\vdash ?^\bullet F, G, \Gamma}\ (?\mathrm{c}^\bullet) \qquad \vdash G^\perp, \Delta}{\vdash ?^\bullet F, \Gamma, \Delta}\ (cut) \quad \longrightarrow^3 \quad \dfrac{\dfrac{\vdash ?^\bullet F, ?^\bullet F, G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash ?^\bullet F, ?^\bullet F, \Gamma, \Delta}\ (cut)}{\vdash ?^\bullet F, \Gamma, \Delta}\ (?\mathrm{c}^\bullet)$$

$$\dfrac{\dfrac{\vdash G, \Gamma}{\vdash ?^\bullet F, G, \Gamma}\ (?\mathrm{w}^\bullet) \qquad \vdash G^\perp, \Delta}{\vdash ?^\bullet F, \Gamma, \Delta}\ (cut) \quad \longrightarrow^3 \quad \dfrac{\dfrac{\vdash G, \Gamma \qquad \vdash G^\perp, \Delta}{\vdash \Gamma, \Delta}\ (cut)}{\vdash ?^\bullet F, \Gamma, \Delta}\ (?\mathrm{w}^\bullet)$$

$$\dfrac{\dfrac{\vdash F, ?^\bullet G, ?^\bullet \Gamma}{\vdash !^\bullet F, ?^\bullet G, ?^\bullet \Gamma}\ (!\mathrm{p}^\bullet) \qquad \dfrac{\vdash G, ?^\bullet \Delta}{\vdash !^\bullet G^\perp, ?^\bullet \Delta}\ (!\mathrm{p}^\bullet)}{\vdash !^\bullet F, ?^\bullet \Gamma, ?^\bullet \Delta}\ (cut) \quad \longrightarrow^\omega \quad \dfrac{\dfrac{\vdash F, ?^\bullet G, ?^\bullet \Gamma \qquad \dfrac{\vdash G, ?^\bullet \Delta}{\vdash !^\bullet G^\perp, ?^\bullet \Delta}\ (!\mathrm{p}^\bullet)}{\vdash F, ?^\bullet \Gamma, ?^\bullet \Delta}\ (cut)}{\vdash !^\bullet F, ?^\bullet \Gamma, ?^\bullet \Delta}\ (!\mathrm{p}^\bullet)$$

Simulation of $\mu\mathsf{LL}^\infty$ cut-elimination steps

Key-cut rules

$$
\dfrac{\dfrac{\dfrac{\pi}{\vdash F,\Gamma}}{\vdash ?^\bullet F,\Gamma}\,(?\mathsf{d}^\bullet) \qquad \dfrac{\dfrac{\pi'}{\vdash F^\perp, ?^\bullet\Delta}}{\vdash !^\bullet F^\perp, ?^\bullet\Delta}\,(!\mathsf{p}^\bullet)}{\vdash \Gamma, ?^\bullet\Delta}\,(cut)
\longrightarrow^2
\dfrac{\dfrac{\pi}{\vdash F,\Gamma} \qquad \dfrac{\pi'}{\vdash F^\perp, ?^\bullet\Delta}}{\vdash \Gamma, ?^\bullet\Delta}\,(cut)
$$

$$
\dfrac{\dfrac{\dfrac{\pi}{\vdash ?^\bullet F, ?^\bullet F,\Gamma}}{\vdash ?^\bullet F,\Gamma}\,(?\mathsf{c}^\bullet) \qquad \dfrac{\dfrac{\pi'}{\vdash F^\perp, ?^\bullet\Delta}}{\vdash !^\bullet F^\perp, ?^\bullet\Delta}\,(!\mathsf{p}^\bullet)}{\vdash \Gamma, ?^\bullet\Delta}\,(cut)
\longrightarrow^{4int, 4\times\#\Delta ext}
$$

$$
\dfrac{\dfrac{\dfrac{\pi}{\vdash ?^\bullet F, ?^\bullet F,\Gamma} \qquad \dfrac{\pi'}{\vdash !^\bullet F^\perp, ?^\bullet\Delta} \qquad \dfrac{\pi'}{\vdash !^\bullet F^\perp, ?^\bullet\Delta}}{\vdash \Gamma, ?^\bullet\Delta, ?^\bullet\Delta}\,(mcut)}{\vdash \Gamma, ?^\bullet\Delta}\,(?\mathsf{c}^\bullet)^{\star}
$$

$$
\dfrac{\dfrac{\dfrac{\pi}{\vdash \Gamma}}{\vdash ?^\bullet F,\Gamma}\,(?\mathsf{w}^\bullet) \qquad \dfrac{\dfrac{\pi'}{\vdash F^\perp, ?^\bullet\Delta}}{\vdash !^\bullet F^\perp, ?^\bullet\Delta}\,(!\mathsf{p}^\bullet)}{\vdash \Gamma, ?^\bullet\Delta}\,(cut)
\longrightarrow^{3int, 3\times\#\Delta ext}
\dfrac{\dfrac{\pi}{\vdash \Gamma}}{\vdash \Gamma, ?^\bullet\Delta}\,(?\mathsf{w}^\bullet)^{\star}
$$

Cut-elimination for μLL$^\infty$

1. Consider a fair cut-reduction sequence $\sigma = (\pi_i)_{i \in \omega}$ in μLL$^\infty$ from π.
2. σ converges to a cut-free μLL$^\infty$ pre-proof. By contradiction: Otherwise, a suffix τ of σ would contain only key-cut steps. The encoding of τ in μMALL$^\infty$, $\tau^\bullet$ would either be unproductive or would produce an infinite tree of encodings of $?\mathrm{w}, ?\mathrm{c}$ containing no ν inference. This would contradict μMALL$^\infty$ cut-elimination theorem.

Cut-elimination for μLL^∞

1. Consider a fair cut-reduction sequence $\sigma = (\pi_i)_{i \in \omega}$ in μLL^∞ from π.
2. σ converges to a cut-free μLL^∞ pre-proof.
3. As σ is productive and since reduction only occurs above cuts, it strongly converges to some μLL^∞ cut-free pre-proof π'.
4. $\sigma^\bullet$ is a transfinite sequence from $\pi^\bullet$ strongly converging to $\pi'^\bullet$: because $\pi'^\bullet$ – the encoding of π' – is cut-free and because only ! commutations and reductions above a promotion create infinite reductions: boxes are simulated by strongly converging sequences.

Cut-elimination for $\mu\mathsf{LL}^\infty$

1. Consider a fair cut-reduction sequence $\sigma = (\pi_i)_{i \in \omega}$ in $\mu\mathsf{LL}^\infty$ from π.
2. σ converges to a cut-free $\mu\mathsf{LL}^\infty$ pre-proof.
3. As σ is productive and since reduction only occurs above cuts, it strongly converges to some $\mu\mathsf{LL}^\infty$ cut-free pre-proof π'.
4. $\sigma^\bullet$ is a transfinite sequence from $\pi^\bullet$ strongly converging to $\pi'^\bullet$.
5. The compression lemma applies: there exists ρ an ω-indexed $\mu\mathsf{MALL}^\infty$ cut-reduction sequence converging to $\pi'^\bullet$.
6. Fairness of σ transfers (almost) to ρ: ρ can be turned into a fair $\mu\mathsf{MALL}^\infty$ cut-red sequence.

Cut-elimination for μLL^∞

1. Consider a fair cut-reduction sequence $\sigma = (\pi_i)_{i \in \omega}$ in μLL^∞ from π.
2. σ converges to a cut-free μLL^∞ pre-proof.
3. As σ is productive and since reduction only occurs above cuts, it strongly converges to some μLL^∞ cut-free pre-proof π'.
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5. The compression lemma applies: there exists ρ an ω-indexed $\mu MALL^\infty$ cut-reduction sequence converging to $\pi'^\bullet$.
6. Fairness of σ transfers (almost) to ρ: ρ can be turned into a fair $\mu MALL^\infty$ cut-red sequence.
7. Therefore, by $\mu MALL^\infty$ cut-elimination thm, ρ has a limit, $\pi'^\bullet$, which is a valid cut-free $\mu MALL^\infty$ proof.
8. Using preservation of validity, π' is a valid cut-free μLL^∞-proof. $\square$

About Seely isomorphisms

$$!A \otimes !B \dashv\vdash !(A \,\&\, B)$$

About Seely isomorphisms

$!A \otimes !B \dashv\vdash !(A \& B)$

$$\pi_S = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(ax)}{!A \vdash A}\,(?)}{!A, !B \vdash A}\,(w) \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}\,(ax)}{!B \vdash B}\,(?)}{!A, !B \vdash B}\,(w)}{!A, !B \vdash A \& B}\,(\&)}{!A, !B \vdash !(A \& B)}\,(!)}{!A \otimes !B \vdash !(A \& B)}\,(⅋)$$

$$\pi'_S = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(ax)}{A \& B \vdash A}\,(\oplus_1)}{!(A \& B) \vdash A}\,(?)}{!(A \& B) \vdash !A}\,(!) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,(ax)}{A \& B \vdash B}\,(\oplus_2)}{!(A \& B) \vdash B}\,(?)}{!(A \& B) \vdash !B}\,(!)}{!(A \& B), !(A \& B) \vdash !A \otimes !B}\,(\otimes)}{!(A \& B) \vdash !A \otimes !B}\,(c)$$

$$\dfrac{\pi_S \qquad \pi'_S}{!A \otimes !B \vdash !A \otimes !B}\,(cut) \qquad \to^{\star}_{cut}$$

About Seely isomorphisms

$$!A \otimes !B \dashv\vdash !(A \& B)$$

$$\pi_S = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(ax)}{!A \vdash A}\,(?)}{!A, !B \vdash A}\,(w) \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}\,(ax)}{!B \vdash B}\,(?)}{!A, !B \vdash B}\,(w)}{!A, !B \vdash A \& B}\,(\&)}{!A, !B \vdash !(A \& B)}\,(!)}{!A \otimes !B \vdash !(A \& B)}\,(\parr)$$

$$\pi'_S = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(ax)}{A \& B \vdash A}\,(\oplus_1)}{!(A \& B) \vdash A}\,(?)}{!(A \& B) \vdash !A}\,(!) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,(ax)}{A \& B \vdash B}\,(\oplus_2)}{!(A \& B) \vdash B}\,(?)}{!(A \& B) \vdash !B}\,(!)}{!(A \& B), !(A \& B) \vdash !A \otimes !B}\,(\otimes)}{!(A \& B) \vdash !A \otimes !B}\,(c)$$

$$\dfrac{\pi_S \qquad \pi'_S}{!A \otimes !B \vdash !A \otimes !B}\,(cut) \quad \to^\star_{cut} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(ax)}{!A \vdash A}\,(?)}{!A, !B \vdash A}\,(w)}{!A, !B \vdash !A}\,(!) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,(ax)}{!B \vdash B}\,(?)}{!A, !B \vdash B}\,(w)}{!A, !B \vdash !B}\,(!)}{!A, !B, !A, !B \vdash !A \otimes !B}\,(\otimes)}{!A, !B \vdash !A \otimes !B}\,(c)^2}{!A \otimes !B \vdash !A \otimes !B}\,(\parr)$$

About Seely isomorphisms

What about the fixed-point encoding?

$$\dfrac{(\pi_S)^\bullet \qquad (\pi'_S)^\bullet}{!^\bullet A \otimes !^\bullet B \vdash !^\bullet A \otimes !^\bullet B}\ (cut)$$

The left occurrences of A, B require two unfoldings of the fixed-point, while the right occurrences of A, B require only one unfolding of the fixed-point.

The fixed-point unfolding structure tracks the history of the structural rules.

About Seely isomorphisms

What about the fixed-point encoding?

$$
\dfrac{(\pi_S)^\bullet \qquad (\pi'_S)^\bullet}{!^\bullet A \otimes !^\bullet B \vdash !^\bullet A \otimes !^\bullet B}\ (cut)
\quad \to^{\omega}_{cut} \quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\ (ax)}{!^\bullet A \vdash A}\ (?d^\bullet)}{!^\bullet A, !^\bullet B \vdash A}\ (?w^\bullet)}{!^\bullet A, ! B \vdash ! A}\ (!p^\bullet)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\ (ax)}{!^\bullet B \vdash B}\ (?d^\bullet)}{!^\bullet A, !^\bullet B \vdash B}\ (?w^\bullet)}{!^\bullet A, !^\bullet B \vdash !^\bullet B}\ (!p^\bullet)}{!^\bullet A, !^\bullet B, !^\bullet A, !^\bullet B \vdash !^\bullet A \otimes !^\bullet B}\ (\otimes)}{!^\bullet A, !^\bullet B \vdash !^\bullet A \otimes !^\bullet B}\ (?c^\bullet)^2}{!^\bullet A \otimes !^\bullet B \vdash !^\bullet A \otimes !^\bullet B}\ (\parr)
$$

The left occurrences of A, B require two unfoldings of the fixed-point, while the right occurrences of A, B require only one unfolding of the fixed-point.

The fixed-point unfolding structure tracks the history of the structural rules.

Cut-elimination for μLK^∞, μLJ^∞

The usual call-by-value embedding of LJ in ILL (intuitionnistic LL) can be lifted to μLJ^∞: indeed, the translation of proofs does not introduce cuts. For μLK^∞, it is slightly trickier as the well-known T/Q-translations introduce cuts breaking validity. An alternative translation which does not introduce cuts can be used.

Moreover, one gets the skeleton of a μLL^∞ (resp. μILL^∞) proof which is a μLK^∞ (resp. μLJ^∞) proof, simply by erasing the exponentials (connectives and inferences), preserving validity.
The skeleton of a μLL^∞ (resp. μILL^∞) cut-reduction sequence is a μLK^∞ (resp. μLJ^∞) cut-reduction sequence. As a result, one has:

Theorem

If π is an μLK^∞ (resp. μLJ^∞) proof of $\vdash \Gamma$ (resp. $\Gamma \vdash F$), there exists a μLL^∞ (resp. μILL^∞) proof of the translated sequents.

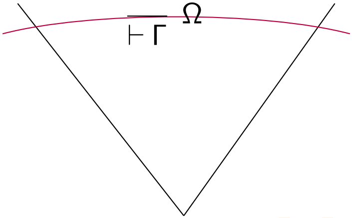
Theorem

There are productive cut-reduction strategies producing cut-free μLK^∞ (resp. μLJ^∞) proofs.

NUTS as a denotational model of μLL$^\infty$

$$\left[\!\!\left[\dfrac{\dfrac{\pi}{\vdash \Gamma, F[\mu X.F/X]}}{\vdash \Gamma, \mu X.F}\ (\mu) \right]\!\!\right] = [\![\pi]\!] \qquad \left[\!\!\left[\dfrac{\dfrac{\pi}{\vdash \Gamma, F[\nu X.F/X]}}{\vdash \Gamma, \nu X.F}\ (\nu) \right]\!\!\right] = [\![\pi]\!]$$

Build a semantics by finite approximants:



$$[\![\pi]\!]_{\mathbf{REL}} = \bigcup_{\rho \in \mathrm{fin}(\pi)} [\![\rho]\!]_{\mathbf{REL}}$$

Soundness of μLL$_\infty$

Lemma

Let (π_i) be a Cauchy sequence. Then $[\![\lim_{n\to\infty}\pi_i]\!]_{\mathbf{REL}} = \bigcup_i \bigcap_{j>i} [\![\pi_j]\!]_{\mathbf{REL}}$.

Corollary

If π and π' are proofs of $\vdash \Gamma$ and π reduces to π' by the cut-elimination rules of μLL$_\infty$, then $[\![\pi]\!]_{\mathbf{REL}} = [\![\pi']\!]_{\mathbf{REL}}$.

Soundness of μLL$_\infty$

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Let (π_i) be a Cauchy sequence. Then
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Corollary

If π and π' are proofs of $\vdash \Gamma$ and π reduces to π' by the cut-elimination rules of μLL$_\infty$, then $[\![\pi]\!]_{\mathbf{REL}} = [\![\pi']\!]_{\mathbf{REL}}$.

Theorem

If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!] \in \mathscr{T}[\![\Gamma]\!]$.

Inductive vs circular linear logic proofs

Consider $\text{Unfold} : \mu\text{LL} \to \mu\text{LL}^\infty$, the translation considered earlier which unfolds a finitary proofs into a circular proof.

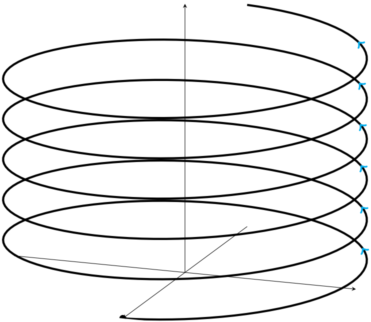
Theorem: Invariance of the semantics by circular unfolding.

Let π be a μLL proof. Then we have $[\![\pi]\!] = [\![\text{Unfold}(\pi)]\!]$ where the interpretation is given in a model $(\mathscr{L}, \overrightarrow{\mathscr{L}})$ of μLL.

Current work:

- Benefit from the inductive structure of circular proofs to define an interpretation inductively.
- When considering strongly valid circular proofs – which can be finitized – will finitization also preserve the semantics?

Circular or Helical reasoning?



Proof theory of least and greatest fixed points

	μMALL/μLL
Proof objects	Finite trees
Inferences	Induction rules
MALL rules +	$\dfrac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F}\ (\mu)$ $\dfrac{\vdash \Gamma, S \quad \vdash S^{\perp}, F[S/X]}{\vdash \Gamma, \nu X.F}\ (\nu)$
Logical correctness	local
Cut-elimination	sort of: (ν) hides a cut
Subformula prop.	NO (if there are ν)
Focusing	$\checkmark$, but μ/ν have arbitrary polarities
Categorical sem.	$\checkmark$
Denotational sem.	$\checkmark$

Proof theory of least and greatest fixed points

	μMALL/μLL
Proof objects	Finite trees
Inferences MALL rules +	Induction rules $\dfrac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F}\ (\mu)$ $\dfrac{\vdash \Gamma, S \quad \vdash S^{\perp}, F[S/X]}{\vdash \Gamma, \nu X.F}\ (\nu)$
Logical correctness	local
Cut-elimination	sort of: (ν) hides a cut
Subformula prop.	NO (if there are ν)
Focusing	$\checkmark$, but μ/ν have arbitrary polarities
Categorical sem. Denotational sem.	$\checkmark$ $\checkmark$

Proof theory of least and greatest fixed points

	μMALL/μLL	μMALL$^\infty$/μLL$^\infty$ + circ. frag.
Proof objects	Finite trees	Non well-founded trees
Inferences	Induction rules	Fixed points unfoldings (+ validity conditions)
MALL rules +	$\dfrac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F}\ (\mu)$ $\dfrac{\vdash \Gamma, S \quad \vdash S^\perp, F[S/X]}{\vdash \Gamma, \nu X.F}\ (\nu)$	$\dfrac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F}\ (\mu)$ $\dfrac{\vdash \Gamma, F[\nu X.F/X]}{\vdash \Gamma, \nu X.F}\ (\nu)$
Logical correctness	local	global straight/bouncing threads
Cut-elimination	sort of: (ν) hides a cut	✓
Subformula prop.	NO (if there are ν)	✓
Focusing	✓, but μ/ν have arbitrary polarities	✓ μ pos. and ν neg.
Categorical sem.	✓	NO
Denotational sem.	✓	✓

Conclusion

- To sum up:
  - Fixed-point logics with circular or non-wellfounded proofs equipped with a parity condition to discriminate valid/invalid proofs;
  - Syntactic cut elimination for various nwf sequent calculi: $\mu\mathsf{MALL}^\infty$, $\mu\mathsf{LL}^\infty$, $\mu\mathsf{LJ}^\infty$, $\mu\mathsf{LK}^\infty$;
  - Bouncing validity condition with a better management of cuts. (jww Baelde, Doumane & Kuperberg)
- Not covered here:
  - Provability / Phase semantics
  - Infinets
- Ongoing and future work:
  - Relax further the bouncing validity condition; (jww Bauer)
  - More canonical proof-objects (circular natural deduction and circular λ-calculus, proof-nets); (jww De, Pellissier)
  - Provability and denotational semantics of circular proofs; (jww De, Ehrhard and Jafarrahmani)
  - Understand how to interface with other approaches to productivity (guarded recursion, sized types, etc.)?

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Questions?