M2 LMFI – SOFIX
Quantification du second-ordre et points fixes en logique
Second lecture: Second-order logic and arithmetics
(preliminary version of the 16/01/23)

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16th january 2023

Contents

In this chapter, one introduces second-order logique, its proof theory in the form of a natural deduction system as well as the theory of second-order arithmetics

Compared to other logical frameworks considered previously (propositional logic, predicate calculus, etc.), second-order logique distinguishes by allowing statements which are quantified not over elements of the interpretation domain, but over predicates of the domain. In particular, wich unary predicates comes the ability to quantify over sets of elements, with binary predicates comes the ability to quantify over relations, etc. Such an extension brings a fabulous gain of expressiveness... as well as some additional complexities and associated technical issues.

1 Definition of second-order logic

1.1 Second-order formulas

When working with system F, one shall only consider the propositional fragment of second-order logic, when working with MSO, one shall only consider the monadic, that is quantifying only over unary predicates (or sets of elements), but for now, one shall consider the ability to quantify over predicate of any arity, restricted to the fragment with implication and first-order universal quantification:

Definition 1.1 (*Formulas of second-order logic*)

Let us assume given as countable set of first-order variables ($\mathcal{V}^1$), a first-order signature($\mathcal{L}$) and the *associated set of terms* T *defined as usual:*

$$t, u ::= x \mid f(t_1, \dots, t_n)$$

where f *is a function symbol of arity* n*. One also assumes that function symboles of arity* n *are in bijection with primitive recursive functions of* $\mathbb{N}^n \longrightarrow \mathbb{N}$ *(in particular, symbols* $\mathsf{0}$ *and* S *are available).*

One considers a countable set of second-order variables $(X, Y, Z, \dots \in \mathcal{V}^2)$ *structured by arity* $(\mathcal{V}^2 = \cup_{i \in \omega} \mathcal{V}^2_i)$ *and disjoint from* $\mathcal{L}$*. One defines the set of second-order formula by induction as follows:*

$$A, B, C ::= t = u \mid R(t_1, \dots, t_n) \mid X(t_1, \dots, t_n) \mid A \Rightarrow B \mid \forall x.A \mid \forall X.A.$$

with t, u *and the* t_i *being first-order terms,* R *is a n-ary relational symbol from* $\mathcal{L}$*and* $X \in \mathcal{V}^2_n$*. One shall also write* $(t_1, \dots, t_n) \in X$ *in place of* $X(t_1, \dots, t_n)$*.*

Quantifiers bind respectively first- and second-order variables.

1.2 Second-order natural deduction

Definition 1.2 (*Second-order substitution*)

One defines a substitution operation for second-order as follows:

If A *and* B *are second-order formulas,* X *a* n*-ary second-order variable and* $x_1, \dots x_n$ *first-order variables, one writes* $A\{B/X(x_1, \dots, x_n)\}$ *for the formula obtained by replacing, in* A*, every free occurrence of* $X(t_1, \dots, t_n)$ *with* $B\{t_1/x_1, \dots, t_n/x_n\}$*.*

Example 1.3

For instance, $\mathsf{0} \in X\{(\mathsf{S0} = x)/X(x)\} \triangleq (\mathsf{S0} = \mathsf{0})$*.*

One then defines second-order natural deduction (One considers here the formulation of natural deduction as trees of formulas and discharged formulas, not sequent-based natural deduction):

Definition 1.4 (*Second-order natural deduction*)

Second-order natural deduction, NK^2*, is obtained by considering the usual classical natural deduction system for first-order logic,* NK*, over formulas of second-order logique, extended with the two following inference rules:*

$$\frac{A}{\forall X.A}\ (\forall^2 i) \quad (\star)$$

$$\frac{\forall X.A}{A\{B/X(x_1, \dots, x_n)\}}\ (\forall^2 e(B))$$

$(\star)$ *The inference rule* $\forall^2 i$ *can only be applied if* X *does not occur free in the non-discharged hypotheses of the deduction of conclusion* A*.*

As we do not have $\neg$*, the negation connective, nor* $\bot$ *at hand, one shall find another way to express classical reasoning. For this, we shall consider Peirce law in the form of the following axiom:*

$$\frac{}{\forall X.\forall Y.(((X \Rightarrow Y) \Rightarrow X) \Rightarrow X)}\ \mathsf{Peirce}$$

Remark 1.5

Note then it is significant to write explicitely **in the rule lable** *the formula B by which one instanciates the variable in the second-order quantifier elimination rule. We shall come back to this point later.*

Example 1.6

$$\dfrac{\dfrac{[X]^\alpha}{X \Rightarrow X}\ (\Rightarrow i)^\alpha}{\forall X.X \Rightarrow X}\ (\forall\ i)$$

In addition to implicative and first-order cuts, imported from first-order natural deduction, one shall now also consider cuts for second-order universal quantifier:

Definition 1.7 (*Universal cuts and simplification rules for universal cuts*)

A universal cut, d, is a deduction of formula $A\{B/X(x_1, \ldots, x_n)\}$ of the form:

$$d : \dfrac{\dfrac{d' : A}{\forall X.A}\ (\forall^2 i)}{A\{B/X(x_1, \ldots, x_n)\}}\ (\forall^2 e)(B)$$

The reduct/contractum of this cut is the deduction:

$$d'\{B/X(x_1, \ldots, x_n)\} : A\{B/X(x_1, \ldots, x_n)\}$$

where $d'\{B/X(x_1, \ldots, x_n)\}$ represents deduction d' in which every occurrence of variable X in d', of the form $X(t_1, \ldots, t_n)$ has been replaced by formula $B\{t_1/x_1, \ldots, t_n/x_n\}$.

The deduction so obtained is a valid deduction precisely because X does not occur free in the undischarged hypothese of d' and this also explains the need to make explicit the name of formula B in the deduction.)

1.3 Second-order arithmetics

One can now define the theory of second-order arithmétics (written PA_2 in the classical case, and HA_2 in the intuitionistic case).

Definition 1.8 (*Axioms of second-order arithmetics, $\mathsf{HA_2}, \mathsf{PA_2}$*)

One considers the formula $Int(x) \triangleq \forall X.\mathsf{0} \in X \Rightarrow (\forall y.y \in X \Rightarrow \mathsf{S}(y) \in X) \Rightarrow x \in X$.

The theory of second-order arithmetics (called HA_2 or PA_2 depending on whether one considers intuitionistic of classical logic) is given by the following axioms:

- $E_{\mathsf{Refl}} \triangleq \forall x.x = x$
- $E_{\mathsf{Leibniz}} \triangleq \forall x, y.x = y \Rightarrow \forall Z.(x \in Z \Rightarrow y \in Z)$
- $P_4 \triangleq \forall x.\mathsf{S}(x) \neq \mathsf{0}$
- $I \triangleq \forall x.Int(x)$
- *as well as, for every symbol of primitive recursive function, a universally closed formula expressing the definition of this function, this will be noted E_{PrimRec}, the primitive recursive axiom schema.*

(In fact, for axiom P_4, our restricted syntax forces us to consider the following formula: $\forall x.(\mathsf{S}(x) = \mathsf{0} \Rightarrow \forall X.X)$)

Example 1.9

The following formulas are two examples of axioms in E_{PrimRec}:

- $\forall x.\mathsf{Pred}(\mathsf{S}(x)) = x$;
- $(\forall x.x + 0 = x) \wedge (\forall x.\forall y.x + \mathsf{S}(y) = \mathsf{S}(x + y))$.

Proposition 1.10

The following formulas which are axioms of first-order arithmetics, are now provable from the other axioms:

- $P_3 \triangleq \forall x.\forall y.\mathsf{S}(x) = \mathsf{S}(y) \Rightarrow x = y$;
- $E_{\mathsf{Sym}} \triangleq \forall x.\forall y.x = y \Rightarrow y = x$;
- $E_{\mathsf{Trans}} \triangleq \forall x.\forall y.\forall z.x = y \Rightarrow (y = z \Rightarrow x = z)$;
- $E^{u}_{\mathsf{Subst}} \triangleq \forall x.\forall y.x = y \Rightarrow u\{x/z\} = u\{y/z\}$, *for any term* u.

Proof: One proves that P_3 is provable in PA_2:

$$
\dfrac{\dfrac{\dfrac{E_{\mathsf{Trans}} \quad \dfrac{E_{\mathsf{Sym}} \quad \dfrac{\forall z.\mathsf{Pred}(\mathsf{S}(z)) = z}{\mathsf{Pred}(\mathsf{S}(x)) = x}\,(\forall^1 e)}{x = \mathsf{Pred}(\mathsf{S}(x))}\,(\Rightarrow e) \quad \dfrac{E_{\mathsf{Trans}} \quad \dfrac{\forall z.\mathsf{Pred}(\mathsf{S}(z)) = z}{\mathsf{Pred}(\mathsf{S}(y)) = y}\,(\forall^1 e) \quad \dfrac{\dfrac{E^{\mathsf{Pred}(\mathsf{S}(z))}_{\mathsf{Subst}}}{\mathsf{S}(x) = \mathsf{S}(y) \Rightarrow \mathsf{Pred}(\mathsf{S}(x)) = \mathsf{Pred}(\mathsf{S}(y))}\,(\forall^1 e)^2 \quad [\mathsf{S}(x) = \mathsf{S}(y)]^{\alpha}}{\mathsf{Pred}(\mathsf{S}(x)) = \mathsf{Pred}(\mathsf{S}(y))}\,(\Rightarrow e)}{\mathsf{Pred}(\mathsf{S}(x)) = y}\,(\Rightarrow e)}{x = y}\,(\Rightarrow e)^2}{\mathsf{S}(x) = \mathsf{S}(y) \Rightarrow x = y}\,(\Rightarrow i)^{\alpha}}{\forall x.\forall y.\mathsf{S}(x) = \mathsf{S}(y) \Rightarrow x = y}\,(\forall^1 i)^2
$$

The other axioms are left as exercises.

□

Exercise 1.1

Déduire dans NK_2 les formules suivantes: E_{Sym}, E_{Trans}, E^{u}_{Sub}.

Proposition 1.11

In the same way, the following rule is derivable in PA_2:

$$
\dfrac{t = u \quad A\{t/x\}}{A\{u/x\}}\,(= e)
$$

Proof: Equality elimination is actually derivable in PA_2 (as well as its symmetric rule):

$$
\dfrac{\dfrac{\dfrac{\dfrac{E_{\mathsf{Leibniz}}}{t = u \Rightarrow (\forall Z.Z(t) \Rightarrow Z(u))}\,(\forall^1 e)^2 \quad t = u}{\forall Z.Z(t) \Rightarrow Z(u)}\,(\Rightarrow e)}{A\{t/x\} \Rightarrow A\{u/x\}}\,(\forall^2 e) \quad A\{t/x\}}{A\{u/x\}}\,(\Rightarrow e)
$$

□

2 Definability of other logical connectives

Second-order logic is built from a very constraint grammar of formulas as only implications and universal quantifications are available. In fact, all other connectives are definable from those, including negation. Even though it may not seem surprising from the point of view of classical logic where connectives were already definable from a small basis of connectives, it is a striking novelty from the point of view of intuitinistic logic in which connectives were not interdefinable. One shall see that this definability property is actually quite powerful and very fine-grained in the sense that not only are the connectives interdefinable, but together with there definition comes the usual inference rules. Moreover, the dynamics of cut-elimination through the second-order encoding of the connectives is indeed the expected dynamics: this interdefinability is meaningful at the level of formulas, but also at the level of proofs (and therefore of programs built from this logical framework) with the exception of commutative cuts.

In the remaining of this section, on consider the implicative fragment of second-order propositional logic for simplicity, because this corresponds to system F and because the result is more precise stated in this way and transfer to classical logic of course. The omission of first-order quantifier is simply to make things simpler as the results also holds in this case.

Let us firt recall the elimination and cut-reduction rules associated with second-order quantifiers (for implication, it is simply the usual proof transformation from natural deduction):

Let d be a universal cut, that is a deduction of formula $A\{B/X\}$ of the form:

$$d : \dfrac{\dfrac{d' : A}{\forall X.A}\,(\forall^2 i)}{A\{B/X\}}\,(\forall^2 e)(B)$$

The reduct/contractum of this cut is the deduction:

$$d'\{B/X\} : A\{B/X\}$$

where $d'\{B/X\}$ represents deduction d' in which every occurrence of variable X has been replaced by formula B. The deduction so obtained is a valid deduction precisely because X does not occur free in the undischarged hypothese of d' and this also explains the need to make explicit the name of formula B in the deduction.)

One can now define the various connectives as well as their second-order inferences.

Definition 2.1 (*Second-order encoding of logical connectives*)

One defines the following formulas:

- $\bot \triangleq \forall X.X$;
- $A \wedge B \triangleq \forall X.(A \Rightarrow (B \Rightarrow X)) \Rightarrow X$;
- $A \vee B \triangleq \forall X.(A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X$;
- $\exists X.A \triangleq \forall Y.(\forall X.(A \Rightarrow Y)) \Rightarrow Y$;
- $\exists x.A \triangleq \forall Y.(\forall x.(A \Rightarrow Y)) \Rightarrow Y$.

The encodings given above admit the expected inference rules, which are derivable in NJ_2 (and also in NK_2 of course):

Proof: • The elimination rule for the absurdity is definable:

$$\dfrac{\bot}{A}\ (\forall^2 e(A))$$

One thus has intuitionistic logic for free from minimal logic.

- Introduction and elimination rules for conjunctions are derivable:

$$\dfrac{\dfrac{d : A \wedge B}{(A \Rightarrow (B \Rightarrow A)) \Rightarrow A}\ (\forall^2 e(A) \qquad \dfrac{\dfrac{[A]^\alpha}{B \Rightarrow A}\ (\Rightarrow i)^\beta}{A \Rightarrow (B \Rightarrow A)}\ (\Rightarrow i)^\alpha}{A}\ (\Rightarrow e)$$

$$\dfrac{\dfrac{d : A \wedge B}{(A \Rightarrow (B \Rightarrow B)) \Rightarrow B}\ (\forall^2 e(B)) \qquad \dfrac{\dfrac{[B]^\beta}{B \Rightarrow B}\ (\Rightarrow i)^\beta}{A \Rightarrow (B \Rightarrow B)}\ (\Rightarrow i)^\alpha}{B}\ (\Rightarrow e)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[A \Rightarrow (B \Rightarrow X)]^\alpha \quad d_A : A}{B \Rightarrow X}\ (\Rightarrow e) \qquad d_B : B}{X}\ (\Rightarrow e)}{(A \Rightarrow (B \Rightarrow X)) \Rightarrow X}\ (\Rightarrow i)^\alpha}{A \wedge B}\ (\forall^2 i)$$

- Introduction and elimination rules for disjunction are derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{[A \Rightarrow X]^\alpha \quad d_A : A}{X}\ (\Rightarrow e)}{(B \Rightarrow X) \Rightarrow X}\ (\Rightarrow i)^\beta}{(A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X}\ (\Rightarrow i)^\alpha}{A \vee B}\ (\forall^2 i)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[B \Rightarrow X]^\beta \quad d_B : B}{X}\ (\Rightarrow e)}{(B \Rightarrow X) \Rightarrow X}\ (\Rightarrow i)^\beta}{(A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X}\ (\Rightarrow i)^\alpha}{A \vee B}\ (\forall^2 i)$$

$$\dfrac{\dfrac{\dfrac{d : A \vee B}{(A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C}\ (\forall^2 e(C)) \qquad \dfrac{\begin{matrix}[A]^\alpha \ldots [A]^\alpha \\ \vdots\ d_A \\ C\end{matrix}}{A \Rightarrow C}\ (\Rightarrow i)^\alpha}{(B \Rightarrow C) \Rightarrow C}\ (\Rightarrow e) \qquad \dfrac{\begin{matrix}[A]^\beta \ldots [A]^\beta \\ \vdots\ d_B \\ C\end{matrix}}{B \Rightarrow C}\ (\Rightarrow i)^\beta}{C}\ (\Rightarrow e)$$

The other connectives are left to the reader.

□

Proposition 2.2

Through the second-order encoding, cuts on $\wedge, \vee, \exists$ can be reduced as expected.

Proof: Left as exercise.

□