SOFIX 2023:

Non-wellfounded, linear proof-theory of the $\mu$-calculus: on the virtues of circular reasoning

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SOFIX lecture – March 22nd, 2024
1 Introduction
   - Infinite descent
   - Circular LL
   - On the non-wellfounded proof-theory of fixed-point logics

2 \( \mu \text{LL} \): Least and greatest fixed-points in LL
   - \( \mu \text{MALL} \) & \( \mu \text{LL} \) languages and finitary proof systems
   - \( \mu \text{LL} \) cut-reduction

3 \( \mu \text{LL}^\infty \): circular and non-wellfounded proofs for \( \mu \text{LL} \)
   - Non-wellfounded proof system \( \mu \text{LL}^\infty \)
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Introduction
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- Circular LL
- On the non-wellfounded proof-theory of fixed-point logics

μLL: Least and greatest fixed-points in LL
- μMALL & μLL languages and finitary proof systems
- μLL cut-reduction

μLL∞: circular and non-wellfounded proofs for μLL
- Non-wellfounded proof system μLL∞
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Cut-elimination for circular and non-wellfounded proofs
- μMALL∞ Cut elimination
- μLL∞ Cut elimination

Conclusion
Proposition 31

Any composite number is measured by some prime number.

Let \(A\) be a composite number;
I say that \(A\) is measured by some prime number.

For, since \(A\) is composite,
some number will measure it.
Let a number measure it, and let it be \(B\).
Now, if \(B\) is prime, what was enjoined will have been done.
But if it is composite, some number will measure it.
Let a number measure it, and let it be \(C\).
Then, since \(C\) measures \(B\),
and \(B\) measures \(A\),
therefore \(C\) also measures \(A\).
And, if \(C\) is prime, what was enjoined will have been done.
But if it is composite, some number will measure it.
Thus, if the investigation be continued in this way, some prime number will be found which will measure the number before it, which will also measure \(A\).
For, if it is not found, an infinite series of numbers will measure the number \(A\), each of which is less than the other:
which is impossible in numbers.

Therefore some prime number will be found which will measure the one before it, which will also measure \(A\).
Therefore any composite number is measured by some prime number.

Q. E. D.
PROPOSITION 31

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For, since $A$ is composite,
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Therefore any composite number is measured by some prime number.
Q. E. D.
Infinite descent and Circular proofs
An old mathematical story: Fermat identifies a powerful heuristics
Pierre de Fermat studied in depth infinite descent and used it extensively. See letter of August 1659 to Carcavaci where Fermat listed 10 theorems he “proved” using infinite descent:

1  Aucun nombre de la forme, moindre de l’unité qu’un multiple de 3, ne peut être composé d’un carré et du triple d’un autre carré.

2  Aucun triangle rectangle en nombres n’a une aire carrée.

3  Tout nombre premier qui surpasse de l’unité un multiple de 4 est somme de deux carrés.

(...)

9  Toutes les puissances carrées de 2, augmentées de 1, sont des nombres premiers.

10  Il n’y a que 1 et 7 qui sont moindres de 1 qu’un double carré et aient un carré de même nature.
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(...)

9. *Toutes les puissances carrées de 2, augmentées de 1, sont des nombres premiers.*

10. *Il n’y a que 1 et 7 qui sont moindres de 1 qu’un double carré et aient un carré de même nature.*

... even for proving wrong statements! Property 9 asserts that every Fermat number is prime, which was later disproved by Euler who factorized $F_5 = 2^{25} + 1$ as $641 \times 6,700,417$. 
Infinite descent and Circular proofs

For any integer \( m \), \( \sqrt{m} \) is either an integer, or irrational.

Proof

Let \( m \in \mathbb{N} \) and for the sake of contradiction, assume \( \sqrt{m} \in \mathbb{Q} \setminus \mathbb{N} \).

1. Choose \( a_0, b_0, q \in \mathbb{N} \) st. \( \sqrt{m} = a_0/b_0 \) and \( q < \sqrt{m} < q + 1 \).
2. One has \( b_0\sqrt{m} = a_0 \in \mathbb{N} \) and \( a_0\sqrt{m} = mb_0 \in \mathbb{N} \).
3. Therefore by setting \( a_1 \triangleq b_0m - a_0q = a_0(\sqrt{m} - q) \) and \( b_1 \triangleq a_0 - b_0q = b_0(\sqrt{m} - q) \), we have:
   - \( a_1, b_1 \) are integers,
   - \( 0 < a_1 < a_0, \ 0 < b_1 < b_0 \) and
   - \( \sqrt{m} = a_1/b_1 \).
4. In a similar way, one can build \((a_i)_{i \in \mathbb{N}} \) and \((b_i)_{i \in \mathbb{N}} \) infinite sequences of integers, “each of which is less than the other”. This is impossible.
5. Therefore \( \sqrt{m} \) is either integer or irrational. \( \square \)
Infinite descent and Circular proofs
Towards sequent calculus: Irrationality of $\sqrt{2}$

\[
\begin{array}{c}
0 < x_0, x_0^2 = 2x_1^2 \vdash \\
0 < x_1 < x_0 \land \\
\exists x_2. x_0 = 2x_2 \\
\hline
0 < x_1, x_1^2 = 2x_2^2 \vdash \\
0 < x_2 < x_1 \land \\
\exists x_3. x_1 = 2x_3 \\
\hline
0 < x_2, x_2^2 = 2x_3^2 \vdash \\
0 < x_3 < x_2 \land \\
\exists x_4. x_2 = 2x_4 \\
\hline
0 < x_3, x_4^2 = 2x_4^2 \vdash \\
x_3 < x_2, 0 < x_3, 4x_4^2 = 2x_3^2 \vdash \\
x_2 < x_1, 0 < x_2, 4x_3^2 = 2x_2^2 \vdash \\
x_1 < x_0, 0 < x_1, 4x_2^2 = 2x_1^2 \vdash \\
0 < x_0, x_0^2 = 2x_1^2, 0 < x_1 < x_0 \land \exists x_2. x_0 = 2x_2 \vdash \\
\hline
0 < x_0, x_0^2 = 2x_1^2 \vdash 
\end{array}
\]
Infinite descent and Circular proofs

Inductive and coinductive cases

Inductive case:

\[ \vdash \text{nat } 0 \]
\[ \vdash \text{even } y \rightarrow \text{nat } y \]
\[ \vdash \text{even } y \rightarrow \text{nat } (s \ y) \]
\[ \vdash \text{nat } 0 \rightarrow \text{even } y \rightarrow \text{nat } (s (s \ y)) \]
\[ \vdash \text{even } x \vdash \text{nat } x \]

The infinite branch unfolds the inductive predicate even infinitely often on the left: valid!
Infinite descent and Circular proofs

Inductive and coinductive cases

**Inductive case:**

\[
\begin{align*}
\text{even } y & \vdash \text{nat } y \\
\text{even } y & \vdash \text{nat } (s \, y) \\
\text{nat } 0 & \vdash \text{even } y \vdash \text{nat } (s \, (s \, y)) \\
\text{even } x & \vdash \text{nat } x
\end{align*}
\]

The infinite branch unfolds the *inductive* predicate `even` infinitely often on the left: valid!

**Coinductive case:**

\[
\begin{align*}
\text{step } p \, \alpha \, q & \vdash \text{step } p \, \alpha \, q \quad \vdash \text{sim } q \, q \\
\text{step } p \, \alpha \, q & \vdash \text{step } p \, \alpha \, q \land \text{sim } q \, q \\
\forall \alpha \forall q. \quad \text{step } p \, \alpha \, q & \supset \exists q'. \quad \text{step } p \, \alpha \, q' \land \text{sim } q \, q'
\end{align*}
\]

\[
\vdash \text{sim } p \, p
\]

The infinite branch unfolds the *coinductive* predicate `sim` infinitely often on the right: valid!
Mixing inductive and coinductive definitions

A matter of priority

\[ p \triangleq \text{ind} \ q \qquad q \triangleq \text{coind} \ p \]
Mixing inductive and coinductive definitions

A matter of priority

\[ p \triangleq \text{ind} \]
\[ q \triangleq \text{coind} \]

Choose *which matters most* between \( p \) and \( q \):

<table>
<thead>
<tr>
<th></th>
<th>( p \prec q )</th>
<th>( q \prec p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( \mu X. \nu Y.X )</td>
<td>( \mu X.q )</td>
</tr>
<tr>
<td>( q )</td>
<td>( \nu Y.p )</td>
<td>( \nu Y.\mu X.Y )</td>
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**5 Conclusion**
Circular LL?
In which case is it safe to allow infinite branches in a LL proof?

1. Applying infinitely many MALL rules?

2. Applying infinitely many cut rules?

3. Applying infinitely many structural rules?

Or...
Circular LL?
In which case is it safe to allow infinite branches in a LL proof?

1. Applying infinitely many MALL rules?
   No deductive progress

2. Applying infinitely many cut rules?

3. Applying infinitely many structural rules?
   Or...

Impossible! They strictly reduce the size of the sequent.
The length of a (cut-free) branch is bounded by the size of the conclusion sequent.
In which case is it safe to allow infinite branches in a LL proof?

1. Applying infinitely many MALL rules?
   No deductive progress

2. Applying infinitely many cut rules?
   No deductive progress

3. Applying infinitely many structural rules?

Or...

What the Tortoise said to Achilles
(1895, L. Carroll), revisited by J-YG:

Achilles’ goal: proving $A \multimap B, A \vdash B$

The Tortoise rejects $(\multimap_L)$ but accepts all the $T_i, i \geq 2$:

$T_0 \triangleq A$
$T_{k+1} \triangleq (\bigotimes_{i=0}^{k} T_i) \multimap B$

<table>
<thead>
<tr>
<th>$T_2$</th>
<th>$T_0, T_1, T_2 \vdash B$</th>
<th>(Cut)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_3$</td>
<td>$T_0, \ldots, T_3 \vdash B$</td>
<td>(Cut)</td>
</tr>
<tr>
<td>$T_4$</td>
<td>$T_0, \ldots, T_4 \vdash B$</td>
<td>(Cut)</td>
</tr>
<tr>
<td>$T_5$</td>
<td>$T_0, \ldots, T_5 \vdash B$</td>
<td>(Cut)</td>
</tr>
<tr>
<td>\vdots</td>
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Circular LL
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Or...

**Infinite structural trees:**

\[ ?F, ?F, ?F \]
\[ \vdash ?F, ?F \]
\[ ?F \]

\[ ?F \]
\[ \vdash ?F, ?F \]
\[ ?F \]

\[ ?w \]
\[ \vdash ?F, ?F \]
\[ ?c \]
Circular LL  ![image]
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4. A non-uniform promotion

A promotion must react to any (finite) structural tree.
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\[ \vdash F, ?\Delta, !F, ?\Delta, !F, ?\Delta \]

\[ \vdash !F, ?\Delta \]  \((!p_{nu})\)
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\[\Gamma, \Delta \vdash F, \Delta \quad \Gamma, \Delta \vdash F, \Delta \quad \Gamma, \Delta \vdash F, \Delta\]

\[\vdash \Gamma, \Delta \vdash F, \Delta (\text{Cut})\]

\[\pi \quad \Gamma \vdash F, \Delta \quad \Gamma \vdash F, \Delta \quad \Gamma \vdash F, \Delta\]

\[\vdash \Gamma, \Delta \vdash F, \Delta (\text{Cut})\]

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\hline
\vdash !F, ?\Delta
\end{align*}
\]

(!p^nu)

Usual promotion is derivable as:

\[
\begin{align*}
\vdash F, ?\Delta & \quad \vdash !F, ?\Delta & \quad \vdash !F, ?\Delta \\
\hline
\vdash !F, ?\Delta
\end{align*}
\]

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Condition to admit a nwf branch:
a !-formula occurrence must be principal infinitely often along the branch.
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5. \( \mu LL^\infty \)

\( \mu LL^\infty \):

In the following, we will extend the language of LL formulas to admit sort of “infinite” formulas, defined by fixed-point constructions:

\[ \mu X.F, \nu X.F. \]

In some cases (use of an inductive hypothesis, production of a coinductive conclusion), one can allow non-wellfounded branches.
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- μMALL\textsuperscript{∞} Cut elimination
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Conclusion
Circular & non-wellfounded proofs in the literature

- **As verification device**: Complete deduction system giving algorithms for checking validity (Tableaux, sequent calculi),
  
  \[ \text{Success} \rightarrow \text{Validity} \]

  \[\begin{align*}
  \mu\text{-calculus formula} & \rightarrow \text{Proof search} \\
  \text{Failure} & \rightarrow \text{Invalidity}
  \end{align*}\]

- **Completeness arguments**: Intermediate objects between syntax and semantics for modal \(\mu\)-calculus (Kozen, Kaivola, Walukiewicz)

  \[\begin{align*}
  \mu\text{-calculus formula} & \rightarrow \text{Circular proof} \\
  & \rightarrow \text{Finite axiomatization}
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Circular & non-wellfounded proofs in the litterature

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\( \mu \)-calculus formula \( \rightarrow \) Circular proof \( \rightarrow \) Finite axiomatization

- **But rarely as proof–program objects in themselves**:
  - pioneering works by Santocanale; Studer; Brotherston & Simpson; Dax, Hoffman & Lange.
  - develop such a proof-theoretical study, from a Curry-Howard perspective: study the dynamics of cut-elimination.
Circular & non-wellfounded proofs in the literature

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  - develop such a proof-theoretical study, from a Curry-Howard perspective: study the dynamics of cut-elimination.

- Recently, development of numerous circular/cyclic proof systems (Afshari & Leigh, Das, Doumane & Pous, Cohen & Rowe, Tatsuta et al. etc.)
Proof theory of fixed-point logics

- Various deductive frameworks for (co)inductive reasoning (Martin-Löf’s inductive definitions, \( \mu \)-calculi, ...), suitable to represent and reason about (co)inductive data structures.

- **Structural proof-theory, Curry-Howard-oriented**: not only to express statements and their provability relation, but stressing the proof objects themselves, in particular in the substructural setting.

- **LL with fixed points**, considered with proofs as finite trees (\( \mu \LL \)) or proofs as infinite, non-wellfounded trees (\( \mu \LL^\infty \)) with a special fragment of interest, circular proofs.
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- **LL with fixed points**, considered with proofs as finite trees ($\mu LL$) or proofs as infinite, non-wellfounded trees ($\mu LL^\infty$) with a special fragment of interest, circular proofs.

- Can we extend the proof–program correspondence to circular proofs?

- E.g., in Coq proof assistant, syntactic productivity conditions are required: assert progress after every step. Many productive programs are rejected by Coq type-checker.

![All programs

Productive programs

Programs accepted by Coq](image)
Some valid and invalid definitions

CoInductive stream := Cons : (nat * stream) → stream.
CoInductive bstream := BCons : (bool * bstream) → bstream.

Definition hdinc (s: stream) : stream := match s with
  | Cons (a, s') ⇒ Cons (S a, s') end.
CoFixpoint enum (n:nat) : stream := Cons (n, (enum (S n))).
CoFixpoint drop (s : stream) : stream := match s with
  | Cons (a, Cons (b, s')) ⇒ Cons (b, (drop s')) end.
CoFixpoint incdrop (s : stream) : stream := match s with
  | Cons (a, Cons (b, s')) ⇒ hdinc (Cons (b, incdrop s')) end.

Definition hdneg (s: bstream) : bstream := match s with
  | BCons (a, s') ⇒ BCons (negb a, s') end.
CoFixpoint filter1everyk (m : nat) (s : bstream) :
  bstream := match (m,s) with
  | (0, BCons (a, s')) ⇒ BCons (a, filter1everyk k s')
  | (S m', BCons (a, s')) ⇒ hdneg (filter1everyk m' s') end.
Aim of this talk

- Our goal: investigate productivity conditions which are proof-theoretically grounded, by considering circular and non-wellfounded linear proofs in $\mu$-calculi.
- Ideally, accept such proof objects:
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- Our goal: investigate productivity conditions which are proof-theoretically grounded, by considering circular and non-wellfounded linear proofs in $\mu$-calculi.
- Ideally, accept such proof objects:

- Report on some progress in designing more flexible validity conditions for circular and non-wellfounded proofs in linear logic with fixed-points as well as some proof invariants.
- Based on joint works with Baelde, Bauer, Chardonnet, Das, De, Doumane, Ehrhard, Jaber, Jafarrahmani, Kuperberg, Nollet, Pellissier and Tasson.
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- $\mu$MALL & $\mu$LL languages and finitary proof systems
- $\mu$LL cut-reduction

3. $\mu$LL$^\infty$: circular and non-wellfounded proofs for $\mu$LL

- Non-wellfounded proof system $\mu$LL$^\infty$
- Validity condition
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4. Cut-elimination for circular and non-wellfounded proofs

- $\mu$MALL$^\infty$ Cut elimination
- $\mu$LL$^\infty$ Cut elimination

5. Conclusion
μLL formulas and sequent calculus
(Baelde & Miller 2007, Baelde 2012)

μLL formulas

LL formula grammar extended with fixed-point constructs:

\[ F ::= \ldots | X | \mu X.F | \nu X.F \]

- \( \mu \) and \( \nu \) are binders, consider closed formulas only.
- \( \mu \) and \( \nu \) are dual.
- One-sided sequents: \( \vdash A_1, \ldots, A_n \). 

\( (\Gamma \vdash \Delta \text{ is a short for } \vdash \Gamma^\perp, \Delta) \)

Ex: \((\nu X.X \otimes X) \perp = \mu X.X \bowtie X\).
\[ \mu LL \text{ formulas and sequent calculus} \]

(Baelde & Miller 2007, Baelde 2012)

\[ \mu LL \text{ formulas} \]

LL formula grammar extended with fixed-point constructs:

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- \( \mu \) and \( \nu \) are dual. \( (\nu X. X \otimes X) \perp = \mu X. X \bowtie X. \)
- One-sided sequents: \( \Gamma \vdash A_1, \ldots, A_n. \) (\( \Gamma \vdash \Delta \) is a short for \( \vdash \Gamma \perp, \Delta \))
- Data types encodings:
  \[
  \begin{align*}
  \text{Nat} & \triangleq \mu X. 1 \oplus X & \text{Nat} \perp & = \nu X. \perp \& X \\
  \text{List}(A) & \triangleq \mu X. 1 \oplus (A \otimes X) & \text{List}(A) \perp & = \nu X. \perp \& (A \perp \bowtie X) \\
  \text{Stream}(A) & \triangleq \nu X. 1 \& (A \otimes X) & \text{Stream}(A) \perp & = \mu X. \perp \oplus (A \perp \bowtie X)
  \end{align*}
  \]
**µLL formulas and sequent calculus**  
(Baelde & Miller 2007, Baelde 2012)

**µLL formulas**

LL formula grammar extended with fixed-point constructs:

\[ F ::= \ldots \mid X \mid \mu X . F \mid \nu X . F \]

- \( \mu \) and \( \nu \) are binders, consider closed formulas only.
- \( \mu \) and \( \nu \) are dual. Ex: \((\nu X . X \otimes X) \perp = \mu X . X \bowtie X)\).
- One-sided sequents: \( \vdash A_1, \ldots, A_n \) (\( \Gamma \vdash \Delta \) is a short for \( \vdash \Gamma \perp \), \( \Delta \))
- Data types encodings:
  
  \[
  \begin{align*}
  \text{Nat} & \triangleq \mu X . 1 \oplus X & \text{Nat}^\perp & = \nu X . \perp \& X \\
  \text{List}(A) & \triangleq \mu X . 1 \oplus (A \otimes X) & \text{List}(A)^\perp & = \nu X . \perp \& (A^\perp \bowtie X) \\
  \text{Stream}(A) & \triangleq \nu X . 1 \& (A \otimes X) & \text{Stream}(A)^\perp & = \mu X . \perp \oplus (A^\perp \bowtie X)
  \end{align*}
  \]

**µLL Sequent Calculus**

- LL inference rules together with
- Inference rules for \( \mu \) and \( \nu \)  

\( \Rightarrow \) See following slides
Knaster-Tarski fixed-point theorem

Let $C$ be a complete lattice and $F$ a monotonic operator on $C$.

**Theorem**

$F$ has a **least** F.P. $\mu F$.

$\mu F$: **least prefixed**-point:
- $F(\mu F) \subseteq \mu F$ and
- $\forall S, F(S) \subseteq S \Rightarrow \mu F \subseteq S$.

**Theorem**

$F$ has a **greatest** F.P. $\nu F$.

$\nu F$ **greatest postfixed**-point:
- $\nu F \subseteq F(\nu F)$ and
- $\forall S, S \subseteq F(S) \Rightarrow S \subseteq \nu F$. 
Knaster-Tarski fixed-point theorem

Let $C$ be a complete lattice and $F$ a monotonic operator on $C$.

**Theorem**

$F$ has a **least** F.P. $\mu F$.

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- $F(\mu F) \subseteq \mu F$ and
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**Proof by induction:**

To prove that $\mu F \subseteq P$, it is sufficient to find some $S \subseteq P$ and to prove that $\forall x \in F(S), x \in S$.

**Theorem**

$F$ has a **greatest** F.P. $\nu F$.

$\nu F$ **greatest postfixed**-point:

- $\nu F \subseteq F(\nu F)$ and
- $\forall S, S \subseteq F(S) \Rightarrow S \subseteq \nu F$.

**Proof by coinduction:**

To prove that $P \subseteq \nu F$, it is sufficient to find some $S \supseteq P$ and to prove that $\forall x \in S, x \in F(S)$. 
Knaster-Tarski fixed-point theorem

Let $C$ be a complete lattice and $F$ a monotonic operator on $C$.

**Theorem**

$F$ has a **least** F.P. $\mu F$.

$\mu F$: **least prefixed**-point:

- $F(\mu F) \sqsubseteq \mu F$ and
- $\forall S, F(S) \sqsubseteq S \Rightarrow \mu F \sqsubseteq S$.

**Proof by induction:**

To prove that $\mu F \subseteq P$, it is sufficient to find some $S \subseteq P$ and to prove that $\forall x \in F(S), x \in S$.

$$
\begin{align*}
H \vdash F[\mu X.F/X] & \quad (\mu_r) \\
H \vdash \mu X.F & \quad \mu X.F \vdash S \quad (\mu_l)
\end{align*}
$$

**Theorem**

$F$ has a **greatest** F.P. $\nu F$.

$\nu F$ **greatest postfixed**-point:

- $\nu F \sqsubseteq F(\nu F)$ and
- $\forall S, S \sqsubseteq F(S) \Rightarrow S \sqsubseteq \nu F$.

**Proof by coinduction:**

To prove that $P \subseteq \nu F$, it is sufficient to find some $S \supseteq P$ and to prove that $\forall x \in S, x \in F(S)$.

$$
\begin{align*}
F[\nu X.F/X] \vdash H & \quad (\nu_l) \\
\nu X.F \vdash H & \quad S \vdash F[S/X] \quad (\nu_r)
\end{align*}
$$
Inferences for fixed-points

- One-sided version
- The inferences of the previous slides do not have cut-elimination:

\[
\begin{align*}
\Gamma, 0, \top \quad & \vdash \top \\
\Gamma, 0, \top \quad & \vdash 0, \nu X.X \\
\hline
\Gamma, 0, 0, \nu X.X \\
\end{align*}
\]

(Cut)

- Consider branching \(\nu\)-rule:

\[
\begin{align*}
\Gamma, S \quad & \vdash S^\bot, F[S/X] \\
\hline
\Gamma, \nu X.F
\end{align*}
\]

(\(\nu\))
Inferences for fixed-points

- One-sided version

- The inferences of the previous slides do not have cut-elimination:

\[
\begin{align*}
\Gamma &\vdash 0, \top \\
\Gamma &\vdash 0, 0, \top \\
\Gamma &\vdash 0, 0, \nu X.X \\
\Gamma &\vdash \nu X.X
\end{align*}
\]

Consider branching \(\nu\)-rule:

\[
\begin{align*}
\Gamma, S &\vdash S^\perp, F[S/X], \Delta \\
\Gamma, \nu X.F, \Delta &\vdash \Gamma, \nu X.F, \Delta
\end{align*}
\]
Inferences for fixed-points

- One-sided version

The inferences of the previous slides do not have cut-elimination:

\[ \vdash 0, 0, \top \quad (\top) \]
\[ \vdash 0, \top \]
\[ \vdash 0, 0, \nu X.X \quad (\text{Cut}) \]
\[ \vdash 0, 0, \nu X.X \]

- Consider branching \( \nu \)-rule:

\[ \vdash \Gamma, S \quad \vdash S^\bot, F[S/X] \]
\[ \vdash \Gamma, \nu X.F \quad (\nu) \]

- Cut-elimination holds in \( \mu \text{MALL} \) (Baelde, 2012).
1 Introduction
- Infinite descent
- Circular LL
- On the non-wellfounded proof-theory of fixed-point logics

2 µLL: Least and greatest fixed-points in LL
- µMALL & µLL languages and finitary proof systems
- µLL cut-reduction

3 µLL∞: circular and non-wellfounded proofs for µLL
- Non-wellfounded proof system µLL∞
- Validity condition
- Decidability of the validity condition
- Expressiveness of circular proofs
- µLL∞ focusing

4 Cut-elimination for circular and non-wellfounded proofs
- µMALL∞ Cut elimination
- µLL∞ Cut elimination

5 Conclusion
The key step $(\mu) - (\nu)$ in $\mu$LL cut-elimination

Lemma: Functoriality

For any $\mu$LL pre-formula $F$ with one free fixed-point variable,

$$
\vdash \Delta, A, B \quad (\exists F) \quad \text{is cut-free derivable in } \mu\text{LL.}
$$

(\text{By induction on the maximal depth of free occurrences of } X \text{ in } F.)
The key step $(\mu) - (\nu)$ in $\mu LL$ cut-elimination

Lemma: Functoriality

For any $\mu LL$ pre-formula $F$ with one free fixed-point variable,

$$
\vdash \Delta, A, B \\
\vdash \Delta, F \perp [A/X], F [B/X]
$$

is cut-free derivable in $\mu LL$.

(By induction on the maximal depth of free occurrences of $X$ in $F$.)

Key $(\mu)/(\nu)$ cut-reduction case (slightly simplified):

$$
\begin{array}{c}
\frac{\Pi}{\vdash \Delta, S, \nu X. (N \perp)} \\
\frac{\Theta}{\vdash \Delta, S, N \perp [S \perp / X]} \\
\vdash \Delta, S \perp
\end{array}
\quad \rightarrow \\
\begin{array}{c}
\frac{\Pi}{\vdash \Delta, S, \nu X. (N \perp)} \\
\frac{\Theta}{\vdash \Delta, S, N \perp [S \perp / X]} \\
\vdash \Delta, S \perp
\end{array}
$$

$$
\begin{array}{c}
\frac{\Pi}{\vdash \Delta, S, \nu X. (N \perp)} \\
\frac{\Theta}{\vdash \Delta, S, N \perp [S \perp / X]} \\
\vdash \Delta, S \perp
\end{array}
\quad \rightarrow \\
\begin{array}{c}
\frac{\Pi}{\vdash \Delta, S, \nu X. (N \perp)} \\
\frac{\Theta}{\vdash \Delta, S, N \perp [S \perp / X]} \\
\vdash \Delta, S \perp
\end{array}
$$

$\vdash \Delta, ?\Delta, S \perp$
1 Introduction
- Infinite descent
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- On the non-wellfounded proof-theory of fixed-point logics

2 $\mu LL$: Least and greatest fixed-points in LL
- $\mu MALL$ & $\mu LL$ languages and finitary proof systems
- $\mu LL$ cut-reduction

3 $\mu LL^\omega$: circular and non-wellfounded proofs for $\mu LL$
- Non-wellfounded proof system $\mu LL^\omega$
- Validity condition
- Decidability of the validity condition
- Expressiveness of circular proofs
- $\mu LL^\omega$ focusing

4 Cut-elimination for circular and non-wellfounded proofs
- $\mu MALL^\omega$ Cut elimination
- $\mu LL^\omega$ Cut elimination

5 Conclusion
Non-Wellfounded Sequent Calculus

Consider your favourite logic $\mathcal{L}$ & add fixed-points as in the $\mu$-calculus

Pre-proofs are the trees coinductively generated by:

- $\mathcal{L}$ inference rules
- inference for $\mu, \nu$:

$$
\begin{align*}
\frac{\Gamma, F[\mu X.F/X] \vdash \Delta}{\Gamma, \mu X.F \vdash \Delta} & (\mu_i) \\
\frac{\Gamma \vdash F[\nu X.F/X], \Delta}{\Gamma \vdash \mu X.F, \Delta} & (\mu_r) \\
\frac{\Gamma, F[\nu X.F/X] \vdash \Delta}{\Gamma, \nu X.F \vdash \Delta} & (\nu_i) \\
\frac{\Gamma \vdash F[\mu X.F/X], \Delta}{\Gamma \vdash \nu X.F, \Delta} & (\nu_r)
\end{align*}
$$

Circular (pre-)proofs: the regular fragment of infinite (pre-)proofs, ie finitely many sub-(pre)proofs.

Pre-proofs are unsound!!

Need for a correctness criterion!

$$
\begin{align*}
\vdash \mu X.X & (\mu) \\
\vdash \mu X.X & (\mu) \\
\vdash \nu X.X, F & (v) \\
\vdash \nu X.X, F & (v) \\
\vdash F & (cut)
\end{align*}
$$
\( \mu LL^\infty \) Non-Wellfounded Sequent Calculus

Consider your favourite logic LL & add fixed-points as in the \( \mu \)-calculus

\( \mu LL^\infty \) Pre-proofs are the trees \textit{coinductively} generated by:

- LL inference rules
- inference for \( \mu, \nu \):

\[
\frac{\Gamma, F[X/F] \vdash \Delta}{\Gamma, \mu X.F \vdash \Delta} \quad (\mu_r) \quad \frac{\Gamma, F[X/F] \vdash \Delta}{\Gamma, \nu X.F \vdash \Delta} \quad (\nu_r)
\]

Circular (pre-)proofs: the regular fragment of infinite (pre-)proofs, ie finitely many sub-(pre)proofs.

Pre-proofs are unsound!!

\textit{Need for a correctness criterion!}

One-sided sequents as lists: \( \vdash A_1, \ldots, A_n \).

\( \mu \) and \( \nu \) are dual binders.

\( (\Gamma \vdash \Delta \text{ is a short for } \vdash \Gamma^\perp, \Delta) \)

Ex: \( (\nu X.X \otimes X)^\perp = \mu X.X @ X \)
\[ \mu \mathsf{LL}^\infty \text{ Inferences} \]

\[ \mu \mathsf{LL}^\infty \text{ Inference Rules} \]

\[ \frac{\vdash F, F \bot}{\vdash F, F \bot} \quad (\text{ax}) \]

\[ \frac{\vdash \Gamma, F \quad \vdash F \bot, \Delta}{\vdash \Gamma, \Delta} \quad (\text{cut}) \]

\[ \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta} \quad (\text{ex}) \]

\[ \frac{\vdash F, \Gamma}{\vdash ?F, \Gamma} \quad (?) \]

\[ \frac{\vdash \Gamma}{\vdash ?F, \Gamma} \quad (w) \]

\[ \frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma} \quad (c) \]

\[ \frac{\vdash F, ?\Gamma}{\vdash !F, ?\Gamma} \quad (!) \]

\[ \frac{\vdash \Gamma}{\vdash \bot, \Gamma} \quad (\bot) \]

\[ \frac{\vdash F, \Gamma}{\vdash !F \otimes G, \Gamma} \quad (\otimes) \]

\[ \frac{\vdash 1}{\vdash 1} \quad (1) \]

\[ \frac{\vdash 1}{\vdash T, \Gamma} \quad (\top) \]

\[ \frac{\vdash F, \Gamma}{\vdash F \& G, \Gamma} \quad (\&) \]

\[ \frac{\vdash A_i, \Gamma}{\vdash A_1 \oplus A_2, \Gamma} \quad (\oplus_i) \quad \text{(no rule for 0)} \]

\[ \frac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma} \quad (\nu) \]

\[ \frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma} \quad (\mu) \]
μLL^∞ Inferences

μLL^∞ Inference Rules (with ancestor relation)

(ax) \[ \vdash F, F^\perp \]

(cut) \[ \vdash \Gamma, \Delta \]

(ex) \[ \vdash \Gamma, F, G, \Delta \]

(?) \[ \vdash F, \Gamma \]

(w) \[ \vdash ?F, \Gamma \]

(c) \[ \vdash ?F, \Gamma \]

(!) \[ \vdash !F, \Gamma \]

(⊥) \[ \vdash \bot, \Gamma \]

(\otimes) \[ \vdash F \otimes G, \Gamma, \Delta \]

(1) \[ \vdash 1 \]

(\top) \[ \vdash \top, \Gamma \]

(\&) \[ \vdash F, \Gamma \]

(⊕) \[ \vdash A_1 \oplus A_2, \Gamma \]

(v) \[ \vdash \nu X. G, \Gamma \]

(μ) \[ \vdash \mu X. F, \Gamma \]

How to distinguish valid nwf proofs from invalid ones?
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- μLL∞ Cut elimination

5 Conclusion
Validity condition: Reachable formulas

Example: \( F = \nu X.(((a \otimes a^\perp) \otimes \!(X \otimes \mu Y.X))) \)

\( FL(F) \) is the least set of formulas such that:

- \( F \in FL(F) \);
- \( G_1 \star G_2 \in FL(F) \Rightarrow G_1, G_2 \in FL(F) \) for \( \star \in \{\oplus, \& , `\} \);
- \( \sigma X.G \in FL(F) \Rightarrow G[\sigma X.G / X] \in FL(F) \) for \( \sigma \in \{\mu, \nu\} \);
- \( mG \in FL(F) \Rightarrow G \in FL(F) \) for \( m \in \{\!\!, \?\} \).

\( FL(F) \) is a finite set for any formula \( F \).
Validity condition: Reachable formulas

Example: \( F = \nu X.((a \otimes a^\perp) \otimes (\!X \otimes \mu Y.X)) \)

\[ FL(F) = \{ F \}, \]
Validity condition: Reachable formulas

Example: \( F = \nu X.((a \otimes a') \otimes (!X \otimes \mu Y.X)) \)

\[
FL(F) = \{ F, (a \otimes a') \otimes (!F \otimes \mu Y.F), \}
\]
Validity condition: Reachable formulas

Example: $F = \nu X.((a \bowtie a^\perp) \otimes (!X \otimes \mu Y.X))$

$$FL(F) = \{ F , (a \bowtie a^\perp) \otimes (!F \otimes \mu Y.F), a \bowtie a^\perp , !F \otimes \mu Y.F \}$$
Validity condition: Reachable formulas

Example: $F = \nu X.((a \otimes a) \otimes (!X \otimes \mu Y.X))$

$$FL(F) = \{ F , (a \otimes a) \otimes (!F \otimes \mu Y.F) , a \otimes a , a , a \otimes a \otimes (!F \otimes \mu Y.F) , \mu Y.F , !F \}$$
Validity condition: Reachable formulas

Example: \( F = \nu X.((a \otimes a^\bot) \otimes (!X \otimes \mu Y.X)) \)

\[
FL(F) = F \Rightarrow (a \otimes a^\bot) \otimes (!F \otimes \mu Y.F) \]

\[
\begin{align*}
& a \otimes a^\bot \Rightarrow a \\
& !F \otimes \mu Y.F \Rightarrow \mu Y.F \\
& !F \Rightarrow !F
\end{align*}
\]
Validity condition: Reachable formulas

Example: \( F = \nu X.((a \otimes a^\perp) \otimes (!X \otimes \mu Y.X)) \)

\[
\text{FL}(F) = F \rightarrow (a \otimes a^\perp) \otimes (!F \otimes \mu Y.F) \rightarrow a \otimes a^\perp \rightarrow a \rightarrow a^\perp
\]

FL\((F)\) is the least set of formulas such that:

- \( F \in \text{FL}(F) \);
- \( G_1 \star G_2 \in \text{FL}(F) \Rightarrow G_1, G_2 \in \text{FL}(F) \) for \( \star \in \{\oplus, \& , \otimes, \otimes\} \);
- \( \sigma X.G \in \text{FL}(F) \Rightarrow G[\sigma X.G/X] \in \text{FL}(F) \) for \( \sigma \in \{\mu, \nu\} \);
- \( mG \in \text{FL}(F) \Rightarrow G \in \text{FL}(F) \) for \( m \in \{!, ?\} \).

FL\((F)\) is a finite set for any formula \( F \).
Infinite threads, validity

\[ F = \nu X.((a \otimes a^\perp) \otimes (X \otimes \mu Y.X)). \]
\[ G = \mu Y.F \]

A \textbf{thread} along an infinite branch \((\Gamma_i)_{i \in \omega}\) is an infinite sequence of formula occurrences \((F_i)_{i \geq k}\) such that for any \(i \geq k\), \(F_i \in \Gamma_i\) and \(F_{i+1}\) is an immediate ancestor of \(F_i\).
Infinite threads, validity

\[ F = \nu X.((a \land a \perp) \otimes (X \otimes \mu Y.X)). \]
\[ G = \mu Y.F \]

A **thread** along an infinite branch \((\Gamma_i)_{i \in \omega}\) is an infinite sequence of formula occurrences \((F_i)_{i \geq k}\) such that for any \(i \geq k\), \(F_i \in \Gamma_i\) and \(F_{i+1}\) is an immediate ancestor of \(F_i\).

A thread is **valid** if it unfolds infinitely many \(\nu\). More precisely, if the **minimal recurring** principal formula of the thread is a \(\nu\)-formula.

A proof is **valid** if every infinite branch contains a valid thread.
Infinite threads, validity

\[ F = \nu X. \left( (a \otimes a^\perp) \otimes (X \otimes G) \right) \]

\[ G = \mu Y. \nu X. \left( (a \otimes a^\perp) \otimes (X \otimes Y) \right) \]

A **thread** along an infinite branch \((\Gamma_i)_{i \in \omega}\) is an infinite sequence of formula occurrences \((F_i)_{i \geq k}\) such that for any \(i \geq k\), \(F_i \in \Gamma_i\) and \(F_{i+1}\) is an immediate ancestor of \(F_i\).

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Infinite threads, validity

\[ F = \nu X.((a \otimes a^\perp) \otimes (X \otimes \mu Y.X)). \]
\[ G = \mu Y.F \]

A \textit{thread} along an infinite branch \((\Gamma_i)_{i \in \omega}\) is an infinite sequence of formula occurrences \((F_i)_{i \geq k}\) such that for any \(i \geq k\), \(F_i \in \Gamma_i\) and \(F_{i+1}\) is an immediate ancestor of \(F_i\).

A thread is \textit{valid} if it unfolds infinitely many \(\nu\). More precisely, if the \textit{minimal recurring} principal formula of the thread is a \(\nu\)-formula.

A proof is \textit{valid} if every infinite branch contains a valid thread.

\textbf{Theorem (Baelde, Doumane \\ & S, 2016)}
\(\mu\text{MALL}^\infty\) is sound, and admits cut-elimination.

\textbf{Theorem (Doumane 2017 + Nollet, Tasson \\ & S, 2019)}
 Validity of \(\mu\text{LL}^\omega\) (circular) pre-proofs is decidable and PSPACE-complete.
1 Introduction
• Infinite descent
• Circular LL
• On the non-wellfounded proof-theory of fixed-point logics

2 $\mu\text{LL}$: Least and greatest fixed-points in LL
• $\mu\text{MALL} & \mu\text{LL}$ languages and finitary proof systems
• $\mu\text{LL}$ cut-reduction

3 $\mu\text{LL}^\infty$: circular and non-wellfounded proofs for $\mu\text{LL}$
• Non-wellfounded proof system $\mu\text{LL}^\infty$
• Validity condition
• Decidability of the validity condition
• Expressiveness of circular proofs
• $\mu\text{LL}^\infty$ focusing

4 Cut-elimination for circular and non-wellfounded proofs
• $\mu\text{MALL}^\infty$ Cut elimination
• $\mu\text{LL}^\infty$ Cut elimination

5 Conclusion
Parity automata

Definition

A parity automaton is a finite state word automaton, whose states are ordered and given a parity bit \( \nu/\mu \), which accepts runs \( (q_i)_{i \in \omega} \) such that \( \min(\inf((q_i)_i)) \) has parity \( \nu \).
Definition

A *parity automaton* is a finite state word automaton, whose states are ordered and given a parity bit $\nu/\mu$, which accepts runs $(q_i)_{i \in \omega}$ such that $\min(\inf((q_i)_i))$ has parity $\nu$.

Remarks

- States are usually given a color in $\mathbb{N}$, equivalently.
- Only co-accessible states need to be ordered.
Parity automata

Definition

A *parity automaton* is a finite state word automaton, whose states are ordered and given a parity bit $\nu/\mu$, which accepts runs $(q_i)_{i \in \omega}$ such that $\min(\inf((q_i)_{i}))$ has parity $\nu$.

Remarks

- States are usually given a color in $\mathbb{N}$, equivalently.
- Only co-accessible states need to be ordered.

Properties

- PA can be determinized,
- PA are closed by complementation and intersection,
- The emptiness problem is decidable,
- (Thus) inclusion of parity automata is decidable.
Theorem: The validity of circular pre-proofs is decidable.

Proof.

Consider a pre-proof $\Pi$ i.e. a graph with nodes $s_i = (F^j_i)_{j \in [1;n_i]}$.

*The proof goes as follows:*

- One builds a parity automaton recognizing the language $L_B$ of infinite branches of $\Pi$;
- One builds a parity automaton recognizing the language $L_T$ the valid branches of $\Pi$.
- Validity amounts to the inclusion of $L_B$ in $L_T$, that is showing that $L_B \setminus L_T = \emptyset$ which is decidable.

*Branch automaton:* Let $A_B$ be the branch automaton with states $s_i$, transitions $s_i \rightarrow^k s_j$ when $s_j$ is the $k$-th premise of $s_i$, and which accepts all runs.

(...)

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Theorem: The validity of circular pre-proofs is decidable.

Proof.

Consider a pre-proof $\Pi$ i.e. a graph with nodes $s_i = (F^j_i)_{j\in[1;n_i]}$.

(...)

Thread automaton: Let $\mathcal{A}_T$ be the thread automaton with states $F^+_i, F^-_i$ or $s_i$, with transitions:

- $s_i \rightarrow^k s_p$ and $s_i \rightarrow^k F^q_p$ when $s_p$ is the $k$-th premise of $s_i$
- $F^+_i \rightarrow^k F^q_p \varepsilon$ ($\varepsilon \in \{+,-\}$) when $s_i \rightarrow^k s_p$ and $F^+_i$ is active in the rule of conclusion $s_i$ and has ancestor $F^q_p$
- $F^-_i \rightarrow^k F^q_p \varepsilon$ ($\varepsilon \in \{+,-\}$) when $s_i \rightarrow^k s_p$ and $F^-_i$ is passive in the rule of conclusion $s_i$ and has ancestor $F^q_p$

acceptance based on subformula ordering with the active/passive distinction: only active $\nu$-formulas have coinductive parity.

Validity of $\Pi$ equivalent to $\mathcal{L}(\mathcal{A}_B) \backslash \mathcal{L}(\mathcal{A}_T) = \emptyset$, thus decidable.
Introduction
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Cut-elimination for circular and non-wellfounded proofs
• µMALL∞ Cut elimination
• µLL∞ Cut elimination

Conclusion
Examples of circular proofs

\[ N = \mu X. 1 \oplus X \]

**double**: \[ N \rightarrow N \]

\[ \text{double}(0) = 0 \]

\[ \text{double}(\text{succ}(m)) = \text{succ}(\text{succ}(\text{double}(m))) \]

\[
\begin{align*}
\pi_0 &= \frac{\vdash 1}{\vdash N} \quad (1) \\
\pi_k &= \frac{\pi_k}{\vdash 1 \oplus N} \quad (\oplus_1) \\
\pi_{k+1} &= \frac{\pi_k}{\vdash 1 \oplus N} \quad (\mu) \\
\pi_{\text{succ}} &= \frac{N \vdash N}{N \vdash 1 \oplus N} \quad (\oplus_2) \\
\end{align*}
\]
Examples of circular proofs

\[ N = \mu X. 1 \oplus X \]

\[ double : N \rightarrow N \]

\[ double(0) = 0 \]

\[ double(succ(m)) = succ(succ(double(m))) \]

\[ \pi_0 = \frac{\frac{\vdash 1}{\vdash 1 \oplus N}}{(\mu)} \]

\[ \pi_{k+1} = \frac{\frac{\pi_k}{\vdash 1 \oplus N}}{(\mu)} \]

\[ \pi_{\text{succ}} = \frac{\frac{N \vdash N}{\vdash N}}{(\mu)} \]

\[ \pi_{\text{double}} = \frac{\frac{\frac{\vdash 1}{\vdash 1 \oplus N}}{(\mu)} \frac{\vdash N}{\vdash N} \frac{1 \vdash N}{\vdash 1 \oplus N} \frac{\vdash N}{\vdash 1 \oplus N}}{(\&)} \frac{\vdash N}{\vdash N} \]

\[ \frac{\frac{\pi_k \pi_{\text{succ}}}{\vdash N}}{(\text{cut})} \rightarrow^* \pi_{k+1} \]

\[ \frac{\pi_k \pi_{\text{double}}}{\vdash N} \]

\[ \rightarrow^* \pi_{2k} \]
Examples of circular proofs

\[ N = \mu X.1 \oplus X \]

\[ \pi_{\text{dup}} = \quad \frac{\vdash N_1 \quad (\mu), (\oplus_1), (1) \quad \vdash N_2 \quad (\mu), (\oplus_1), (1) \quad 1 \vdash N_1 \otimes N_2 \quad (\bot), (\otimes) \quad \pi_{\text{succ}} \quad \pi_{\text{succ}}}{\vdash N_1 \otimes N_2} \]

\[ N' \vdash N_1' \otimes N_2' \quad \frac{N_1' \otimes N_2' \vdash N_1 \otimes N_2}{(\otimes), (\otimes) \quad \text{(Cut)}} \]

\[ N' \vdash N_1 \otimes N_2 \quad (v), (\&) \]

\[ \text{WNat}(\pi) = \quad \frac{\vdash \Gamma \quad (\bot) \quad N \vdash \Gamma}{1 \vdash \Gamma} \]

\[ \frac{1 \oplus N \vdash \Gamma \quad (\&)}{N \vdash \Gamma} \quad (v) \]
Examples of circular proofs

\[ \mu X.1 \oplus X \]

\[ \pi_{\text{dup}} = \frac{\ \frac{\ \frac{\ \frac{\ \frac{N_1 \vdash N_2}{(\mu, (\oplus_1), (1)}}{(\mu, (\oplus_1), (1))}}{(\bot), (\otimes)}}{(\bot), (\otimes)}}{N \vdash N_1 \otimes N_2} }{N' \vdash N_1' \otimes N_2' \vdash N_1 \otimes N_2} \]

\[ \pi_{\text{suc}} \]

\[ \pi_{\text{suc}} \]

\[ \pi_{\text{cut}} \]

\[ \pi \]

\[ \text{WNat}(\pi) = \frac{\ \frac{\ \frac{\ \frac{\ \frac{\pi}{(\bot)}}{N \vdash \Gamma}}{(\&)}}{1 \oplus N \vdash \Gamma} }{N \vdash \Gamma} \]

\[ \pi_{k} \]

\[ \pi_{\text{dup}} \]

\[ \frac{(\otimes)}{N \vdash N \otimes N} \]

\[ \frac{(\bot)}{1 \vdash N \otimes N} \]

\[ \frac{(\&)}{N \vdash \Gamma} \]

\[ \pi_{k} \]

\[ \pi_{\text{cut}} \]

\[ \frac{(\otimes)}{N \vdash N \otimes N} \]

\[ \frac{(\&)}{N \vdash \Gamma} \]

\[ \frac{(\otimes)}{N \vdash N \otimes N} \]

\[ \frac{(\&)}{N \vdash \Gamma} \]
Examples of circular proofs

\[ S = \nu X.(1 \& (N \otimes X)) \]

\[ \text{enum} : \ Nat \to \ Stream \]

\[ \text{enum}(n) = n :: \text{enum}(\text{succ}(n)) \]

\[ \pi_{\text{enum}} = \]

\[ \frac{\frac{\frac{1 \vdash 1}{!N \vdash 1}}{!N \vdash 1}^1 \quad \frac{\frac{\frac{N \vdash N}{!N \vdash N}}{!N \vdash N}^? \quad \frac{\frac{\frac{N \vdash N}{!N \vdash !N}}{!N \vdash !N}^?}{!N \vdash !N}^1 \quad \frac{\frac{\frac{!N \vdash N}{!N \vdash S}}{!N \vdash !N}^\otimes \quad \frac{\frac{\frac{N \vdash N}{!N \vdash N \otimes S}}{!N \vdash !N \otimes S}}{!N \vdash N \otimes S}^{\otimes} \quad \frac{\frac{\frac{!N \vdash 1 \& (N \otimes S)}{!N \vdash !N \otimes S}}{!N \vdash S}^{\otimes}}{!N \vdash S}^v}^{\otimes} \]

\[ \pi_{\text{succ}} = \]

\[ \frac{\frac{\frac{N \vdash N}{!N \vdash N}}{!N \vdash N}^? \quad \frac{\frac{\frac{N \vdash N}{!N \vdash !N}}{!N \vdash !N}^?}{!N \vdash !N}^1 \quad \frac{\frac{\frac{!N \vdash N}{!N \vdash S}}{!N \vdash !N}^\otimes \quad \frac{\frac{\frac{N \vdash N}{!N \vdash N \otimes S}}{!N \vdash !N \otimes S}}{!N \vdash N \otimes S}^{\otimes} \quad \frac{\frac{\frac{!N \vdash 1 \& (N \otimes S)}{!N \vdash !N \otimes S}}{!N \vdash S}^{\otimes}}{!N \vdash S}^v}^{\otimes} \]
Examples of circular proofs

\[ S = \nu X. (1 \& (N \otimes X)) \]

\begin{align*}
\text{enum} & : \ Nat \to \ Stream \\
\text{enum}(n) & = n :: \text{enum}(\text{succ}(n))
\end{align*}

\[ \pi_{\text{enum}} \]
Fixed-point encoding the exponentials

Consider the following encoding of LL exponentials:

\[ ? \cdot F \triangleq \mu X.F \oplus (\perp \oplus (X \otimes X)) \]
\[ ! \cdot F \triangleq \nu X.F \& (1 \& (X \otimes X)) \]
Fixed-point encoding the exponentials

Consider the following encoding of LL exponentials:

\[
\begin{align*}
?\cdot F & \triangleq \mu X. F \oplus (\perp \oplus (X \lozenge X)) \\
!\cdot F & \triangleq \nu X. F \& (1 \& (X \otimes X))
\end{align*}
\]

The exponential inferences can be derived, circularly:

**Dereliction (?d•):**

\[
\begin{array}{c}
\vdash F, \Delta \\
\hline
\vdash F \oplus (\perp \oplus (?\cdot F \lozenge ?\cdot F)), \Delta \\
\hline
\vdash ?\cdot F, \Delta
\end{array}
\]  

**Contraction (?c•):**

\[
\begin{array}{c}
\vdash ?\cdot F, ?\cdot F, \Delta \\
\hline
\vdash ?\cdot F \lozenge ?\cdot F, \Delta \\
\hline
\vdash \perp \oplus (\perp \oplus (?\cdot F \lozenge ?\cdot F)), \Delta
\end{array}
\]

**Weakening (?w•):**

\[
\begin{array}{c}
\vdash \Delta \\
\hline
\vdash \perp, \Delta \\
\hline
\vdash \perp \oplus (\perp \oplus (?\cdot F \lozenge ?\cdot F)), \Delta
\end{array}
\]

**Promotion (!p•):**

\[
\begin{array}{c}
\vdash 1 \\
\hline
\vdash 1, ?\cdot \Delta
\end{array}
\]

\[
\begin{array}{c}
\vdash !\cdot F, ?\cdot \Delta \\
\vdash !\cdot F, ?\cdot \Delta \\
\hline
\vdash !\cdot F \otimes !\cdot F, ?\cdot \Delta, ?\cdot \Delta
\end{array}
\]

\[
\begin{array}{c}
\vdash !\cdot F \otimes !\cdot F, ?\cdot \Delta, ?\cdot \Delta \\
\hline
\vdash !\cdot F \otimes !\cdot F, ?\cdot \Delta
\end{array}
\]
Fixed-point encoding the exponentials

Consider the following encoding of LL exponentials:

\[ \begin{align*}
?\cdot F & \triangleq \mu X. F \oplus (\bot \oplus (X \otimes X)) \\
!\cdot F & \triangleq \nu X. F \& (1 \& (X \otimes X))
\end{align*} \]

Preservation of validity

\( \pi \) is a valid \( \mu \text{LL}^\infty \) pre-proof of \( \vdash \Gamma \) iff

\( \pi^\bullet \) is a valid \( \mu \text{MALL}^\infty \) pre-proof of \( \vdash \Gamma^\bullet \).

Preservation of provability

If \( \vdash \Gamma \) is provable in \( \mu \text{LL}^\infty \) (resp. \( \mu \text{LL}^\omega \)),
then \( \vdash \Gamma^\bullet \) is provable in \( \mu \text{MALL}^\infty \) (resp. \( \mu \text{MALL}^\omega \)).
Fixed-point encoding the exponentials

Consider the following encoding of LL exponentials:

\[
\begin{align*}
\text{\texttt{?\_}\_F} & \triangleq \mu X. F \oplus (\bot \oplus (X \& X)) \\
\text{\texttt{!\_}\_F} & \triangleq \nu X. F \& (1 \& (X \otimes X))
\end{align*}
\]

Preservation of validity

\(\pi\) is a valid \(\mu\_LL^\infty\) pre-proof of \(\vdash \Gamma\) iff
\(\pi^*\) is a valid \(\mu\_MALL^\infty\) pre-proof of \(\vdash \Gamma^*\).

Preservation of provability

If \(\vdash \Gamma\) is provable in \(\mu\_LL^\infty\) (resp. \(\mu\_LL^\omega\)),
then \(\vdash \Gamma^*\) is provable in \(\mu\_MALL^\infty\) (resp. \(\mu\_MALL^\omega\)).

Shortcomings of this encoding

No soundness result for the encoding: converse result for the preservation of provability. Loss of Seely isomorphisms, etc.
Circular & finitary proofs

From finitary to circular proofs

Theorem

Finitary proofs can be transformed to (valid) circular proofs.

The key translation step is the following:

\[ \pi_1 \vdash \Gamma, S \]
\[ \vdash \Gamma, \nu X. F \]
\[ \pi_2 \vdash \Gamma, S \]
\[ \vdash S^\perp, F[S] \]
\[ [\pi_2] \vdash S^\perp, F[S] \]
\[ \vdash S^\perp, \nu X. F \]
\[ [\pi_1] \vdash S^\perp, F[S] \]
\[ \vdash S^\perp, \nu X. F \]
\[ \vdash S^\perp, F[\nu X. F] \]
\[ \vdash S^\perp, F[S] \]
\[ \vdash S^\perp, F[S] \]
\[ \vdash S^\perp, \nu X. F \]
\[ \vdash \Gamma, \nu X. F \]

From circular to finitary proofs

Open problem for \( \mu \text{LL}^\omega \) in general. Solved positively for strongly valid circular proofs.
1 Introduction
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   - $\mu\text{MALL}$ & $\mu\text{LL}$ languages and finitary proof systems
   - $\mu\text{LL}$ cut-reduction

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   - Non-wellfounded proof system $\mu\text{LL}^\infty$
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   - Expressiveness of circular proofs
   - $\mu\text{LL}^\infty$ focusing

4 Cut-elimination for circular and non-wellfounded proofs
   - $\mu\text{MALL}^\infty$ Cut elimination
   - $\mu\text{LL}^\infty$ Cut elimination

5 Conclusion
Intuitive Idea of Focusing

**Idea of focusing:** Reduce the proof search space by

- **Reversibility** of negatives: no choice to make, provability of conclusion entails provability of premisses.
- **Focusing** the positives: involves choice, but proofs can proceed in a stubborn way by committing hereditarily to a positive *focus* and its subformulas.

<table>
<thead>
<tr>
<th>$\Gamma$ contains a negative formula</th>
<th>$\Gamma$ contains no negative formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>choose the leftmost negative formula and apply the unique negative rule available.</td>
<td>choose some positive formula and decompose it hereditarily until atoms or negative subformulas are reached.</td>
</tr>
</tbody>
</table>

**Various proof methods:**
by cut-elimination, inference permutations, etc.
Here application of a proof method designed jointly with Miller.
µMALL∞ focusing

By adapting the proof of focusing using Focalization graphs by Miller and S., 2007:

1. Reversibility of the negatives;
2. Focusing for positive sequents:
   1. Weak commutation properties among the positives;
   2. Positive Trunks;
   3. Focalization graph;
   4. Existence of a potential focus in a positive sequent.
3. Productivity of the focusing process;
4. Validity of the produced proof.

Polarity of fixed points:

ν must be negative and μ must be positive.
Reversibility of negative sequents:
Similar to MALL except that it cannot be treated with local rule permutations as shown by the following example:

\[ \vdash F, P \nabla Q, \pi \vdash F, P, Q \]
\[ \vdash F, P \nabla Q \]
\[ \vdash F, P \nabla Q, \pi \vdash F, P \nabla Q \]
\[ \vdash F \& F, P \nabla Q \]
\[ \vdash (F \& F) \oplus 0, P \nabla Q \]
\[ \vdash F, P \nabla Q \]

(reminiscent to what happens with ↑ in LL focusing)

Focusing of positive sequents:
- Positive trunks are finite trees (due to the polarization of fixed points formulas);
- The rest of the proof goes as for MALL.
**µMALL**∞ focusing

Productivity and validity of the focusing process

- Productivity of the focusing process is essentially direct from MALL case:
  - Reversibility is productive by construction;
  - The positive focusing takes place in a finite subtree (finite positive trunks): just as in MALL.

- Preservation of validity relies on an analysis of the kind of permutations involved in focusing. Since a positive never permutes below a positive, valid thread cannot be infinitely postponed.

*The extension to µLL is achieved exactly as for LL.*
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- $\mu$MALL$^\omega$ Cut elimination
- $\mu$LL$^\omega$ Cut elimination

5 Conclusion
\( \mu \text{MALL}^{\infty} \) Cut Elimination Theorem

Theorem (Baelde, Doumane & S, 2016)

Fair \( \mu \text{MALL}^{\infty} \) cut-reduction sequences converge to cut-free \( \mu \text{MALL}^{\infty} \) proofs.

Previous result by Santocanale and Fortier for the purely additive fragment of \( \mu \text{MALL}^{\infty} \).
Proof uses a locative treatment of occurrences.

- **Strategy**: “push” the cuts away from the root.

- **Cut-Cut**:

\[
\begin{align*}
\vdash \Gamma, F & \quad \vdash F^{\perp}, \Delta, G \\
\vdash \Gamma, \Delta, G & \quad \text{(cut)} \\
\vdash \Gamma, \Delta, \Sigma & \quad \vdash G^{\perp}, \Sigma \\
& \quad \text{(cut)} \\
\vdash \Gamma, \Delta, \Sigma & \quad \leftrightarrow \\
\vdash F^{\perp}, \Delta, \Sigma & \quad \text{(cut)}
\end{align*}
\]
Theorem (Baelde, Doumane & S, 2016)

*Fair μMALL∞ mcut-reduction sequences converge to cut-free μMALL∞ proofs.*

Previous result by Santocanale and Fortier for the purely additive fragment of μMALL∞. Proof uses a locative treatment of occurrences.

- **Strategy:** “push” the cuts away from the root.

- **Cut-Cut:**

\[
\frac{\vdash \Gamma, F \quad \vdash F^\perp, \Delta, G}{\vdash \Gamma, \Delta, G} \quad (cut) \quad \vdash \Gamma, \Delta, G \quad \vdash G^\perp, \Sigma \quad (cut) \quad \vdash \Gamma, \Delta, G^\perp, \Sigma \quad (mcut)
\]
Cut elimination procedure

External phase: Cut-commutation cases

\[ \vdash \Delta, F, G \quad (\exists) \]
\[ \vdash \Delta, F \nrightarrow G \quad \ldots \quad (mcut) \]
\[ \vdash \Sigma, F \nrightarrow G \]
\[ \vdash \Delta, F, G \quad \ldots \quad (mcut) \]
\[ \vdash \Sigma, F, G \quad (\exists) \]
\[ \vdash \Sigma, F \nrightarrow G \]

\[ \vdash \Delta, F \quad \vdash \Delta, G \quad (\&) \]
\[ \vdash \Delta, F \& G \quad \ldots \quad (mcut) \]
\[ \vdash \Sigma, F \& G \]
\[ \vdash \Delta, F \quad \ldots \quad (mcut) \]
\[ \vdash \Delta, G \quad \ldots \quad (mcut) \]
\[ \vdash \Sigma, F \quad (\&) \]
\[ \vdash \Sigma, G \]
\[ \vdash \Sigma, F \& G \]

\[ \vdash \Delta, F[\mu X.F/X] \quad (\mu) \]
\[ \vdash \Delta, \mu X.F \quad \ldots \quad (mcut) \]
\[ \vdash \Sigma, \mu X.F \]
\[ \vdash \Delta, F[\mu X.F/X] \quad \ldots \quad (mcut) \]
\[ \vdash \Sigma, F[\mu X.F/X] \quad (\mu) \]
\[ \vdash \Sigma, \mu X.F \]

+ additional cases

Cut-commutation steps are productive
Cut elimination procedure

Internal Phase: Key cases

\[ \vdash \Delta, F_2 \quad \vdash \Delta, F_1 \quad \vdash \Gamma, F_i \quad \vdash \Gamma, F_\perp \]
\[ \vdash \Delta, F_2 & F_1 \quad \vdash \Gamma, F_1 \perp F_2 \perp \quad \vdash \Gamma, F_i \perp \]
\[ \vdash \Sigma \]

\[ \Rightarrow \quad \vdash \Delta, F_i \quad \vdash \Gamma, F_i \perp \]
\[ \vdash \Sigma \]

\[ \vdash \Delta, F[\mu X.F/X] \quad \vdash \Gamma, F[\nu X.F_/X] \]
\[ \vdash \Delta, \mu X.F \quad \vdash \Gamma, \nu X.F \perp \]
\[ \vdash \Delta, \mu X.F \quad \vdash \Gamma, \nu X.F \perp \]
\[ \vdash \Sigma \]

\[ \Rightarrow \quad \vdash \Delta, F[\mu X.F/X] \quad \vdash \Gamma, F[\nu X.F_/X] \]
\[ \vdash \Sigma \]

+ additional cases

Key cases are not productive
Cut elimination algorithm

- **Internal phase:** Perform key case reductions as long as you cannot do anything else.
- **External phase:** Build a part of the output tree by applying cut-commutation steps as soon possible, being fair.
- Repeat.
Cut elimination algorithm

- **Internal phase**: Perform key case reductions as long as you cannot do anything else.
- **External phase**: Build a part of the output tree by applying cut-commutation steps as soon possible, being fair.
- Repeat.

**Remark**: We consider a **fair** strategy i.e. every reduction which is available at some point will be performed eventually.

**Theorem**

Internal phases always halt. Cut-elimination produces a pre-proof.

**Theorem**

The pre-proof obtained by the cut elimination algorithm is valid.

**\(\mu LL^\omega\)** is not stable by cut-elimination

Eliminating cuts from a \(\mu LL^\omega\) proof (circular) may result in a \(\mu LL^\infty\), non circular, proof.
Validity sensitive to cut-introduction in cycles

Circular derivations corresponding to:

\[
\begin{align*}
N \vdash N & \quad (\text{Ax}) \\
S \vdash S & \\
N, S \vdash N \otimes S & \quad (\otimes_1) \\
N \otimes S \vdash N \otimes S & \quad (\otimes_1) \\
S \vdash N \otimes S & \quad (\nu_1) \\
S \vdash S & \\
N, S \vdash S & \quad (\text{WNat}_1) \\
N \otimes S \vdash S & \quad (\otimes_1) \\
S \vdash S & \quad (\nu_1)
\end{align*}
\]
Validity sensitive to cut-introduction in cycles

Circular derivations corresponding to:

\[
\begin{align*}
N \vdash N & \quad \text{(Ax)} \quad S \vdash S \\
N, S \vdash N \otimes S & \quad \text{(\(\otimes_l\))} \\
N \otimes S \vdash N \otimes S & \quad \text{(\(\otimes_r\))} \\
S \vdash N \otimes S & \quad \text{(\(\nu_l\))} \\
S \vdash S & \quad \text{(\(\nu_r\))} \\
N, S \vdash S & \quad \text{(WNat\(_l\))} \\
N \otimes S \vdash S & \quad \text{(WNat\(_r\))} \\
S \vdash S & \quad \text{(\(\nu_l\))} \\
S \vdash S & \quad \text{(\(\nu_r\))} \\
N, S \vdash S & \quad \text{(\(\otimes_l\))} \\
N \otimes S \vdash S & \quad \text{(\(\otimes_r\))} \\
S \vdash S & \quad \text{(Cut)}
\end{align*}
\]
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Cut-elimination for $\mu LL^\infty$

Theorem

Fair $\mu LL^\infty$ mcut-reduction sequences converge to cut-free $\mu LL^\infty$ proofs.
Cut-elimination for $\mu\text{LL}^\infty$

**Theorem**

*Fair $\mu\text{LL}^\infty$ mcut-reduction sequences converge to cut-free $\mu\text{LL}^\infty$ proofs.*

**Idea.** The proof goes by:

- considering the following encoding of LL exponential modalities:
  
  $\mu F = \mu X.F \oplus (\bot \oplus (X \otimes X))$
  
  $\nu F = \nu X.F \& (1 \& (X \otimes X))$

- translating $\mu\text{LL}^\infty$ sequents and proofs in $\mu\text{MALL}^\infty$,

- simulating $\mu\text{LL}^\infty$ cut-reduction sequences in $\mu\text{MALL}^\infty$ and

- applying $\mu\text{MALL}^\infty$ cut-elimination theorem.

Extends to the circular version of LL with $(!p^{nu})$ (even with fixed-points):

**Theorem (Cut-elimination for circular LL)**

Circular LL (with $(!p^{nu})$ for promotion) eliminates cuts (even with fixed-points).
Ancoding $\mu LL^\infty$ in $\mu MALL^\infty$

$$?^*F = \mu X.F \oplus (\bot \oplus (X \otimes X)) \quad !^*F = \nu X.F \& (1 \& (X \otimes X))$$

$\mu MALL^\infty$ derivability of the exponential rules ($?d^*, ?c^*, ?w^*, !p^*$):

**Dereliction:**

<table>
<thead>
<tr>
<th>Rule</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vdash F, \Delta$</td>
<td>$\vdash F \oplus (\bot \oplus (?^*F \otimes ?^*F)), \Delta$ (\oplus_1)</td>
</tr>
<tr>
<td></td>
<td>$\vdash ?^*F, \Delta$ (\oplus)</td>
</tr>
</tbody>
</table>

**Contraction:**

<table>
<thead>
<tr>
<th>Rule</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vdash ?^*F \otimes ?^*F, \Delta$</td>
<td>$\vdash ?^*F, \Delta$ (\oplus_2)</td>
</tr>
<tr>
<td>$\vdash \bot \oplus (\bot \oplus (?^*F \otimes ?^*F)), \Delta$</td>
<td>$\vdash ?^*F \otimes ?^*F, \Delta$ (\oplus_2)</td>
</tr>
</tbody>
</table>

**Weakening:**

<table>
<thead>
<tr>
<th>Rule</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vdash \Delta$</td>
<td>$\vdash \bot, \Delta$ (\bot)</td>
</tr>
<tr>
<td>$\vdash \bot \oplus (\bot \oplus (?^*F \otimes ?^*F)), \Delta$</td>
<td>$\vdash ?^*F, \Delta$ (\oplus_2)</td>
</tr>
<tr>
<td>$\vdash F \oplus (\bot \oplus (?^*F \otimes ?^*F)), \Delta$</td>
<td>$\vdash ?^*F, \Delta$ (\oplus_2)</td>
</tr>
</tbody>
</table>

**Promotion:**

<table>
<thead>
<tr>
<th>Rule</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vdash F, ?^*\Delta$</td>
<td>$\vdash 1$ (1)</td>
</tr>
<tr>
<td>$\vdash 1, ?^*\Delta$ (\oplus_1)</td>
<td>$\vdash !^<em>F \otimes !^<em>F, ?^</em>\Delta, ?^</em>\Delta$ (\otimes_2)</td>
</tr>
<tr>
<td>$\vdash !^<em>F \otimes !^<em>F, ?^</em>\Delta, ?^</em>\Delta$ (\otimes)</td>
<td>$\vdash !^<em>F, ?^</em>\Delta$</td>
</tr>
</tbody>
</table>

**Preservation of validity**

$\pi$ is a valid $\mu LL^\infty$ pre-proof of $\vdash \Gamma$ iff $\pi^*$ is a valid $\mu MALL^\infty$ pre-proof of $\vdash \Gamma^*$. 
Simulation of $\mu LL^\infty$ cut-elimination steps

$\mu LL^\infty$ cut-elimination steps can be simulated by the previous encoding.

For instance, the following reduction can be simulated by applying the external reduction rule $(\mu)/(cut)$ followed by the external reduction rule $(\oplus)/(cut)$.

\[
\frac{\vdash F, G, \Gamma}{\vdash ? \cdot F, G, \Gamma} \quad (?d^\bullet) \quad \frac{\vdash G^\perp, \Delta}{\vdash ? \cdot F, \Gamma, \Delta} \quad (cut) \quad \frac{\vdash F, G, \Gamma}{\vdash G^\perp, \Delta} \quad (cut) \quad \frac{\vdash F, G, \Gamma}{\vdash F, \Gamma, \Delta} \quad (cut) \quad \frac{\vdash ? \cdot F, \Gamma, \Delta}{\vdash ? \cdot F, \Gamma, \Delta} \quad (?d^\bullet)
\]

Challenge: to show that the simulation of derivation also holds

(i) for the reductions involving $[!p]$ as well as
(ii) for reductions occurring above a promotion rule (aka. in a box) since the encoding of $[!p]$ uses an infinite, circular derivation.
Simulation of $\mu LL_\infty$ cut-elimination steps

Cut-commutation rules

$$\vdash F, G, \Gamma$$  
$$\vdash ?\cdot F, G, \Gamma$$ (d*)  
$$\vdash G \perp, \Delta$$  
$$\vdash ?\cdot F, \Gamma, \Delta$$ (cut)

$$\vdash ?\cdot F, ?\cdot F, G, \Gamma$$ (c*)  
$$\vdash G \perp, \Delta$$  
$$\vdash ?\cdot F, \Gamma, \Delta$$ (cut)

$$\vdash G, \Gamma$$ (w*)  
$$\vdash ?\cdot F, G, \Gamma$$  
$$\vdash G \perp, \Delta$$  
$$\vdash ?\cdot F, \Gamma, \Delta$$ (cut)

$$\vdash F, ?\cdot G, ?\cdot \Gamma$$ (p*)  
$$\vdash ?\cdot G \perp, \Delta$$  
$$\vdash G, ?\cdot \Delta$$  
$$\vdash ?\cdot F, ?\cdot G, ?\cdot \Gamma$$ (cut)

$$\vdash !\cdot F, ?\cdot G, ?\cdot \Gamma$$ (p*)  
$$\vdash !\cdot G \perp, ?\cdot \Delta$$  
$$\vdash !\cdot F, ?\cdot \Gamma, ?\cdot \Delta$$ (cut)

$$\vdash G, ?\cdot \Delta$$ (p*)  
$$\vdash F, ?\cdot \Gamma, ?\cdot \Delta$$  
$$\vdash !\cdot G \perp, ?\cdot \Delta$$ (cut)

$$\vdash F, ?\cdot \Gamma, ?\cdot \Delta$$  
$$\vdash !\cdot F, ?\cdot \Gamma, ?\cdot \Delta$$ (p*
Simulation of $\mu LL^\infty$ cut-elimination steps

Key-cut rules
Cut-elimination for $\mu LL^\infty$

1. Consider a fair cut-reduction sequence $\sigma = (\pi_i)_{i \in \omega}$ in $\mu LL^\infty$ from $\pi$.

2. $\sigma$ converges to a cut-free $\mu LL^\infty$ pre-proof. By contradiction: Otherwise, a suffix $\tau$ of $\sigma$ would contain only key-cut steps. The encoding of $\tau$ in $\mu MALL^\infty$, $\tau^\bullet$ would either be unproductive or would produce an infinite tree of encodings of $?w, ?c containing no $\nu$ inference. This would contradict $\mu MALL^\infty$ cut-elimination theorem.
Cut-elimination for $\mu LL^\infty$

1. Consider a fair cut-reduction sequence $\sigma = (\pi_i)_{i \in \omega}$ in $\mu LL^\infty$ from $\pi$.
2. $\sigma$ converges to a cut-free $\mu LL^\infty$ pre-proof.
3. As $\sigma$ is productive and since reduction only occurs above cuts, it strongly converges to some $\mu LL^\infty$ cut-free pre-proof $\pi'$.
4. $\sigma^\bullet$ is a transfinite sequence from $\pi^\bullet$ strongly converging to $\pi'^\bullet$: because $\pi'^\bullet$ — the encoding of $\pi'$ — is cut-free and because only $!$ commutations and reductions above a promotion create infinite reductions: boxes are simulated by strongly converging sequences.
Cut-elimination for $\mu LL^\infty$

1. Consider a fair cut-reduction sequence $\sigma = (\pi_i)_{i \in \omega}$ in $\mu LL^\infty$ from $\pi$.

2. $\sigma$ converges to a cut-free $\mu LL^\infty$ pre-proof.

3. As $\sigma$ is productive and since reduction only occurs above cuts, it strongly converges to some $\mu LL^\infty$ cut-free pre-proof $\pi'$.

4. $\sigma^\bullet$ is a transfinite sequence from $\pi^\bullet$ strongly converging to $\pi'^\bullet$.

5. The compression lemma applies: there exists $\rho$ an $\omega$-indexed $\mu MALL^\infty$ cut-reduction sequence converging to $\pi'^\bullet$.

6. Fairness of $\sigma$ transfers (almost) to $\rho$: $\rho$ can be turned into a fair $\mu MALL^\infty$ cut-red sequence.
Cut-elimination for $\mu\text{LL}^\infty$

1. Consider a fair cut-reduction sequence $\sigma = (\pi_i)_{i \in \omega}$ in $\mu\text{LL}^\infty$ from $\pi$.

2. $\sigma$ converges to a cut-free $\mu\text{LL}^\infty$ pre-proof.

3. As $\sigma$ is productive and since reduction only occurs above cuts, it strongly converges to some $\mu\text{LL}^\infty$ cut-free pre-proof $\pi'$.

4. $\sigma^\bullet$ is a transfinite sequence from $\pi^\bullet$ strongly converging to $\pi'^\bullet$.

5. The compression lemma applies: there exists $\rho$ an $\omega$-indexed $\mu\text{MALL}^\infty$ cut-reduction sequence converging to $\pi'^\bullet$.

6. Fairness of $\sigma$ transfers (almost) to $\rho$: $\rho$ can be turned into a fair $\mu\text{MALL}^\infty$ cut-red sequence.

7. Therefore, by $\mu\text{MALL}^\infty$ cut-elimination thm, $\rho$ has a limit, $\pi'^\bullet$, which is a valid cut-free $\mu\text{MALL}^\infty$ proof.

8. Using preservation of validity, $\pi'$ is a valid cut-free $\mu\text{LL}^\infty$-proof.
About Seely isomorphisms

\[ !A \otimes !B \vdash ! (A \& B) \]
About Seely isomorphisms

\[ !A \otimes !B \vdash ! (A \& B) \]

\[ \pi_S = \frac{\frac{A \vdash A}{!A \vdash A}}{!A!B \vdash A} \quad (ax) \quad \frac{\frac{B \vdash B}{!B \vdash B}}{!A!B \vdash B} \quad (ax) \]

\[ \pi_S' = \frac{\frac{\vdash A}{A \vdash A}}{A \& B \vdash A} \quad (\oplus_1) \quad \frac{\vdash B}{A \& B \vdash B} \quad (\oplus_2) \]

\[ \frac{\frac{A \& B \vdash !A}{!A, !B \vdash !A} \quad (w) \quad \frac{\vdash B}{!A!B \vdash B} \quad (w) \quad \frac{\vdash !B}{!A, !B \vdash !B}}{!A \otimes !B \vdash ! (A \& B)} \quad (!) \quad \frac{!(A \& B) \vdash !A}{!A \otimes !B \vdash ! (A \& B)} \quad (\&') \quad \frac{!(A \& B) \vdash !B}{!A \otimes !B \vdash ! (A \& B)} \quad (c) \]

\[ \pi_S \quad \pi_S' \quad \frac{\vdash !A \otimes !B}{!A \otimes !B \vdash !A \otimes !B} \quad (cut) \quad \rightarrow_{\text{cut}}^* \]
About Seely isomorphisms

\[ !A \otimes !B \vdash !(A \& B) \]

\[
\pi_S = \frac {A \vdash A} {!A \vdash !A} \quad (ax) \quad \frac {B \vdash B} {!B \vdash !B} \quad (ax) \\
\frac {\vdash A, !B \vdash A} {!A, !B \vdash A} \quad (w) \quad \frac {\vdash B, !B \vdash B} {!A, !B \vdash B} \quad (w) \\
\frac {!A, !B \vdash A \& B} {!A \otimes !B \vdash !(A \& B)} \quad (\&) \quad \frac {!A \otimes !B \vdash A \& B} {!(A \& B) \vdash !A} \quad (!) \quad \frac {!(A \& B) \vdash !B} {!(A \& B) \vdash !A \otimes !B} \quad (\otimes)
\]

\[
\pi'_S = \frac {A \vdash A} {A \& B \vdash A} \quad (\oplus_1) \quad \frac {B \vdash B} {A \& B \vdash B} \quad (\oplus_2) \\
\frac {A \& B \vdash A} {!(A \& B) \vdash !A} \quad (?1) \quad \frac {A \& B \vdash B} {!(A \& B) \vdash !B} \quad (?2) \\
\frac {!(A \& B), !(A \& B) \vdash !(A \otimes !B)} {!(A \& B) \vdash !A \otimes !B} \quad (c)
\]

\[
\pi_S \quad \pi'_S \\
\frac {!A \otimes !B \vdash !A \otimes !B} {!A \otimes !B \vdash !A \otimes !B} \quad (cut) \quad \rightarrow^*_{cut}
\]

\[
\frac {A \vdash A} {!A \vdash !A} \quad (w) \quad \frac {B \vdash B} {!B \vdash !B} \quad (w) \\
\frac {\vdash A, !B \vdash A} {!A, !B \vdash A} \quad (\!) \quad \frac {\vdash B, !B \vdash B} {!A, !B \vdash B} \quad (\!) \\
\frac {!A, !B \vdash A \& B} {!A \otimes !B \vdash !(A \& B)} \quad (\otimes) \quad \frac {!A, !B \vdash A \& B} {!A \otimes !B \vdash A \otimes !B} \quad (c)^2 \\
\frac {!A, !B \vdash A \& B} {!A \otimes !B \vdash !A \otimes !B} \quad (\otimes) \quad \frac {!A, !B \vdash A \& B} {!A \otimes !B \vdash !A \otimes !B} \quad (\otimes)
\]

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About Seely isomorphisms

What about the fixed-point encoding?

\[(\pi_S) \star \ (\pi'_S) \star \]
\[!A \otimes !B \vdash !A \otimes !B \] (cut)

The left occurrences of \(A, B\) require two unfoldings of the fixed-point, while the right occurrences of \(A, B\) require only one unfolding of the fixed-point.

The fixed-point unfolding structure tracks the history of the structural rules.
About Seely isomorphisms

What about the fixed-point encoding?

The left occurrences of $A, B$ require two unfoldings of the fixed-point, while the right occurrences of $A, B$ require only one unfolding of the fixed-point. The fixed-point unfolding structure tracks the history of the structural rules.
Cut-elimination for $\mu LK^\infty$, $\mu LJ^\infty$

The usual call-by-value embedding of LJ in ILL (intuitionistic LL) can be lifted to $\mu LJ^\infty$: indeed, the translation of proofs does not introduce cuts. For $\mu LK^\infty$, it is slightly trickier as the well-known T/Q-translations introduce cuts breaking validity. An alternative translation which does not introduce cuts can be used.

Moreover, one gets the skeleton of a $\mu LL^\infty$ (resp. $\mu ILL^\infty$) proof which is a $\mu LK^\infty$ (resp. $\mu LJ^\infty$) proof, simply by erasing the exponentials (connectives and inferences), preserving validity.

The skeleton of a $\mu LL^\infty$ (resp. $\mu ILL^\infty$) cut-reduction sequence is a $\mu LK^\infty$ (resp. $\mu LJ^\infty$) cut-reduction sequence. As a result, one has:

**Theorem**

*If $\pi$ is an $\mu LK^\infty$ (resp. $\mu LJ^\infty$) proof of $\vdash \Gamma$ (resp. $\Gamma \vdash F$), there exists a $\mu LL^\infty$ (resp. $\mu ILL^\infty$) proof of the translated sequents.*

**Theorem**

*There are productive cut-reduction strategies producing cut-free $\mu LK^\infty$ (resp. $\mu LJ^\infty$) proofs.*
Introduction
- Infinite descent
- Circular LL
- On the non-wellfounded proof-theory of fixed-point logics

\(\mu LL\): Least and greatest fixed-points in LL
- \(\mu MALL \& \mu LL\) languages and finitary proof systems
- \(\mu LL\) cut-reduction

\(\mu LL^\infty\): circular and non-wellfounded proofs for \(\mu LL\)
- Non-wellfounded proof system \(\mu LL^\infty\)
- Validity condition
- Decidability of the validity condition
- Expressiveness of circular proofs
- \(\mu LL^\infty\) focusing

Cut-elimination for circular and non-wellfounded proofs
- \(\mu MALL^\infty\) Cut elimination
- \(\mu LL^\infty\) Cut elimination

Conclusion
Circular or Helical reasoning?
## Proof theory of least and greatest fixed points

<table>
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<th>Proof objects</th>
<th>( \mu \text{MALL}/\mu \text{LL} )</th>
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<td>MALL rules +</td>
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<td></td>
<td>( \vdash \Gamma, \mu X.F )</td>
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<td></td>
<td>( \vdash \Gamma, S \vdash S_{\perp}, F[S/X] ) ( (\nu) )</td>
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<tr>
<td>Logical</td>
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<tr>
<td>Cut-elimination</td>
<td>sort of: ( (\nu) ) hides a cut</td>
</tr>
<tr>
<td>Subformula prop.</td>
<td>NO (if there are ( \nu ))</td>
</tr>
<tr>
<td>Focusing</td>
<td>( \checkmark ), but ( \mu/\nu ) have arbitrary polarities</td>
</tr>
<tr>
<td>Categorical sem.</td>
<td>( \checkmark )</td>
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# Proof theory of least and greatest fixed points

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</table>
| MALL rules +  | \[
\frac{\Gamma, F[\mu X.F / X]}{\Gamma, \mu X.F} \] (\( \mu \))
\[
\frac{\Gamma, S \quad \Gamma, S^\perp, F[S/X]}{\Gamma, \nu X.F} \] (\( \nu \))
| Logical correctness | local |
| Cut-elimination | sort of: (\( \nu \)) hides a cut |
| Subformula prop. | NO (if there are \( \nu \)) |
| Focusing       | \( \checkmark \), but \( \mu / \nu \) have arbitrary polarities |
| Categorical sem. | \( \checkmark \) |
| Denotational sem. | \( \checkmark \) |
Proof theory of least and greatest fixed points

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<td>Fixed points unfoldings ( + validity conditions)</td>
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<td>straight/bouncing threads</td>
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</tr>
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</tr>
<tr>
<td></td>
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<td></td>
</tr>
<tr>
<td>Categorical sem.</td>
<td>✓</td>
<td>NO</td>
</tr>
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<td>✓</td>
<td>✓</td>
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</tbody>
</table>
Conclusion

To sum up:

- Fixed-point logics with circular or non-wellfounded proofs equipped with a parity condition to discriminate valid/invalid proofs;
- Syntactic cut elimination for various nwf sequent calculi: $\mu$MALL$^\infty$, $\mu$LL$^\infty$, $\mu$LJ$^\infty$, $\mu$LK$^\infty$;
- Bouncing validity condition with a better management of cuts. (jww Baelde, Doumane & Kuperberg)

Not covered here:
- Provability / Phase semantics
- Infinets

Ongoing and future work:
- Relax further the bouncing validity condition; (jww Bauer)
- More canonical proof-objects (circular natural deduction and circular $\lambda$-calculus, proof-nets); (jww De, Pellissier)
- Provability and denotational semantics of circular proofs; (jww De, Ehrhard and Jafarrahmani)
- Understand how to interface with other approaches to productivity (guarded recursion, sized types, etc.)?
References

Questions?