M2 LMFI – SOFIX 2024 – **Homework – Part I**

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The clarity of reasonings and explanations will be taken into account. In particular, when writing a natural deduction derivation or typing derivation, write the rule names as seen in class. Stars describe the difficulty of a question as well as its length: you should not have to spend 20 minutes on a ($\star$) question while it is reasonable for a ($\star\star\star$) question.

This is a graded homework: we expect the highest degree of integrity of each of you. In particular:

1. Obviously, you shall not look for solutions online or in books.
2. You are allowed to discuss and get some explanations on the homework topic from colleague students attending the course but only in the following terms: no such discussion should be by electronic means: they must take place in face to face discussions and preferably working on a board. In the end of such a discussion, you shall not keep any written notes: erase the board or trash draft notes: you **must** be able to redo alone the whole reasoning and
3. You will start your homework copy with a first paragraph asserting whether or not you had some discussions with another student and summarising the topics of those discussions, whether you provided explanations or received some help (this should be very brief: about 2-3 lines...).

1 Strong normalization of System $\mathsf{F}\Omega$ with reducibility candidates.

Let us consider Church style system F, extended with a constant $\Omega : \forall X.X$, with no reduction rule associated with Ω. The reduction relation over $\mathsf{F}\Omega$ is written $\longrightarrow_{\mathsf{F}\Omega}$. This problem consists in proving Strong normalization for this calculus $\mathsf{F}\Omega$ using the reducibility candidate technique studied in class.

Definition 1.1 (*Neutral Term*)

A **neutral term** *is a term which does not start with a abstraction, that is a term of the form:*

$$N ::= x \mid (t)u \mid (t)U \mid \Omega.$$

Definition 1.2 (*Reducibility Candidate*)

A **reducibility candidate** *of type U is a set $\mathcal{R}$ of terms of type U satisfying the three conditions:*

CR1 *elements of $\mathcal{R}$ are strongly normalisable;*

CR2 *$\mathcal{R}$ is closed* **by reduction***;*

CR3 *If t is neutral and if all the one-step reducts of t are in $\mathcal{R}$ (ie $\forall u, t \longrightarrow_{\mathsf{F}\Omega} u \Rightarrow u \in \mathcal{R}$), then $t \in \mathcal{R}$.*

We write $\mathsf{CR}(T)$ for the set of all reducibility candidates of type T.

$\triangleright$ **Question 1.1.** ($\star$) Prove that if $\mathcal{R}$ is a reducibility candidate, then it contains all terms (of the corresponding type) that are neutral and normal. Conclude that a reducibility candidate is never empty.

▷ **Question 1.2.** ($\star$) If $\mathcal{U}$, $\mathcal{V}$ are respectively sets of terms of type U and V, one defines $\mathcal{U} \to \mathcal{V}$ as follows: $\mathcal{U} \to \mathcal{V} = \{t : U \to V \mid \forall u \in \mathcal{U}, (t)u \in \mathcal{V}\}$. Prove that if $\mathcal{U}$, $\mathcal{V}$ are reducibility candidates, then $\mathcal{U} \to \mathcal{V}$ is so.

One defines a parameterized notion of reducibility, similarly to the case of the weak normalization proof:

Definition 1.3

A **valuation** *ρ is a partial function from type variables to reducibility whose domain, written $\mathsf{dom}(\rho)$, is finite. We shall write $\rho[X := R]$ for the valuation ρ' of domain $\mathsf{dom}(\rho) \cup \{X\}$ such that:*

$$\rho'(X) = R \qquad \rho'(Y) = \rho(Y) \text{ if } Y \neq X.$$

We say that a valuation ρ **covers** *a type T (resp. a term $t : T$) if its domain contains all the type variables which are free in T (resp. in T and in all the types of the free variables of t).*

Definition 1.4 ($\mathsf{RED^{SN}}_\rho(T)$)

$\mathsf{RED^{SN}}_\rho(T)$ is defined by induction on T (if ρ covers T) as follows:

- $\mathsf{RED^{SN}}_\rho(X) = \rho(X)$;
- $\mathsf{RED^{SN}}_\rho(U \to V) = \{t : (U \to V)^\rho / \forall u \in \mathsf{RED^{SN}}_\rho(U), (t)u \in \mathsf{RED^{SN}}_\rho(V)\}$;
- $\mathsf{RED^{SN}}_\rho(\forall X.U) = \{t : (\forall X.U)^\rho / \forall V \in Type, \forall R \in CR(V), (t)V \in \mathsf{RED^{SN}}_{\rho[X:=R]}(U)\}$.

▷ **Question 1.3.** ($\star\star$) Prove that $\mathsf{RED^{SN}}_\rho(T)$ is a reducibility candidate of type T^ρ.

▷ **Question 1.4.** Prove that if V, W are types and ρ is a valuation covering V, W, we have:

$$\mathsf{RED^{SN}}_\rho(V\{W/Y\}) = \mathsf{RED^{SN}}_{\rho[Y:=\mathsf{RED^{SN}}_\rho(W)]}(V).$$

▷ **Question 1.5.** ($\star$) Prove that if for any type U and V and any candidate $\mathcal{S}$ of type U, one has $w[U/Y] \in \mathsf{RED^{SN}}_{\rho[Y:=\mathcal{S}]}(V)$, then $\Lambda Y.w \in \mathsf{RED^{SN}}_\rho(\forall Y.V)$.

▷ **Question 1.6.** ($\star$) Prove that if $t \in \mathsf{RED^{SN}}_\rho(\forall Y.V)$, then for all type U, $(t)U \in \mathsf{RED^{SN}}_\rho(V[U/Y])$.

▷ **Question 1.7.** ($\star\star$) Prove that if for all $u \in \mathsf{RED^{SN}}_\rho(U)$ one has $t\{u/x\} \in \mathsf{RED^{SN}}_\rho(V)$, then $\lambda x.t \in \mathsf{RED^{SN}}_\rho(U \to V)$.

▷ **Question 1.8.** ($\star\star\star$) Prove the Adequacy Lemma: for all $t : T$ in $\mathsf{F}\Omega$ of free variables in $(x_i^{U_i})_{1\leq i\leq n}$, all valuation ρ covering t and all $u_i \in \mathsf{RED^{SN}}_\rho(U_i)$, $1 \leq i \leq n$, one has $t^\rho\,\{u_i/x_i, 1 \leq i \leq n\} \in \mathsf{RED^{SN}}_\rho(T)$. Deduce strong normalization of $\mathsf{F}\Omega$.

2 Deducing an upside-down induction!?

The induction axiom in second-order arithmetic is formulated as $\forall x.\mathsf{Nat}(x)$ with

$$\mathsf{Nat}(x) = \forall X.(X(0) \Rightarrow \forall y.(X(y) \Rightarrow X(S(y))) \Rightarrow X(x)).$$

Let us consider the following formula: $\mathsf{Nat}'(x) = \forall Y.(Y(x) \Rightarrow \forall y.(Y(S(y)) \Rightarrow Y(y)) \Rightarrow Y(0))$.

▷ **Question 2.1.** ($\star$) Derive in NJ_2 the formula $\forall x.\mathsf{Nat}(x) \Rightarrow \mathsf{Nat}'(x)$.

▷ **Question 2.2.** ($\star\star\star$) Derive in NJ_2 the formula $\forall x.\mathsf{Nat}'(x) \Rightarrow \mathsf{Nat}(x)$.