M2 LMFI – Cours Fondamental de Logique
Théorie de la Démonstration
Introduction to natural deduction and cut elimination

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1 Natural deduction

1.1 Natural deduction inference rules

▷ **Question 1.1.** Let T be a theory, F and G two formulas. Prove that if $T \vdash F \vee G$ and $T \vdash \neg F$, then $T \vdash G$.

▷ **Question 1.2.** Give a natural deduction derivation for each axiom of Hilbert system for predicate logic

▷ **Question 1.3.** Prove that, if one forgets to check the restrictions on the application of $(\forall \mathsf{intro})$ and $(\exists \mathsf{elim})$ inferences, one can deduce $\exists x.(x = 0) \Rightarrow \forall x.(x = 0)$ and $\exists x.A \wedge \exists x.B \Rightarrow \exists x.(A \wedge B)$. (Try to use only one faulty inference for each statement.)

▷ **Question 1.4.** Prove that the following rules are derivable in natural deduction :

—
$$\dfrac{\begin{array}{c}[A]\\ \vdots \\ B\end{array} \qquad A}{B}\ (\mathsf{Cut})$$

—
$$\dfrac{\begin{array}{c}[\neg A]\\ \vdots \\ A\end{array}}{A}\ (\mathsf{Peirce})$$

1.2 Classical reasoning

▷ **Question 1.5.** Prove the following statement in natural deduction :

— $A \vee \neg A$;
— $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$;
— $\neg\neg A \Rightarrow A$.

▷ **Question 1.6.** Prove that one can deduce $\neg\neg(\neg\neg A \Rightarrow A)$ in natural deduction ***without using the classical absurdity rule*** (that is, in intuitionistic natural deduction, NJ).

Similarly, prove that one can derive $\neg\neg\neg A \Rightarrow \neg A$ in NJ.

▷ **Question 1.7.** Let us consider the following formula $\mathsf{Drinker} = \exists x.(\neg D(x) \vee \forall y.D(y))$ where D is a unary predicate symbol. Give a natural deduction proof (with no free hypothesis) for $\mathsf{Drinker}$ in NK.

1.3 Formal and informal proofs

▷ **Question 1.8.** For each of the following statements, (i) give a usual mathematical proof, (ii) formalize the statement in predicate logic and (iii) give a natural deduction of the statement :

1. Every involution is a bijection.
2. Two injections with distinct support commute.
3. If $f \circ g$ is injective and g is onto, the f is injective.
4. If $f \circ g$ is onto and g is injective, then f is surjective.
5. the composition of two bijections is a bijection.

▷ **Question 1.9.** Prove that equality is an equivalence relation in NJ.

▷ **Question 1.10.** Consider a first-order language containing a constant symbol $\sqrt{2}$, a binary function binaire symbol exp and a unary predicate symbol $\mathbb{Q}$. Prove, in natural deduction, formula $\exists x \exists y(\mathbb{Q}(\mathsf{exp}(x,y)) \wedge (\neg\mathbb{Q}(x) \wedge \neg\mathbb{Q}(y))$ under the hypotheses $\neg\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\mathsf{exp}(\mathsf{exp}(\sqrt{2},\sqrt{2}),\sqrt{2}))$. (equivalently, one shall say that one derive ***judgement***

$$\neg\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\mathsf{exp}(\mathsf{exp}(\sqrt{2},\sqrt{2}),\sqrt{2})) \vdash \exists x \exists y(\mathbb{Q}(\mathsf{exp}(x,y)) \wedge (\neg\mathbb{Q}(x) \wedge \neg\mathbb{Q}(y)).)$$

▷ **Question 1.11.** (Russell's paradox) Consider the language of set theory.

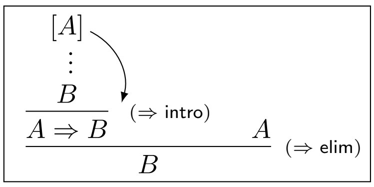
Prove the following statement in natural deduction : $\neg\exists x.\forall y(y \in x \Leftrightarrow y \notin y)$.

1.4 Counting proofs

▷ **Question 1.12.** Consider the purely implicative fragment of proposition logic (ie. $\Rightarrow$ is the only conncetive one can use to build formulas a reason about them). One says that a deduction is ***normal*** when it does not contain any inference ($\Rightarrow$ intro) the conclusion of which is the main premise of a ($\Rightarrow$ elim) inference, , that is of the following form.

Let p be a propositional variable. Describe the set of normal proofs of the following formulas (start by writing all parentheses on those formulas) :

1. $p \Rightarrow p$.
2. $p \Rightarrow p \Rightarrow p$.
3. $p \Rightarrow (p \Rightarrow p) \Rightarrow p$.
4. $(p \Rightarrow p) \Rightarrow p \Rightarrow p$.
5. $(p \Rightarrow p) \Rightarrow p$.



1.5 Peano arithmetics

One considers the language of Peano arithmetics (constant and function symbols 0, S, + and $\times$) and write PA_0 for the theory containing the following formulas :

A_1	$\forall x.S(x) \neq 0$
A_2	$\forall x.(x = 0 \vee \exists y.x = S(y))$
A_3	$\forall x.\forall y.(S(x) = S(y) \Rightarrow x = y)$
A_4	$\forall x.(x + 0 = x)$
A_5	$\forall x.\forall y.(x + S(y) = S(x + y))$
A_6	$\forall x.(x \times 0 = 0)$
A_7	$\forall y.\forall y.(x \times S(y) = (x \times y) + x)$

Consider the induction axiom schema $\mathsf{Rec} = (\mathsf{Rec}_F)_{F \in \mathsf{Form}}$ made of the formulas Rec_F which are the universal closures of

$$(F[x := 0] \wedge (\forall n.F[x := n] \Rightarrow F[x := Sn])) \Rightarrow \forall x.F.$$

Write PA for the theory obtained by extending PA_0 with the induction scheme.

▷ **Question 1.13.** Prove that $\mathsf{PA} \setminus \{A_2\} \vdash A_2$.

2 Cut-elimination and subformula property

Exercice 2.1 (***Subformula property, detailed statement*** **) (⋆⋆)**

Prove that in a deduction π of fragment $\wedge \Rightarrow \neg\forall$ of NJ *that contains no cut, all formulas appearing in the deduction tree are subformulas of the conclusion or of an undischarge hypotheses and moreover, if π ends with an elimination rule, then it contains a* **main branch** *that is a sequence of formulas $F_0, \ldots, F_k$ such that (i) F_0 is an (undischarged) hypothesis (ii) F_n is the conclusion of π and (iii) for any $0 \leq i < k$, F_i is the main premise of an elimination the conclusion of which is F_{i+1}.*

Exercice 2.2 (***Cut elimination for*** **NJ)**

From what has been explained during the lectures, extend the cut-elimination result to the whole of NJ.

Exercice 2.3 (***Subformula property for*** **NJ)**

From what has been explained during the lectures, extend the subformula property to the whole of NJ.

A Hilbert proof systems and natural deduction for predicate logic

MP	$(\{A, A \Rightarrow B\}, B)$
H_1	$A \Rightarrow (B \Rightarrow A)$
H_2	$(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$
H_3	$A \wedge B \Rightarrow A$
H_4	$A \wedge B \Rightarrow B$
H_5	$A \Rightarrow B \Rightarrow A \wedge B$
H_6	$A \Rightarrow A \vee B$
H_7	$B \Rightarrow A \vee B$
H_8	$(A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow (A \vee B) \Rightarrow C$
H_9	$\neg A \Rightarrow (A \Rightarrow \bot)$
H_{10}	$(A \Rightarrow \bot) \Rightarrow \neg A$
H_{11}	$\bot \Rightarrow A$
H_{12}	$A \vee \neg A$

Gen_1	$(\{C \Rightarrow A[x]\}, C \Rightarrow \forall x.A[x])$ pourvu que $x \notin FV(C, \Gamma)$
Gen_2	$(\{A[x] \Rightarrow C\}, \exists x.A[x] \Rightarrow C)$ pourvu que $x \notin FV(C, \Gamma)$
H_{13}	$A[t/x] \Rightarrow \exists x.A[x]$
H_{14}	$\forall x.A[x] \Rightarrow A[t/x]$

FIGURE 1 – Système à la Hilbert pour le calcul des prédicats.

Hypothèse	A	
Connecteurs prop.		
Implication	$\dfrac{\begin{matrix}[A]\\ \vdots\\ B\end{matrix}}{A \Rightarrow B}$ (⇒ intro)	$\dfrac{\begin{matrix}\vdots\\ A \Rightarrow B\end{matrix}\quad \begin{matrix}\vdots\\ A\end{matrix}}{B}$ (⇒ elim)
Conjonction	$\dfrac{\begin{matrix}\vdots\\ A_1\end{matrix}\quad \begin{matrix}\vdots\\ A_2\end{matrix}}{A_1 \wedge A_2}$ (∧intro)	$\dfrac{\begin{matrix}\vdots\\ A_1 \wedge A_2\end{matrix}}{A_i}$ (∧elim$_i$) $, i \in \{1,2\}$
Disjonction	$\dfrac{\begin{matrix}\vdots\\ A_i\end{matrix}}{A_1 \vee A_2}$ (∨intro$_i$) $, i \in \{1,2\}$	$\dfrac{\begin{matrix}\vdots\\ A \vee B\end{matrix}\quad \begin{matrix}[A]\\ \vdots\\ C\end{matrix}\quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}$ (∨elim)
Négation	$\dfrac{\begin{matrix}[A]\\ \vdots\\ \bot\end{matrix}}{\neg A}$ (¬intro)	$\dfrac{\begin{matrix}\vdots\\ \neg A\end{matrix}\quad \begin{matrix}\vdots\\ A\end{matrix}}{\bot}$ (¬elim)
Absurdité		$\dfrac{\begin{matrix}\vdots\\ \bot\end{matrix}}{A}$ (⊥elim) $\quad \dfrac{\begin{matrix}[\neg A]\\ \vdots\\ \bot\end{matrix}}{A}$ (⊥$_c$)
Quantificateurs		
Quantificateur universel	$\dfrac{\begin{matrix}\vdots\\ A[x := y]\end{matrix}}{\forall x.A}$ (∀intro) $(\star)$	$\dfrac{\begin{matrix}\vdots\\ \forall x.A\end{matrix}}{A[x := t]}$ (∀elim)
Quantificateur existentiel	$\dfrac{\begin{matrix}\vdots\\ A[x := t]\end{matrix}}{\exists x.A}$ (∃intro)	$\dfrac{\begin{matrix}\vdots\\ \exists x.A\end{matrix}\quad \begin{matrix}[A[x := y]]\\ \vdots\\ C\end{matrix}}{C}$ (∃elim) $(\star\star)$
Égalité	$\dfrac{}{t = t}$ (= intro)	$\dfrac{\begin{matrix}\vdots\\ t_1 = t_2\end{matrix}\quad \begin{matrix}\vdots\\ A[x := t_j]\end{matrix}}{A[x := t_{3-j}]}$ (= elim$_j$) $\quad j \in \{1,2\}$

$(\star)$ y n'est pas libre dans les hypothèses (non déchargées) de la dérivation de A, ni dans $\forall x.A$.
$(\star\star)$ x n'est pas libre dans les hypothèses (non déchargées) de la dérivation de C (autre que les A déchargés), ni dans C, ni dans $\forall x.A$ elle-même.

Figure 2 – Déduction naturelle de Gentzen