Infinitary and non-wellfounded proof systems for the mu-calculus

2 -- omega-rule, relations with circular proofs

Kozen's axiomatization

Previous inferences plus the following (Koz- is the cut-free fragment):

$$\frac{\Gamma, A(\overline{\Gamma})}{\Gamma, \nu \mathsf{x} A(\mathsf{x})}\,\text{ind} \qquad \mathsf{Ax2}: \nu \mathsf{x} A, \mu \mathsf{x} \bar{A} \qquad \cancel{\frac{\Gamma, A(B), A(C)}{\Gamma, A(B \vee C)}\,\vee_d} \qquad \frac{\Gamma, A \quad \Gamma, \overline{A}}{\Gamma}\,\text{cut}$$

Figure 2: Additional rules present in Koz^-.

Theorem 3.1. Koz *is sound and complete for the μ-calculus.*

Lemma 3.2. *Let $A(\mathsf{x}_0, \ldots, \mathsf{x}_{k-1})$ be a formula with at most the designated variables free.*

If B_i and C_i are closed formulæ for each $i < k$, then

$$\{B_i, C_i\}_{i<k} \vdash_{\mathsf{Koz}^-} \overline{A}(B_0, \ldots, B_{k-1}), A(C_0, \ldots, C_{k-1}).$$

Proof. The proof proceeds by induction on A. We present the case $A = \nu \mathsf{x}_k A_0(\mathsf{x}_0, \ldots, \mathsf{x}_{k-1}, \mathsf{x}_k)$. The remaining cases are straightforward. Let $B_k = \overline{A}(B_0, \ldots, B_{k-1})$. As the sequent $B_k, \overline{B_k}$ is an instance of Ax2, the induction hypothesis implies

$$\{B_i, C_i\}_{i<k} \vdash_{\mathsf{Koz}^-} \overline{A_0}(B_0, \ldots, B_k), A_0(C_0, \ldots, C_{k-1}, \overline{B_k}),$$

whereby an application of μ yields

$$\{B_i, C_i\}_{i<k} \vdash_{\mathsf{Koz}^-} B_k, A_0(C_0, \ldots, C_{k-1}, B_k)$$

and an application of ind completes the proof. $\square$

1 Ax1, Ax2

↳

$$\frac{}{A, \bar{A}}\,\text{Ax.}$$

An

$$\frac{}{\vdash A, \bar{A}}\,\text{Ax.}$$

Π_0 $\vdash B_0, C_0$ ... Π_{k-1} $\vdash B_{k-1}, C_{k-1}$.

$A(x_0, \ldots x_{k-1})$.

$\Pi \vdash A(B_0, B_1 \ldots B_{k-1}), \bar{A}(C_0, \ldots C_{k-1})$

$A, \bar{A}$

Naming convention by Studer:
- Koz
- K^pre(mu)
- K_omega(mu)

Studer

- Koz.
- $K^{pre}(\mu)$
-

$\phi^k_\bot(\bot) = \phi^{k+1}_\bot(\bot)$.

λ limit ordinal.

$\phi^\lambda_\bot(X) = \bigcup_{k<\lambda} \phi^k(X)$.

$\phi^\lambda_\top(X) = \bigcap_{k<\lambda} \phi^k(X)$

$\vdash \Gamma$ valid $\Longrightarrow \vdash_{K_\omega(\mu)} \Gamma \Longrightarrow \vdash_{K^{pre}(\mu)} \Gamma \Longrightarrow \vdash \Gamma$ valid.

$$\frac{\vdash \Gamma, \nu^i x\, A(x), \quad i \geq 1}{\vdash \Gamma, \nu x.\, A(x)}\ \nu\omega$$

$$\frac{\vdash A[\top/x], \Gamma}{\vdash \nu^1 x\, A(x), \Gamma}\ \nu 1 \qquad \frac{\vdash A[\nu^k x A/x], \Gamma}{\vdash \nu x. A, \Gamma}\ \nu^{k+1}$$

$\nu^i x.\, A$

φ, ϕ

$\mu.\, \phi = \bigcup_{k \in On} \phi^k_\bot(\bot)$

$\nu \phi = \bigcap_{k \in On} \phi^k_\top(\top)$.

$\phi^0_\varepsilon(X) = \lambda$

$\phi^{k+1}_\varepsilon(X) = \phi(\phi^k(X))$.

Example 3.1. Recall the valid sequent $\{\nu x \mu y \overline{B}, \nu y \mu x B\}$ from Example 2.1. Let $C =$ $\nu x\, \mu y\, \overline{B}$ $\nu x \mu y \overline{B}$ and $D = \nu y \mu x B$. The following derivation, which we denote π_{koz}, is the Koz-proof of this sequent motivated by the semantic validity argument:

$D = \nu y \underline{\mu x B}.$

B, $x, y \in FV(B)$,

$B(x, y)$

$\overline{C} = \mu x \nu y . B.$

$$\cancel{C, \overline{C}}$$

$$\vdots\ \text{Lemma 3.2}$$

$$\overline{\phantom{\mu y \overline{B}(C, y), \nu y B(\overline{C}, y)}}\ \text{ax.}$$

$$\dfrac{\mu y \overline{B}(C, y), \nu y B(\overline{C}, y)}{C, \nu y B(\overline{C}, y)}\ \nu$$

$$C, \overline{C} \qquad C, \nu y B(\overline{C}, y)$$

$$\vdots\ \text{Lemma 3.2} \qquad k = 2$$

$$\boxed{\vdash C, D\,|}$$

$$\overline{C} \vdash D$$

$$\dfrac{\dfrac{\dfrac{\overline{B}(C, C), B(\overline{C}, \nu y B(\overline{C}, y))}{\overline{B}(C, C), \nu y B(\overline{C}, y)}\ \nu}{\overline{B}(C, C), \overline{C}}\ \mu}{\nu x \overline{B}(x, C), \overline{C}}\ \text{ind}$$

$$\vdots\ \text{Lemma 3.2} \qquad k = 1$$

$\mu x \nu y B \vdash \nu y \mu x B$

$$\dfrac{}{\nu x \overline{B}(x, C), \mu x B(x, \overline{C})}\ \text{Ax} \qquad \dfrac{\mu y \overline{B}(\nu x \overline{B}(x, C), y), \nu y B(\overline{C}, y)}{\mu y \overline{B}(\nu x \overline{B}(x, C), y), \overline{C}}\ \mu$$

$$\vdots\ \text{Lemma 3.2} \qquad k = 0$$

$$\dfrac{\dfrac{\dfrac{\overline{B}(\nu x \overline{B}(x, C), \mu y \overline{B}(\nu x \overline{B}(x, C), y)), B(\mu x B(x, \overline{C}), \overline{C})}{\mu y \overline{B}(\nu x \overline{B}(x, C), y), \mu x B(x, \overline{C})}\ \mu, \mu}{C, \mu x B(x, \overline{C})}\ \text{ind}}{C, D}\ \text{ind}$$

$A(x_0 \ldots x_{k-1})$

$$B(X,Y) \qquad\qquad B[X,\dots X, Y, \dots Y]$$

$$\dfrac{B_1 \vdash B_2}{AA \vdash IO'}\ \mu_L \nu_L \mu_R \qquad \dfrac{B_1 \vdash B_2}{AA' \vdash IO}\ \nu_L \nu_R \mu_B$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B_1 \vdash B[IO/Y, IO'/X] = B_2}\ \mu_B}{B_1 \vdash \mu X\, B[IO/Y] = IO'}\ \mu_R}{B_1 = B[AA/X, AA'/Y] \vdash IO}\ \nu_R}{AA' = \nu Y. B[AA/X] \vdash \nu Y. \mu X. B}\ \nu_L}{AA = \mu X. \nu Y. B \vdash \nu Y \mu X. B_{IO}}\ \mu_L$$

$$B[AA/X, AA'/Y]$$

$$\Downarrow$$

$$AA < AA'$$

$$IO < IO'$$

$$\wedge B_2$$

Non-Wellfounded Sequent Calculus

Consider your favourite logic $\mathscr{L}$ & add fixed points as in the μ-calculus

$\mu X.X$

$\nu X.X$

Pre-proofs are the trees **coinductively** generated by:

- $\mathscr{L}$ inference rules
- inference for μ, ν:

$$\dfrac{\Gamma, F[\mu X.F/X] \vdash \Delta}{\Gamma, \mu X.F \vdash \Delta}\,[\mu_l] \qquad \dfrac{\Gamma, F[\nu X.F/X] \vdash \Delta}{\Gamma, \nu X.F \vdash \Delta}\,[\nu_l]$$

$$\dfrac{\Gamma \vdash F[\mu X.F/X], \Delta}{\Gamma \vdash \mu X.F, \Delta}\,[\mu_r] \qquad \dfrac{\Gamma \vdash F[\nu X.F/X], \Delta}{\Gamma \vdash \nu X.F, \Delta}\,[\nu_r]$$

Circular (pre-)proofs: the regular fragment of infinite (pre-)proofs, ie finitely many sub-(pre)proofs.

Pre-proofs are unsound!! ***Need for a validity condition***

$$\dfrac{\dfrac{\vdash \mu X.X, \nu X.X}{\vdash \mu X.X, \nu X.X}\,\nu}{\vdash \mu X.X, \nu X.X}\,\mu$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X}\,[\mu]}{\vdash \mu X.X}\,[\mu] \qquad \dfrac{\dfrac{\vdots}{\vdash \nu X.X, F}\,[\nu]}{\vdash \nu X.X, F}\,[\nu]}{\vdash F}\,[\mathrm{Cut}] \longrightarrow \text{cut}$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X}\,\mu}{\vdash \mu X.X}\,\mu \qquad \dfrac{\dfrac{\vdots}{\vdash \nu X.X, F}\,\nu}{\vdash \nu X.X, F}\,\nu}{\vdash F}\,\text{cut}$$

Fischer-Ladner subformulas

$FL(F)$ is the least set of formula occurrences such that:

- $F \in FL(F)$;
- $G_1 \star G_2 \in FL(F) \Rightarrow G_1, G_2 \in FL(F)$ for $\star \in \{\vee, \wedge\}$;
- $\sigma X.B \in FL(F) \Rightarrow B[\sigma X.B/X] \in FL(F)$ for $\sigma \in \{\mu, \nu\}$;
- $mG \in FL(F) \Rightarrow G \in FL(F)$ for $m \in \{[a], \langle a \rangle\}$.

Fact

$FL(F)$ is a finite set for any formula F.

Example: $F = \nu X.((a \vee a^{\perp}) \wedge (X \wedge \mu Y.X))$

$$FL(F) = \{F\ ,\ (a \vee a^{\perp}) \wedge (F \wedge \mu Y.F)\ ,\ \begin{array}{cc} a \vee a^{\perp} \quad , & \begin{array}{c} a \\ a^{\perp} \end{array} \\ F \wedge \mu Y.F\ , & \mu Y.F \end{array}\ \}$$

Fischer-Ladner subformulas

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Fact

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Example: $F = \nu X.((a \vee a^\perp) \wedge (X \wedge \mu Y.X))$

$$\begin{tikzcd}
 & & a \vee a^\perp \arrow[r] \arrow[dr] & a \\
FL(F) = F \arrow[r] & (a \vee a^\perp) \wedge (F \wedge \mu Y.F) \arrow[ur] \arrow[dr] & & a^\perp \\
 & & F \wedge \mu Y.F \arrow[r] \arrow[ull, bend left] & \mu Y.F \arrow[ulll, bend left]
\end{tikzcd}$$

Infinite threads, validity

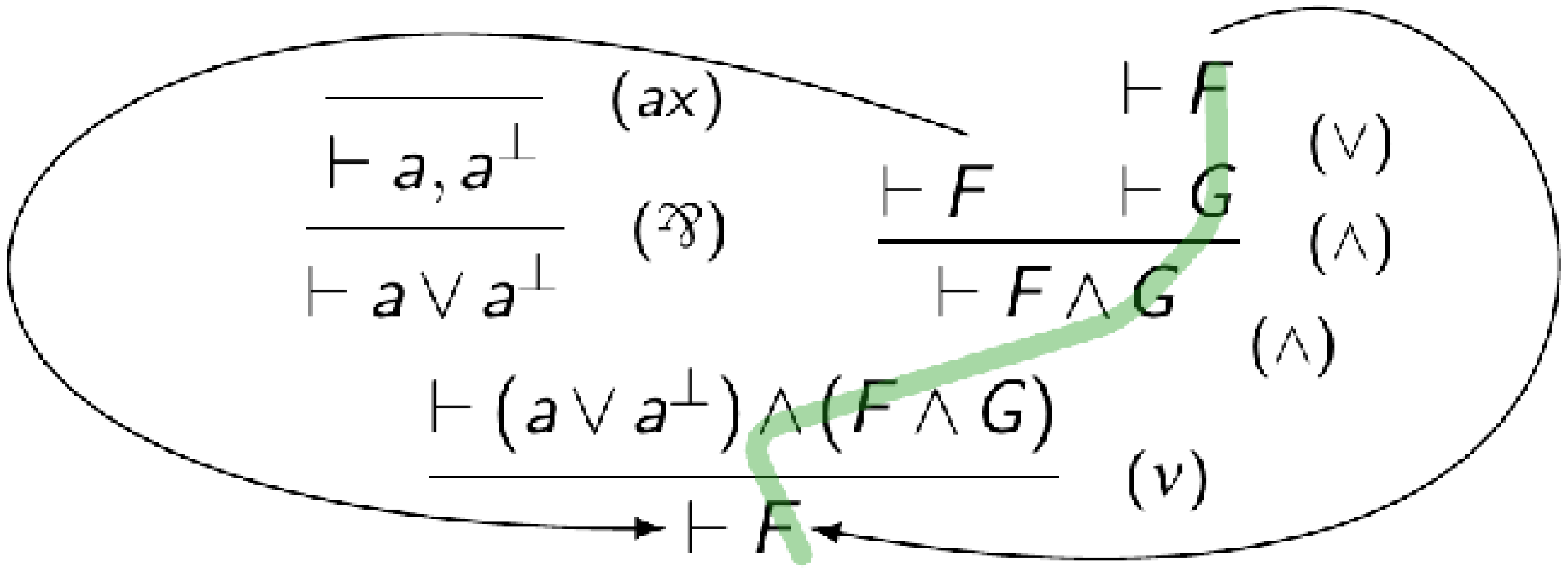
1 Regular proof trees $\to$ Circular proofs
2 Prop: $\vdash \Gamma$ has a non-wellfounded proof, it has a circular proof
3 Prop Validity checking is decidable.
4 Prop Circular provability is sound.

A ***thread*** on an infinite branch $(\Gamma_i)_{i\in\omega}$ is an infinite sequence of formula occurrences $(F_i)_{i\geq k}$ such that for any $i \geq k$, $F_i \in \Gamma_i$ and F_{i+1} is an immediate ancestor of F_i.

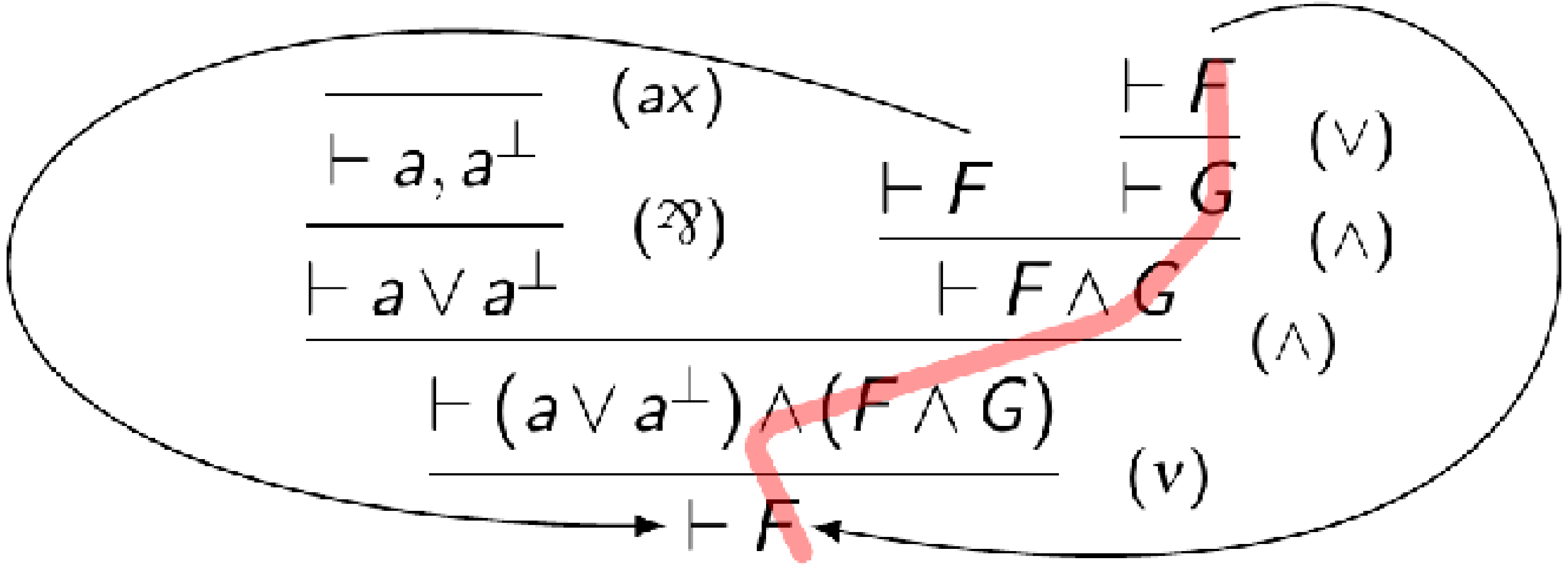
A thread is valid if it unfolds infinitely many ν. More precisely, if the minimal ***recurring*** principal formula of the thread is a ν-formula.

A proof is valid if every infinite branch contains a valid thread.

$$F = \nu X.((a \vee a^\perp) \wedge (X \wedge \mu Y.X)).$$
$$G = \mu Y.F$$



$$F = \nu X.((a \vee a^\perp) \wedge (X \wedge G))$$
$$G = \mu Y.\nu X.((a \vee a^\perp) \wedge (X \wedge Y))$$



Examples of circular proofs

- **Inductive and coinductive definitions**

$$N = \mu X.\top \vee X \qquad\qquad S = \nu X.(N \wedge X)$$

natural numbers   streams of nats.

- **Proofs-programs over these data types**

$double$	:	$N \to N$	
$double(n)$	$=$	0	if $n = 0$
	$=$	$succ(succ(double(m)))$	if $n = succ(m)$

$$\pi_0 = \dfrac{\dfrac{\dfrac{}{\vdash \top}\ (\top)}{\vdash \top \vee N}\ (\vee_r^1)}{\vdash N}\ (\mu)$$

$$\pi_{k+1} = \dfrac{\dfrac{\dfrac{\pi_k}{\vdash N}}{\vdash \top \vee N}\ (\vee_r^2)}{\vdash N}\ (\mu)$$

$$\Pi_{double} = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \top}\ (\top_r)}{\vdash \top \vee N}\ (\vee_r^1)}{\vdash N}\ (\mu_r)}{\top \vdash N}\ (\top_l) \qquad \dfrac{\pi'}{N \vdash N}\ (\mu_r)}{\top \vee N \vdash N}\ (\vee_l)}{N \vdash N}\ (\mu_l)$$

$\pi' =$

$\vdash N$

computational content of π:

$$\dfrac{\dfrac{\pi_{k+1}}{\vdash N} \qquad \dfrac{\pi}{N \vdash N}}{\vdash N}\ \text{cut}$$

$\Downarrow$

$$\dfrac{\dfrac{\pi_k}{\vdash N} \qquad \dfrac{\pi'}{N \vdash N}}{\vdash N}\ \text{cut}$$

$\Rightarrow$. $\mu X. X \Rightarrow X$.

X occurs positively

valid in 2 sided LK

(μLJ).

$$\dfrac{\dfrac{\Pi_0}{\vdash N} \qquad \dfrac{\Pi'}{N \vdash N}}{\vdash N}\,\mathrm{cut} \;\xrightarrow{2}\; \dfrac{\dfrac{}{\vdash T}\,(T_R) \qquad \dfrac{\dfrac{\Pi''}{\vdash N}}{T \vdash N}\,T_\ell}{\vdash N} \;\longrightarrow\; \dfrac{\Pi''}{\vdash N}$$

$$\mathrm{cut}(\Pi', \Pi_{k+1}) \xrightarrow{4} \mathrm{cut}(\Pi', \Pi_k)$$

$$\mathrm{cut}(\Pi', \Pi_0) \xrightarrow{3} \Pi_0 .$$

$$\forall k \in \mathbb{N} \quad \mathrm{cut}(\Pi', \Pi_k) \xrightarrow{*} \Pi_0 . \qquad \boxed{\dfrac{\dfrac{\Pi_0}{\vdash N}}{N \vdash N}\,W_L}$$

Examples of circular proofs

- **Inductive and coinductive definitions**

$$\color{red}{N} = \mu X.\top \vee X \qquad\qquad \color{red}{S} = \nu X.(N \wedge X)$$

- **Proofs-programs over these data types**

$$\begin{array}{lcl} enum & : & N \to S \\ enum(n) & = & n :: enum(succ(n)) \end{array}$$

$$\pi_{\mathrm{succ}} = \dfrac{\dfrac{\dfrac{}{N \vdash N}\,(ax)}{N \vdash \top \vee N}\,(\vee_r^2)}{N \vdash N}\,(\mu_r)$$

$$\Pi_{enum} = \dfrac{\dfrac{\dfrac{\dfrac{}{N \vdash N}\,(ax) \qquad \dfrac{\dfrac{\pi_{\mathrm{succ}}}{N \vdash N} \quad \dfrac{\Pi_{enum}}{N \vdash S}}{N \vdash S}\,(cut)}{N, N \vdash N \wedge S}\,(\wedge_r)}{N \vdash N \wedge S}\,(C_l)}{N \vdash S}\,(\nu)$$

$$\dfrac{\Pi_R \qquad N \vdash S}{\vdash S}\ cut$$

$$\dfrac{\dfrac{\Pi_R}{\vdash N} \qquad \dfrac{\Pi_{k+1} \quad \Pi_{enum}}{\vdash S}\ cut}{\dfrac{\vdash N \wedge S}{\vdash S}\ \nu_\wedge}\ \wedge_r$$

$$\dfrac{\Pi_{k+1} \qquad \vdash S}{\ }\ \nu,\wedge$$

$$\dfrac{\dfrac{\Pi_R \qquad \dfrac{\Pi_{k+1} \quad \vdash S}{\vdash S}\ \nu,\wedge}{\vdash N \wedge S}\ \wedge}{S}\ \nu_\wedge$$

Circular & finitary proofs

From finitary to circular proofs

Theorem

Finitary proofs can be transformed to (valid) circular proofs.

The key translation step is the following:

$$\dfrac{\dfrac{\pi_1}{\vdash \Gamma, S} \qquad \dfrac{\pi_2}{\vdash S^\perp, F[S]}}{\vdash \Gamma, \nu X.F}\ (\nu) \quad \longmapsto \quad \dfrac{\dfrac{[\pi_1]}{\vdash \Gamma, S} \qquad \dfrac{\dfrac{\dfrac{[\pi_2]}{\vdash S^\perp, F[S]} \qquad \dfrac{\boxed{\vdash S^\perp, \nu X.F}}{\vdash (F[S])^\perp, F[\nu X.F]}\ (r_F)}{\vdash S^\perp, F[\nu X.F]}\ (cut)}{\boxed{\vdash S^\perp, \nu X.F}}\ (\nu)}{\vdash \Gamma, \nu X.F}\ (cut)$$

$$\dfrac{\vdash \Gamma, F[\wedge \Gamma^\perp]}{\vdash \Gamma, \nu X.F}\ \nu'$$

Proof systems with the omega rule

$$\frac{\Gamma, \nu^1 \mathsf{x} A \quad \Gamma, \nu^2 \mathsf{x} A \quad \cdots}{\Gamma, \nu \mathsf{x} A}\ \nu_\omega$$

(1) Studer proves completeness of the ω-proof system

(Language $\mathcal{L}_\mu^+$). Let Φ be a countable set of atomic propositions and their negations $\mathsf{p}, {\sim}\mathsf{p}, \mathsf{q}, {\sim}\mathsf{q}, \mathsf{r}, {\sim}\mathsf{r}, \ldots$, let V be a set containing countably many variables and their negations $\mathsf{X}, {\sim}\mathsf{X}, \mathsf{Y}, {\sim}\mathsf{Y}, \mathsf{Z}, {\sim}\mathsf{Z}, \ldots$, let $\mathsf{T} = \{\top, \bot\}$ be a set containing symbols for truth and falsehood and M a set of indices. Define the formulae of the language $\mathcal{L}_\mu$ inductively as follows:

1. If P is an element of $\Phi \cup \mathsf{V} \cup \mathsf{T}$, then P is a formula of $\mathcal{L}_\mu$.
2. If A and B are formulae of $\mathcal{L}_\mu$, then so are $(A \wedge B)$ and $(A \vee B)$.
3. If A is a formula of $\mathcal{L}_\mu$ and $i \in \mathsf{M}$, then so are $\Box_i A$ and $\Diamond_i A$.
4. If A is a formula of $\mathcal{L}_\mu$ and the negated variable ${\sim}\mathsf{X}$ does not occur in A, then $(\mu\mathsf{X})A$ and $(\nu\mathsf{X})A$ are also formulae of $\mathcal{L}_\mu$.
5. If A is a formula of $\mathcal{L}_\mu^+$ and the negated variable ${\sim}\mathsf{X}$ does not occur in A, then for every natural number $k > 0$, $(\nu^k\mathsf{X})A$ is also a formula of $\mathcal{L}_\mu^+$.

$$\frac{}{\Gamma, \mathsf{p}, {\sim}\mathsf{p}}\ (\mathrm{ID1}), \qquad \frac{}{\Gamma, \mathsf{X}, {\sim}\mathsf{X}}\ (\mathrm{ID2}), \qquad \frac{}{\Gamma, \top}\ (\mathrm{ID3}).$$

$$\frac{\Gamma, \mathcal{A}[\top/\mathsf{X}]}{\Gamma, (\nu^1\mathsf{X})\mathcal{A}}\ (\nu.1) \qquad \frac{\Gamma, \mathcal{A}[(\nu^k\mathsf{X})\mathcal{A}]}{\Gamma, (\nu^{k+1}\mathsf{X})\mathcal{A}}\ (\nu.k+1)$$

$$\frac{\Gamma, A, B}{\Gamma, A \vee B}\ (\vee) \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ (\wedge) \qquad \frac{\Gamma, A}{\Diamond_i\Gamma, \Box_i A, \Sigma}\ (\Box) \qquad \frac{\Gamma, \mathcal{A}[(\mu\mathsf{X})\mathcal{A}]}{\Gamma, (\mu\mathsf{X})\mathcal{A}}\ (\mu) \qquad \frac{\Gamma, (\nu^k\mathsf{X})\mathcal{A} \quad \text{for all } k > 0}{\Gamma, (\nu\mathsf{X})\mathcal{A}}\ (\nu.\omega)$$

Embedding the omega-proof system in the circular proof system

Assume we are given the following $\mathsf{T}^{\omega}_{\mu+}$ proof of $(\mu \mathsf{X})\Box\mathsf{X}, (\nu \mathsf{Y})\Diamond\mathsf{Y}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(\mu X)\Box X, \top}\,\top}{\Box((\mu X)\Box X), \Diamond\top}\,\Box}{(\mu X)\Box X, \Diamond\top}\,\mu}{(\mu X)\Box X, (\nu^1 Y)\Diamond Y}\,\nu 1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(\mu X)\Box X, \top}\,\top}{\Box((\mu X)\Box X), \Diamond\top}\,\Box}{(\mu X)\Box X, \Diamond\top}}{(\mu X)\Box X, (\nu^1 Y)\Diamond Y}}{\Box((\mu X)\Box X), \Diamond((\nu^1 Y)\Diamond Y)}\,\Box}{(\mu X)\Box X, \Diamond((\nu^1 Y)\Diamond Y)}}{(\mu X)\Box X, (\nu^2 Y)\Diamond Y} \qquad \cdots}{(\mu X)\Box X, (\nu Y)\Diamond Y}\,\nu.\omega$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\mu X)\Box X, \top}{\Box((\mu X)\Box X), \Diamond\top}}{(\mu X)\Box X, \Diamond\top}}{(\mu X)\Box X, (\nu^1 Y)\Diamond Y}}{\Box((\mu X)\Box X), \Diamond((\nu^1 Y)\Diamond Y)}}{(\mu X)\Box X, \Diamond((\nu^1 Y)\Diamond Y)}}{(\mu X)\Box X, (\nu^2 Y)\Diamond Y}}{(\mu X)\Box X, (\nu Y)\Diamond Y} \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\cancel{(\mu X)\Box X, \top}}{\cancel{\Box((\mu X)\Box X), \Diamond\top}}}{\cancel{(\mu X)\Box X, \Diamond\top}}}{(\mu X)\Box X, (\nu Y)\Diamond Y}}{\Box((\mu X)\Box X), \Diamond((\nu Y)\Diamond Y)}}{(\mu X)\Box X, \Diamond((\nu Y)\Diamond Y)}\,\nu}{(\mu X)\Box X, (\nu Y)\Diamond Y}}{(\mu X)\Box X, (\nu Y)\Diamond Y} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{(\mu X)\Box X, (\nu Y)\Diamond Y}}{\Box((\mu X)\Box X), \Diamond((\nu Y)\Diamond Y)}}{(\mu X)\Box X, \Diamond((\nu Y)\Diamond Y)}}{(\mu X)\Box X, (\nu Y)\Diamond Y}$$

A crucial ingredient to this construction is a cardinality argument which shows that after dropping the iteration numbers, there will be two identical sequents with the same distinguished formula. The following function provides an upper bound on the number of different sequents (taking also into account the different possibilities for the distinguished formula) that may occur in a proof of Γ after dropping the iteration numbers.

Definition 6.1. Let f the function assigning to each $\mathcal{L}_\mu^-$ sequent Γ a natural number as follows:

$$\mathsf{f}(\Gamma) := |\mathbb{FL}(\Gamma^-)| \cdot 2^{|\mathbb{FL}(\Gamma^-)|} + 1$$

where $|\mathbb{FL}(\Gamma^-)|$ is the cardinality of the Fischer-Ladner closure of Γ^-.

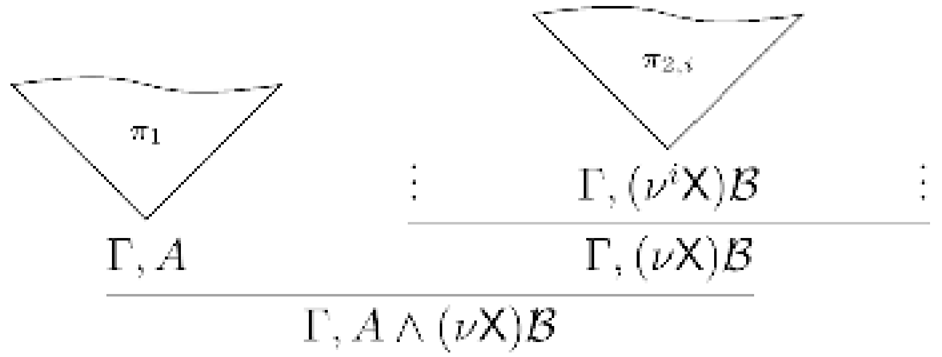
1. The pruned proof tree is a finite tree. When an instance of $(\nu.\omega)$ is treated, then a branch is selected and only that branch contributes to the construction of the pruned proof tree. Therefore there is no infinite branching in the pruned proof tree.

2. In the construction of $\mathcal{PPT}(\pi_2)$, the end-sequent $\Gamma, (\nu\mathsf{X})\mathcal{B}$ has been dropped. The pruned proof tree $\mathcal{PPT}(\pi_2)$ ends with $\Gamma, (\nu^k\mathsf{X})\mathcal{B}$. Therefore, at this point, $(*)$ is not an instance of $(\wedge)$.

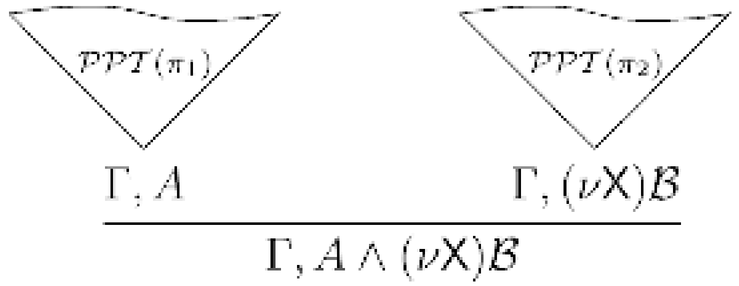
3. In the sequel we will construct a $\mathsf{T}_\mu^{\mathsf{pre}}$ preproof from a given pruned proof tree $\mathcal{PPT}$. In the course of this construction we will drop all the iteration numbers occurring in the sequents of $\mathcal{PPT}$. Note that dropping the iterations number in the above example makes $(*)$ an instance of $(\wedge)$.

4. If we had kept both the end-sequent $\Gamma, (\nu\mathsf{X})\mathcal{B}$ and its premise $\Gamma, (\nu^k\mathsf{X})\mathcal{B}$ in $\mathcal{PPT}(\pi_2)$, then dropping the iteration numbers would leave us with an inference where the premise and the conclusion are equal. Thus we can drop the end-sequent.

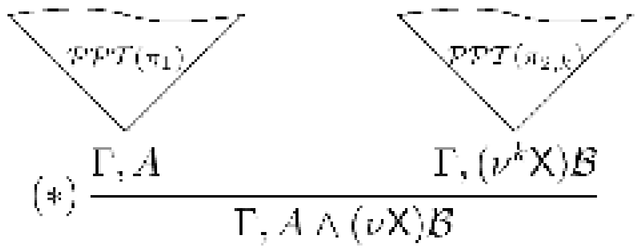
Example 6.3. If π is a $\mathsf{T}_{\mu+}^\omega$ proof, then we denote the pruned proof tree of π by $\mathcal{PPT}(\pi)$. Assume we are given the following $\mathsf{T}_{\mu+}^\omega$ proof:



Let us now construct the corresponding pruned proof tree. In a first step we obtain:



where π_2 is the subproof deriving $\Gamma, (\nu\mathsf{X})\mathcal{B}$. When we construct $\mathcal{PPT}(\pi_2)$, then we get:



where $k = \mathsf{f}(\Gamma, (\nu\mathsf{X})\mathcal{B})$. We make the following observations.

Theorem 6.6. *For all closed $\mathcal{L}_\mu$ formulae D we have*

$$\mathsf{T}^{\omega}_{\mu+} \vdash D \quad\Longrightarrow\quad \mathsf{T}^{\mathsf{pre}}_{\mu} \vdash D.$$

Proof. Given the $\mathsf{T}^{\omega}_{\mu+}$ proof of D, we can construct the corresponding pruned proof tree and from that a preproof of D according to the Definitions 6.2 and 6.5. It remains to show that every infinite path of the preproof contains a ν-thread. First, we notice that an infinite branch can only occur because of Condition 3a in Definition 6.5. Assume that we are given an infinite branch. Let $B_1, B_2, \ldots$ be a thread of this branch that contains a formula of the form $(\nu\mathsf{X})\mathcal{A}$ for which Condition 3a has been applied. Suppose that this thread contains the formula $(\mu\mathsf{Y})\mathcal{B}$ infinitely often. Then this thread must be of the form

$$\ldots, (\nu\mathsf{Z})\mathcal{C}, \ldots, (\mu\mathsf{Y})\mathcal{B}, \ldots, (\nu\mathsf{Z})\mathcal{C}, \ldots \tag{3}$$

such that there is a loop because of Condition 3a for $(\nu\mathsf{Z})\mathcal{C}$. Thus there must be a thread of the form

$$\ldots, (\nu^{i}_{\mathcal{I}}\mathsf{Z})\mathcal{C}, \ldots, (\mu\mathsf{Y})\mathcal{B}, \ldots, (\nu^{j}_{\mathcal{I}}\mathsf{Z})\mathcal{C}, \ldots$$

in the original $\mathsf{T}^{\omega}_{\mu+}$ proof of D (note that this thread need not be the same as (3), there may be different formulae at the $\ldots$ positions). Applying Lemma 4.11 to this thread yields that Z is higher than Y in D. Thus the infinite branch contains a ν-thread. $\square$

$$\boxed{\dfrac{\Gamma, \nu^{k} X.A \quad \{(\Gamma, \nu X A)\}_{k>0}}{\vdash \Gamma, \nu X.A}\ \cancel{\nu\omega}}$$

Kaz $\vdash A$.

Theorem 7.1. *The system $\mathsf{T}^{\mathsf{pre}}_{\mu}$ is sound.*

Corollary 7.2. *Let A be an $\mathcal{L}_\mu$ formula. We have*

$$A \text{ is valid} \quad\Longrightarrow\quad \mathsf{T}^{\omega}_{\mu+} \vdash A \quad\Longrightarrow\quad \mathsf{T}^{\mathsf{pre}}_{\mu} \vdash A \quad\Longrightarrow\quad A \text{ is valid.}$$