M2 LMFI – SOFIX

Quantification du second-ordre et points fixes en logique

Realizability in System F and applications to strong normalization

Today:

- finish the proof of Strong normalization.
- consider some applications of realizability.
- back to second-order logic and second-order arithmetic.

Définition 2.4 (*Pole*)

Given a set of terms Λ_0 containing the variables, a Λ_0**-pole** *is a subset of* P *satisfying following two properties of closure by anti-reduction with respect to* Λ_0*:*

1. *If* $(t\{u/x\}, \pi) \in \perp\!\!\!\perp$ *and* $u \in \Lambda_0$*, then* $(\lambda x.\, t, u \cdot \pi) \in \perp\!\!\!\perp$*.*
2. *If* $(t, u \cdot \pi) \in \perp\!\!\!\perp$ *then* $((t)\, u, \pi) \in \perp\!\!\!\perp$*.*

Définition 2.9 (Π_0, F_{Λ_0})

Given a Λ_0*-pole* $\perp\!\!\!\perp$*,* Π_0 *denotes the set of stacks built from elements of* Λ_0*.*

One shall write F_{Λ_0} *for the set of* **non-empty subsets** *of* Π_0*.*

Définition 2.10 (*Valuation*)

Given a Λ_0*-pole* $\perp\!\!\!\perp$*, a* **valuation** v *is a function from type variables to subsets of* Π_0*.*

Given a valuation v*,* X *a type variable and* $F \subseteq \Pi_0$*,* $\mathsf{v}[X := F]$ *is defined as the valuation equal to* F *on* X *and equal to* v *on any other type variable.*

Définition 2.11 (*Interpretation of a type, falsity value*)

Given a Λ_0*-pole* $\perp\!\!\!\perp$ *and a valuation* v*, one defines inductively* **the interpretation** $\|\_\|_{\mathsf{v}}$ **of F-types** *(taking values in the subsets of* Π_0*) as follows:*

- $\|X\|_{\mathsf{v}} = \mathsf{v}(X)$;
- $\|A \Rightarrow B\|_{\mathsf{v}} = \{t \cdot \pi \mid t \in \|A\|_{\mathsf{v}}^{\perp}, \pi \in \|B\|_{\mathsf{v}}\}$;
- $\|\forall X; A\|_{\mathsf{v}} = \cup_{\emptyset \subsetneq F \subseteq \Pi_0} \|A\|_{\mathsf{v}[X:=F]}$.

$\|T\|_{\rho}$ *will be called* **the falsity value** *of* T*.*

3 Adequation lemma (Adequacy lemma)

Définition 3.1 (***weakly/well adapted valuations***)

A valuation v *is* **weakly adapted** *to a* Λ_0*-pole* $\perp\!\!\!\perp$ *if, for any type* T*,*

$$|T|_{\mathsf{v}} \subseteq \Lambda_0.$$

A valuation v *is* **adapted** *(or well-adapted) to a* Λ_0*-pole* $\perp\!\!\!\perp$ *if, for any type* T*,*

$$\mathcal{V} \subseteq |T|_{\mathsf{v}} \subseteq \Lambda_0.$$

Définition 3.3 (***Admissible set of terms***)

A set $\Lambda_0 \subseteq \Lambda$ *is* **admissible** *if there exists a* Λ_0*-pole* $\perp\!\!\!\perp$ *and a valuation* v *which is well-adapted for* $\perp\!\!\!\perp$*.*

Lemme 3.4 (***Adequation lemma***)

Let v *be a (weakly) adapted valuation for a pole* $\perp\!\!\!\perp$ *and let* t *be a term such that* $x_1 : U_1, \ldots, x_n : U_n \vdash_{\mathsf{F}} t : T$ *is derivable in Curry-Style* F*. Let* $(u_i)_{1\leq i\leq n}$ *be realizers of the* $(U_i)_{1\leq i\leq n}$ *(ie.* $u_i \Vdash_{\mathsf{v}} U_i$ *for* $1 \leq i \leq n$*), then* $t\,\{u_i/x_i, 1 \leq i \leq n\} \Vdash_{\mathsf{v}} T$*.*

Lemme 3.4 (*Adequation lemma*)

Let v *be a (weakly) adapted valuation for a pole* $\perp\!\!\!\perp$ *and let* t *be a term such that* $x_1 : U_1, \ldots, x_n : U_n \vdash_{\mathsf{F}} t : T$ *is derivable in Curry-Style* F*. Let* $(u_i)_{1 \leq i \leq n}$ *be realizers of the* $(U_i)_{1 \leq i \leq n}$ *(ie.* $u_i \Vdash_{\mathsf{v}} U_i$ *for* $1 \leq i \leq n$*), then* $t \{u_i/x_i, 1 \leq i \leq n\} \Vdash_{\mathsf{v}} T$*.*

Démonstration : One proves the lemma by induction on a typing derivation d of $x_i : U_i \vdash_{\mathsf{F}} t : T$. (Note that there may exist several such typing derivations since we work with Curry-Style System F...) One shall write $\Gamma = x_1 : U_1, \ldots, x_n : U_n$ and $t' = t\{u_i/x_i, 1 \leq i \leq n\}$.

- If d is an axiom, the property trivialy holds since $t' = u_i$ for some i which realizes $U_i = T$ by hypothesis.
- If d ends with $\rightarrow I$, one has $t = \lambda x.v$, $T = U \rightarrow V$, and $x_1 : U_1, \ldots x_n : U_n, x : U \vdash_{\mathsf{F}} v : V$ Let $v' = v\{u_i/x_i, 1 \leq i \leq n\}$. We want to prove that t' realizes T for valuation v: one considers a stack $\pi \in \|T\|_{\mathsf{v}}$.

  There are only two possibilities: either no such stack exists and then t' réalizes T trivially, or π has form $u \cdot \pi'$, with $u \Vdash_{\mathsf{v}} U$ and $\pi' \in \|V\|_{\mathsf{v}}$.

  In the second case, we know by induction hypothesis that $v'\{u/x\} \Vdash_{\mathsf{v}} V$ from which $(v'\{u/x\}, \pi') \in \perp\!\!\!\perp$ and by closure by KAM-anti-reduction of $\perp\!\!\!\perp$ (more precisely by property 1.) and since $u \in |U|_{\mathsf{v}} \subseteq \Lambda_0$ by (weak) adaptation of v, one also has that $(t\{u/x\}, u \cdot \pi') \in \perp\!\!\!\perp$ which shows that $t' \Vdash_{\mathsf{v}} T$ since the stack was chosen arbitrarily.

Lemme 3.4 (*Adequation lemma*)

Let v be a (weakly) adapted valuation for a pole $\perp\!\!\!\perp$ and let t be a term such that $x_1 : U_1, \ldots, x_n : U_n \vdash_{\mathsf{F}} t : T$ is derivable in Curry-Style F. *Let $(u_i)_{1\leq i\leq n}$ be realizers of the $(U_i)_{1\leq i\leq n}$ (ie. $u_i \Vdash_{\mathsf{v}} U_i$ for $1 \leq i \leq n$), then $t\{u_i/x_i, 1 \leq i \leq n\} \Vdash_{\mathsf{v}} T$.*

$\forall u\{u_1/x_1 \ldots u_n/x_n\} \Vdash V \to T$ $\quad \|V \to T\|$.

$u_i \Vdash_v U_i$. $\quad \forall v\{u_1/x_1 \ldots u_n/x_n\} \Vdash V$. $\quad (u', v' \cdot \pi) \in \perp\!\!\!\perp$

$t' = t\{u_1/x_1 \ldots u_n/x_n\} \Vdash T$, for $\pi \in \|T\|$. $(t', \pi) \in \perp\!\!\!\perp$, $((u')v', \pi) \in \perp\!\!\!\perp$.

- If d ends with $\to E$, then we have $t = (u)v$ with $x_1 : U_1, \ldots x_n : U_n \vdash_{\mathsf{F}} u : V \to T$ and $x_1 : U_1, \ldots x_n : U_n \vdash_{\mathsf{F}} v : V$ for some type V.

  One can apply the induction hypothesis to both derivation d_u and d_v concluding $x_1 : U_1, \ldots x_n : U_n \vdash_{\mathsf{F}} u : V \to T$ and $x_1 : U_1, \ldots x_n : U_n \vdash_{\mathsf{F}} v : V$ which ensures that $u' = u\{u_i/x_i, 1 \leq i \leq n\}$ and $v' = v\{u_i/x_i, 1 \leq i \leq n\}$ realizes respectively $V \to T$ and V for valuation v.

  To show that t' realizes T, it is enough to consider an arbitrary stack π in $\|T\|_{\mathsf{v}}$ and to remark that $v' \cdot \pi \in \|V \to T\|$ and thus that $(u', v' \cdot \pi) \in \perp\!\!\!\perp$.

  As before $v' \in \Lambda_0$ and, using the second closure property of the pole, one gets $(t', \pi) \in \perp\!\!\!\perp$, which means, since π is any stack in $\|T\|_{\mathsf{v}}$, that $t' \Vdash_{\mathsf{v}} T$.

Lemme 3.4 (*Adequation lemma*)

> *Let v be a (weakly) adapted valuation for a pole $\perp\!\!\!\perp$ and let t be a term such that $x_1 : U_1, \ldots, x_n : U_n \vdash_{\mathsf{F}} t : T$ is derivable in Curry-Style F. Let $(u_i)_{1 \le i \le n}$ be realizers of the $(U_i)_{1 \le i \le n}$ (ie. $u_i \Vdash_{\mathsf{v}} U_i$ for $1 \le i \le n$), then $t\,\{u_i/x_i, 1 \le i \le n\} \Vdash_{\mathsf{v}} T$.*

- If d ends with $\forall I$, then one has $T = \forall X.U$ and $x_1 : U_1, \ldots x_n : U_n \vdash_{\mathsf{F}} t : U$ where X does not occur free in the U_i.

  To show that $t' \Vdash_{\mathsf{v}} \forall X.U$, let us consider $\pi \in \|\forall X.U\|_{\mathsf{v}}$. We know by definition of the realizability interpretation that there exists $F \in F_{\Lambda_0}$ such that $\pi \in \|U\|_{\mathsf{v}[X:=F]} = \mathsf{v}'$

  But since X is not free in the U_i the interpretation of U_i is the same in v and in $\mathsf{v}' = \mathsf{v}[X := F]$, in particular, the lemma hypothesis tells us that $u_i \Vdash_{\mathsf{v}'} U_i$ if $1 \le i \le n$. One can therefore apply the induction hypothesis to the subderivation of conclusion $x_1 : U_1, \ldots x_n : U_n \vdash_{\mathsf{F}} t : U$ with respect to v': $t' \Vdash_{\mathsf{v}'} U$ so that $(t', \pi) \in \perp\!\!\!\perp$ which proves that $t' \Vdash_{\mathsf{v}} \forall X.U$.

- If d ends with $\forall E$, then we have a derivation d' more elementary than d, which concludes with $x_1 : U_1, \ldots x_n : U_n \vdash_{\mathsf{F}} t : \forall X.U$, with $T = U\,\{V/X\}$ for some V.

  Let us consider $\pi \in \|U\,\{V/X\}\|_{\mathsf{v}}$: we need to prove that $(t, \pi) \in \perp\!\!\!\perp$. The substitutivity lemma ensures that $\pi \in \|U\|_{\mathsf{v}[X:=\|V\|_{\mathsf{v}}]}$.

  By applying induction hypothesis to d', we have $t' \Vdash_{\mathsf{v}} \forall X.U$ so for any $F \in F_{\Lambda_0}$, we have that $t' \Vdash_{\mathsf{v}[X:=F]} U$, and in particular when $F = \|V\|_{\mathsf{v}}$.

  We then deduce that $(t', \pi) \in \perp\!\!\!\perp$ which allows to conclude the proof of the lemma.

$\square$

Lemme 3.4 (*Adequation lemma*)

Let v be a (weakly) adapted valuation for a pole $\perp\!\!\!\perp$ and let t be a term such that $x_1 : U_1, \ldots, x_n : U_n \vdash_{\mathsf{F}} t : T$ is derivable in Curry-Style F. Let $(u_i)_{1 \leq i \leq n}$ be realizers of the $(U_i)_{1 \leq i \leq n}$ (ie. $u_i \Vdash_{\mathsf{v}} U_i$ for $1 \leq i \leq n$), then $t\{u_i/x_i, 1 \leq i \leq n\} \Vdash_{\mathsf{v}} T$.

Adequation lemma allows to deduce easily that a typed term realizes its type and that typable terms are in the intersection of all admissible sets:

Théorème 3.5

If Λ_0 is admissible and $\Gamma \vdash_{\mathsf{F}} t : T$, then $t \in \Lambda_0$.

Démonstration : Indeed, if Λ_0 is admissible, then there exists a pole $\perp\!\!\!\perp$ and a valuation v adapted to Λ_0. The adequation lemma can be applied to variables which are realizers of any type and $t = t\{x_i/x_i\} \in |T|_{\mathsf{v}} \subseteq \Lambda_0$.

□

To prove strong normalization of F, it is therefore sufficient to prove that the set of strongly normalizing terms is admissible, that we will do in the following.

4 Application of realizability to strong normalization of system F

One shall now build a Λ_{SN}-pole $\perp\!\!\!\perp$ together with a well-adapted valuation v, that is such that for every type T,

$$\mathcal{V} \subseteq |T|_{\mathsf{v}} \subseteq \Lambda_{SN}.$$

Lemme 4.2

For any λ-terms t, u with u strongly normalizing and π a stack, then if $t\{u/x\}\pi$ is SN, $(\lambda x.\, t)\, u\pi$ is SN.

Démonstration : Let t, u, π as specified in the lemma's statement.

Let us consider $t' = (\lambda x.\, t)\, u\pi$ and $t'' = (t\{u/x\})\,\pi$.

Since t'' is SN, it comes immediately that $t \in \Lambda_{SN}$ and $\pi \in \Pi_{SN}$. Assume, aiming at a contradiction that there exists an infinite reduction sequence from t'. Thanks to the above remark, this reduction cannot be infinitely in t, u or in π.

Therefore one has $t' \longrightarrow_\beta^\star (\lambda x.\, t_0)\, u_0\pi_0 \longrightarrow_\beta (t_0\{u_0/x\})\,\pi_0 \longrightarrow_\beta^\star \ldots$, but we know that $t'' \longrightarrow_\beta^\star (t_0\{u_0/x\})\,\pi_0 \longrightarrow_\beta^\star \ldots$ which contradicts strong normalization of t''.

$\square$

Définition 4.3 ($\perp_{SN}$)

Let $\perp_{SN}$ be $\{(t,\pi) \in \mathsf{P} \mid (t)\,\pi \in \Lambda_{SN}\}$.

Proposition 4.4

$\perp_{SN}$ is a Λ_{SN}-pole.

Démonstration : On shall verify both KAM-anti-reduction closure properties:

- the first is a direct consequence of the previous lemma.
- the second is trivial considering the definition of the pole since processes $((t)u,\pi)$ and $(t, u\cdot\pi)$ correspond to the same λ-term $(t)\,u\pi$.

Lemme 4.5

For any $F \in F_{\Lambda_{SN}}$, we have, for $\perp\!\!\!\perp_{SN}$ orthogonality:

$$\mathcal{V} \subseteq F^{\perp} \subseteq \Lambda_{SN}.$$

Démonstration : Let $F \in F_{\Lambda_{SN}}$.

If $x \in \mathcal{V}$ and $\pi \in F \subseteq F_{\Lambda_{SN}}$, then $(x)\,\pi \in \Lambda_{SN}$ so that $x \in F^{\perp}$ and $\mathcal{V} \subseteq F^{\perp}$.

Of $t \in F^{\perp}$, as F is not empty, let $\pi \in F$. We have $(t)\,\pi \in \Lambda_{SN}$ and therefore it comes that $t \in \Lambda_{SN}$.

One deduce that $F^{\perp} \subseteq \Lambda_{SN}$.

$\square$

Proposition 4.6

Λ_{SN} is admissible.

Démonstration : Consider pole $\perp\!\!\!\perp_{SN}$, one define the valuation v_{SN} such that $\mathsf{v}_{SN}(X) = \Pi_{SN}$ for any type variable X.

It is sufficient to show that for all type T, $\|T\|_{\mathsf{v}_{SN}} \in \mathsf{F}_{\mathsf{SN}}$.

More precisely, one use a stronger induction hypothesis and prove that for any type T, $\|T\|_{\mathsf{v}_{SN}} \in \mathsf{F}_{\mathsf{SN}}$ as soon as v_{SN} takes its values in F_{SN} by induction on type T:

- Case $T = X$. Then $\|X\|_{\mathsf{v}_{\mathsf{SN}}} = \mathsf{v}_{\mathsf{SN}}(X) \in \mathsf{F}_{\mathsf{SN}}$ by hypothesis on v_{SN}.
- Case $T = U \to V$. Then, by induction hypothesis, $\|U\|_{\mathsf{v}_{\mathsf{SN}}}, \|V\|_{\mathsf{v}_{\mathsf{SN}}} \in \mathsf{F}_{\mathsf{SN}}$. By the previous lemma, $|U|_{\mathsf{v}_{\mathsf{SN}}} = \|U\|^{\perp}_{\mathsf{v}_{\mathsf{SN}}}$ contains all variables so that $\|T\|_{\mathsf{v}_{\mathsf{SN}}} = |U|_{\mathsf{v}_{\mathsf{SN}}} \cdot \|V\|_{\mathsf{v}_{\mathsf{SN}}}$ is non-empty and is a subset of Π_{SN} since $|U|_{\mathsf{v}_{\mathsf{SN}}} \subseteq \Lambda_{\mathsf{SN}}$ (by the lemma) and $\|V\|_{\mathsf{v}_{\mathsf{SN}}} \in \Pi_{\mathsf{SN}}$ by induction hypothesis: one has $\|T\|_{\mathsf{v}_{SN}} \in \mathsf{F}_{\mathsf{SN}}$.
- Case $T = \forall X.U$. Then $\|\forall X.U\|_{\mathsf{v}_{\mathsf{SN}}} = \cup_{F \in \mathsf{F}_{\mathsf{SN}}} \|U\|_{\mathsf{v}_{\mathsf{SN}}[X:=F]} \subseteq \mathsf{F}_{\mathsf{SN}}$ since every $\|U\|_{\mathsf{v}_{\mathsf{SN}}[X:=F]} \subseteq \mathsf{F}_{\mathsf{SN}}$ by induction hypothesis.

$\square$

The strong normalization theorem for System F is then a simple corollary of the previous result thanks to adequation lemma for realizability:

Corollaire 4.7

Every typable term in F is strongly normalizing.

Démonstration : We know by the corollary of adequation lemma that typable terms are in the intersection of all admissible sets, so that they are in Λ_{SN} which is admissible by the previous lemma.

$\square$

5 Some more applications of realizability

Proposition 5.2

There is no closed term t such that $\vdash_F t : \bot$. $= \forall X.X$.

Démonstration : Let us apply realizability: there is to show a set of terms Λ_0, a Λ_0-pole and a weakly admissible valuation set for this pole, allowing to use adequation lemma and its consequences.

Λ is of course an admissible set and we know that $\emptyset$ and Λ are Λ-poles (this is a general fact) and that every valuation is weakly admissible for these poles since $\Lambda_0 = \Lambda$ as noted above.

Let us consider $\perp\!\!\!\perp = \emptyset$ We have then $\|\forall X.X\|_v = \cup_{F \in F_\Lambda} F = \Pi$.

Let us reason by contradiction and assume that there exists a term t such that $\vdash t : \forall X.X$. By the theory of realizability, we know that t realize universally $\forall X.X$ ($t \Vdash_v \forall X.X$ for any valuation) this implies that for all $\pi \in \Pi$, we have $(t, \pi) \in \perp\!\!\!\perp$... which is impossible since $\perp\!\!\!\perp$ is empty: as a conclusion, such a term t cannot exist.

Proposition 5.3

If $\vdash_F t : \mathsf{ID}$, then $t \longrightarrow^\star_\beta \lambda x.x$.

$\mathrm{ID} = \forall X(X \to X)$

$t \in |\forall X(X \to X)|_v = \bigcap_{F \in F_\Lambda} |X \to X|_{v[X:F]}$

Démonstration : One shall again consider Λ as admissible set and consider $\perp\!\!\!\perp_x = \{(t, \pi) \mid (t)\pi \longrightarrow^\star x\}$. This is of course a pole since the closure properties are trivially met.

Let us consider $F^\emptyset = \{\emptyset\}$ (*ie.* the singleton made of the empty stack) and $v = [X := F^\emptyset]$. We have therefore $\boxed{x \Vdash_v X}$ (indeed, $(x, \emptyset) \in \perp\!\!\!\perp_x$) and if $\vdash t : \forall X.(X \to X)$ (so that in particular if it is a closed term), we have $\boxed{t \Vdash_v X \to X}$ so $(t, \underbrace{x \cdot \emptyset}) \in \perp\!\!\!\perp_x$ which ensures that $(t)x \longrightarrow^\star x$ by definition du pôle of the pole.

We have $(t)x \longrightarrow^\star (\lambda x.v)x \longrightarrow_\beta v \longrightarrow^\star x$ so that $t \longrightarrow^\star \lambda x.v \longrightarrow^\star \lambda x.x$, QED.

□

$T_R = \forall X.(\underbrace{X \to \cdots \to (X \to X)}_{R})$

Proposition 5.4

If $\vdash_{\mathsf{F}} t : \mathsf{Bool}$, *then* $t \longrightarrow_\beta^\star \lambda x.\, \lambda y.\, x$ *or* $t \longrightarrow_\beta^\star \lambda x.\, \lambda y.\, y$.

$= \forall X.(X \to (X \to X))$

Démonstration : The set $\perp\!\!\!\perp_{x,y} = \perp\!\!\!\perp_x \cup \perp\!\!\!\perp_y$ is a Λ-pole. Let us consider valuation $\mathsf{v} = [X := \{\emptyset\}]$ as before.

We clearly have $x \Vdash_{\mathsf{v}} X$ and $y \Vdash_{\mathsf{v}} X$ and by adequation lemma, if $\vdash_{\mathsf{F}} t : \mathsf{Bool}$, then $t \Vdash_{\mathsf{v}} X \to (X \to X)$ so that $(t)x \Vdash_{\mathsf{v}} X \to X$ and $(t)xy \Vdash_{\mathsf{v}} X$, that is $(t)xy \longrightarrow^\star x$ or $(t)xy \longrightarrow^\star y$. Since t is closed, we have: $(t)xy \longrightarrow^\star (\lambda x.v)xy \longrightarrow (v)y \longrightarrow^\star (\lambda y.w)y \longrightarrow w \longrightarrow^\star z \in \{x, y\}$. from which comes that $t \longrightarrow^\star \lambda x.v \longrightarrow^\star \lambda x.\lambda y.w \longrightarrow^\star \lambda x.\lambda y.z$ with $z \in \{x, y\}$, QED. □

$t \perp z \cdot s \cdot \emptyset$.

$\cancel{\forall X.(X \to X) \to (X \to X)}$.

$\mathsf{Nat} = \forall X.\, X \to (X \to X) \to X$

$(s, \pi) \in \perp\!\!\!\perp_{\mathsf{Nat}}$

Proposition 5.5

If $\vdash_{\mathsf{F}} t : \mathsf{Nat}$, *then there exists an natural* n *such that* $t \longrightarrow_\beta^\star \lambda z.\, \lambda s.\, (s)^n z$.

Démonstration : Let s and z be variables. Let us consider $\perp\!\!\!\perp_{\mathsf{Nat}} = \{(t, \pi) \mid \exists n \geq 0, (t)\pi \longrightarrow^\star (s)^n z\}$.

$\perp\!\!\!\perp_{\mathsf{Nat}}$ is a Λ-pole as before and we can consider the same valuation as before: $\mathsf{v} = [X := \{\emptyset\}]$.

We then have of course $z \Vdash_{\mathsf{v}} X$ (trivial) eand $s \Vdash_{\mathsf{v}} X \to X$. Indeed, if $\pi \in \|X \to X\|_{\mathsf{v}}$, we have $\pi = t' \cdot \emptyset$ with $\underbrace{t' \longrightarrow^\star (s)^k z}$ for some k, so that $(s)t' \longrightarrow^\star (s)^{k+1} z$, that is $(s, \pi) \in \perp\!\!\!\perp_{\mathsf{Nat}}$ and $s \Vdash_{\mathsf{v}} X \to X$.

Let then $\vdash_{\mathsf{F}} t : \mathsf{Nat}$, we have by the adequation lemma, after instanciation, that $t \Vdash_{\mathsf{v}} X \to (X \to X) \to X$ and then that $(t)zs \Vdash_{\mathsf{v}} X \to X$ and $(t)zs \longrightarrow^\star (s)^k z$. It comes that $(t)zs \longrightarrow^\star (\lambda z.v)zs \longrightarrow (v)s \longrightarrow^\star (\lambda s.w)s \longrightarrow w \longrightarrow^\star (s)^k z$ or otherwise said, that $t \longrightarrow^\star \lambda z.\lambda s.(s)^k z$, what needed to be proved. □

Proposition 5.9

There is no closed term t such that $\vdash_{\mathsf{F}} t : \mathsf{DNE}$. $\doteq \forall X\ (((X \to \bot) \to \bot) \to X)$

$x \in F^{\bot}$

$\lambda y.y \in G^{\bot}$

Démonstration : Le us reason by contradiction, assuming t is a closed term such that $\vdash_{\mathsf{F}} t : \mathsf{DNE}$.

Consider Λ as admissible set and consider $\bot_x = \{(t,\pi) \mid (t)\pi \longrightarrow^\star x\}$. Remember also that every valuation is weakly adapted wrt Λ, which is sufficient to apply adequacy lemma.

We know that t realizes universally $\forall X.(((X \to \bot) \to \bot) \to X)$, that is $t \Vdash_{\mathsf{v}} \cancel{\forall X}.(((X \to \bot) \to \bot) \to X)$ for any valuation v. Consider in particular $F = \{\emptyset\}$ and $G = \{x \cdot \emptyset\}$ and $\mathsf{v}_1 = [X := F]$ and $\mathsf{v}_2 = [X := G]$. We have: (i) F, G are non empty; (ii) F, G are disjoint; (iii) F, G have non empty orthogonal sets. We have $t \in |((X \to \bot) \to \bot) \to X|_{\mathsf{v}_i}$ for $i \in \{1,2\}$.

In particular, for any $u \in |(X \to \bot) \to \bot|_{\mathsf{v}_i}$, $(t)u \in |X|_{\mathsf{v}_i} = \mathsf{v}_i(X)^{\bot}$.

For any $v \in |X \to \bot|_{\mathsf{v}_i}$ and $w \in |X|_{\mathsf{v}_i}$, $(v)w \in |\bot|_{\mathsf{v}_i} = \emptyset$. Since $|X|_{\mathsf{v}_i} \neq \emptyset$ (as $x \in |X|_{\mathsf{v}_1}$ and $\lambda x.x \in |X|_{\mathsf{v}_2}$) we have that $|X \to \bot|_{\mathsf{v}_i} = \emptyset$. It follows that $\|(X \to \bot) \to \bot\|_{\mathsf{v}_i} = \emptyset$ and $|(X \to \bot) \to \bot|_{\mathsf{v}_i} = \Lambda$.

Therefore, for any $u \in \Lambda$, $(t)u \in F^{\bot}$ and $(t)u \in G^{\bot}$ which means:

- $(t)u \longrightarrow^\star x$ (using $(t)u \in F^{\bot}$);
- $(t)ux \longrightarrow^\star x$ (using $(t)u \in G^{\bot}$).

But that would imply $(x)x =_\beta x$ which is not, a contradiction.

$\square$

$PA_2 \rightsquigarrow$ $\forall^1 x$ $\forall x . \mathrm{Nat}(x)$.

$\forall^2 X^1$

$\forall^2 X^2$ $\mathbb{N}^3$

$\mathbb{Q}$.

- $\emptyset \subsetneq A \subsetneq \mathbb{Q}$ $\exists x, y, z.\ A(x, y, z)$.
- $\forall x, y \in \mathbb{Q},\ A(x) \wedge y < x \Rightarrow A(y)$
- $\exists z \in \mathbb{Q}\ \forall x \in \mathbb{Q}\ A(x) \Rightarrow x < z$.
- Openness condition to have unique representation.

Ex $(x,y,z) \rightsquigarrow \frac{x-y}{z}$

Reals can be represented as ternary relations in PA_2 using a variant of Dedeking cuts:

$$Real[R] = \cancel{\exists k.R(x)}\ \exists x,y,z.R(x,y,z) \wedge Ini[R] \wedge Bounded[R] \wedge Open[R]$$

together with:

- $Inf[n,m,p,n',m',p'] = (n \times p' + m' \times p) \leqslant (n' \times p + m \times p').$
- $Ini[X] = \forall n,m,p,n',m',p'\ \{X(n,m,p) \rightarrow \mathrm{Inf}[n',m',p',n,m,p] \rightarrow X(n',m',p')\}.$
- $Bounded[X] = \exists n,m,p \forall n',m',p'\ \{X(n',m',p') \rightarrow \mathrm{Inf}[n',m',p',n,m,p]\}.$
- $Open[\cancel{R}X] = \neg\exists n,m,p \forall n',m',p'\ \{X(n',m',p') \leftrightarrow \mathrm{Inf}[n',m',p',n,m,p]\}.$

$\forall R.\ Real(R) \Rightarrow \ldots$