$\underline{PA_2} \rightsquigarrow$ $\forall^1 x$ $\forall x . Nat(x)$.

$\forall^2 X^1$

$\forall^2 X^2$ $\mathbb{N}^3$

$\mathbb{Q}$.

- $\emptyset \subsetneq A \subsetneq \mathbb{Q}$ $\exists x, y, z . A(x, y, z)$.
- $\forall x, y \in \mathbb{Q}, A(x) \wedge y < x \Rightarrow A(y)$
- $\exists z \in \mathbb{Q}\ \forall x \in \mathbb{Q}\ A(x) \Rightarrow x < z$.
- Openness condition to have unique representation.

Ex $(x,y,z) \rightsquigarrow \dfrac{x-y}{z}$

Reals can be represented as ternary relations in PA_2 using a variant of Dedeking cuts:

$$Real[R] = \exists k.R(x) \wedge Ini[R] \wedge Bounded[R] \wedge Open[R]$$

$\exists x,y,z.R(x,y,z)$

together with:

- $Inf[n,m,p,n',m',p'] = (n \times p' + m' \times p) \leqslant (n' \times p + m \times p')$.

- $Ini[X] = \forall n,m,p,n',m',p' \; \{X(n,m,p) \rightarrow \mathrm{Inf}[n',m',p',n,m,p] \rightarrow X(n',m',p')\}$.

- $Bounded[X] = \exists n,m,p \, \forall n',m',p' \; \{X(n',m',p') \rightarrow \mathrm{Inf}[n',m',p',n,m,p]\}$.

- $Open[X] = \neg\exists n,m,p \, \forall n',m',p' \; \{X(n',m',p') \leftrightarrow \mathrm{Inf}[n',m',p',n,m,p]\}$.

$\forall R .\ Real(R) \Rightarrow \ldots$

Expressiveness of second order logic

Finite sets

$$\mathsf{atmost\_n}(U) \triangleq \exists x_1, \ldots \exists x_n . \forall y . (U(y) \Leftrightarrow \forall X . (X(x_1) \Rightarrow \ldots \Rightarrow X(x_n) \Rightarrow X(y)))$$

$$\mathsf{diff\_n}(x_1, \ldots, x_n) \triangleq x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \cdots \wedge x_1 \neq x_n \wedge x_2 \neq x_3 \wedge \cdots \wedge x_2 \neq x_n \wedge \cdots \wedge x_{n-1} \neq x_n .$$

$$\mathsf{exactly\_n}(U) \triangleq \exists x_1, \ldots \exists x_n . \mathsf{diff\_n}(x_1, \ldots, x_n) \wedge \forall y . (U(y) \Leftrightarrow \forall X . (X(x_1) \Rightarrow \ldots \Rightarrow X(x_n) \Rightarrow X(y)))$$

Closure properties of binary relations

$$\mathsf{Refl}(R) \triangleq \forall x. R(x, x)$$

$$\mathsf{Symm}(R) \triangleq \forall x. \forall y (R(x, y) \Rightarrow R(y, x)$$

$$\mathsf{Trans}(R) \triangleq \forall x. \forall y. \forall z. (R(x, y) \Rightarrow R(y, z) \Rightarrow R(x, z))$$

$$\mathsf{SubRel}(R, S) \triangleq \forall x. \forall y. R(x, y) \Rightarrow S(x, y)$$

$$\mathsf{Closure}(R, S, F) \triangleq F(S) \land \forall Z. (\mathsf{SubRel}(R, Z) \land F(Z) \Rightarrow \mathsf{SubRel}(S, Z))$$

$$\mathsf{TransClos}(R, S) \triangleq \mathsf{Closure}(R, S, \mathsf{Trans})$$

$$\mathsf{ReflClos}(R, S) \triangleq \mathsf{Closure}(R, S, \mathsf{Refl})$$

$$\mathsf{Refl}(R) \triangleq \forall x. R(x, x)$$

$$\mathsf{Symm}(R) \triangleq \forall x. \forall y (R(x, y) \Rightarrow R(y, x))$$

$$\mathsf{Trans}(R) \triangleq \forall x. \forall y. \forall z. (R(x, y) \Rightarrow R(y, z) \Rightarrow R(x, z))$$

$$\mathsf{SubRel}(R, S) \triangleq \forall x. \forall y. R(x, y) \Rightarrow S(x, y)$$

$$\mathsf{Closure}(R, S, F) \triangleq F(S) \wedge \forall Z. (\mathsf{SubRel}(R, Z) \wedge F(Z) \Rightarrow \mathsf{SubRel}(S, Z))$$

$$\mathsf{TransClos}(R, S) \triangleq \mathsf{Closure}(R, S, \mathsf{Trans})$$

$$\mathsf{ReflClos}(R, S) \triangleq \mathsf{Closure}(R, S, \mathsf{Refl})$$

$$\mathsf{StrongConf}(R) \triangleq \forall x, y, z \exists w. (R(x, y) \Rightarrow R(x, z) \Rightarrow (R(y, w) \wedge R(z, w)))$$

$$\mathsf{Conf}(R) \triangleq \forall X. \forall Y. (\mathsf{TransClos}(R, X) \Rightarrow \mathsf{ReflClos}(X, Y) \Rightarrow \mathsf{StrongConf}(Y))$$

Fixed points

Proposition 1.1

Let $F(X,x)$ be a second-order formula with a free unary relation variable X and a free first-order variable x.

Let Mon(F) *be the formula:* X,Y, unary F(X) => F(Y)

$$\mathsf{Mon}(F) \triangleq \forall X.\forall Y(\forall x.(X(x) \Rightarrow Y(x)) \Rightarrow \forall x.(F \Rightarrow F\{Y(y)/X(y)\})$$

Let us write $F \subseteq G$ for:

$$F \subseteq G \triangleq \forall x.F(x) \Rightarrow G(x)$$

$$\mathsf{Mon}(F) \vdash \exists Z.(Z \subseteq F\{Z/X\} \wedge F\{Z/X\} \subseteq Z)$$

X(t) replaced with Y(t)

Let Mon(F) *be the formula:*

$$\mathsf{Mon}(F) \triangleq \forall X.\forall Y(\forall x.(X(x) \Rightarrow Y(x)) \Rightarrow \forall x.(F$$

Let us write $F \subseteq G$ *for:*

$$F \subseteq G \triangleq \forall x.F(x) \to G(x)$$

$$\mathsf{Mon}(F) \vdash \exists Z.(Z \subseteq F\{Z/X\} \wedge F\{Z/X\} \subseteq Z)$$

Proof: The idea is that the witness we are looking for (the fixed-

that $F(S) \subseteq S$.

Let $\Phi(x) \triangleq \forall X.(\forall x.(F \Rightarrow X(x)) \Rightarrow X(x))$.

(pre-fixed point of F)

$$\dfrac{[\Phi(x)]^{\alpha}}{F(X) \subseteq X \Rightarrow X(x)}\ \forall^2_e(X) \qquad [F(X) \subseteq X]^{\beta}$$

$$\dfrac{F(X) \subseteq X \Rightarrow X(x) \qquad [F(X) \subseteq X]^{\beta}}{X(x)}\ \Rightarrow_e$$

$$\dfrac{\dfrac{X(x)}{\Phi(x) \Rightarrow X(x)}\ \Rightarrow^{\alpha}_i}{\Phi \subseteq X}\ \forall^1_i$$

$$\dfrac{\dfrac{\mathsf{Mon}(F)}{\forall Y(\forall x.(\Phi(x) \Rightarrow Y(x)) \Rightarrow \forall x.(F(\Phi) \Rightarrow F\{Y(y)/X(y)\})}\ \forall^2_e(\Phi)}{\Phi \subseteq X \Rightarrow F(\Phi) \subseteq F(X)}\ \forall^2_e(X)$$

$$\dfrac{\Phi \subseteq X \Rightarrow F(\Phi) \subseteq F(X) \qquad \Phi \subseteq X}{F(\Phi) \subseteq F(X)}\ \Rightarrow_e$$

$$\dfrac{F(\Phi) \subseteq F(X) \qquad [F(X) \subseteq X]^{\beta}}{F(\Phi) \subseteq X}\ \textit{trans}$$

$$\dfrac{\dfrac{F(\Phi) \subseteq X}{F(\Phi(x), x) \Rightarrow X(x)}\ \forall^1_e \qquad [F(\Phi(x))]^{\gamma}}{X(x)}\ \Rightarrow_e$$

$$\dfrac{\dfrac{\dfrac{\dfrac{X(x)}{(F(X) \subseteq X) \Rightarrow X(x)}\ \Rightarrow^{\beta}_i}{\Phi(x)}\ \forall^2_i}{F(\Phi(x)) \Rightarrow \Phi(x)}\ \Rightarrow^{\gamma}_i}{F(\Phi) \subseteq \Phi}\ \forall^1_i$$

$\forall x\ A \bowtie B$

Proposition 1.1

Let $F(X, x)$ be a second-order formula with a free unary relation variable X and a free first-order variable x.

Let Mon(F) *be the formula:*

$$\mathsf{Mon}(F) \triangleq \forall X.\forall Y(\forall x.(X(x) \Rightarrow Y(x)) \Rightarrow \forall x.(F \Rightarrow F\{Y(y)/X(y)\})$$

Let us write $F \subseteq G$ for:

$$F \subseteq G \triangleq \forall x.F(x) \Rightarrow G(x)$$

$$\mathsf{Mon}(F) \vdash \exists Z.(Z \subseteq F\{Z/X\} \wedge F\{Z/X\} \subseteq Z)$$

Proof: The idea is that the witness we are looking for (the fixed-point of F) is the intersection of all sets S such that $F(S) \subseteq S$.

Let $\Phi(x) \triangleq \forall X.(\forall x.(F \Rightarrow X(x)) \Rightarrow X(x))$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\Phi(x)]^\alpha}{F(F(\Phi)) \subseteq F(\Phi) \Rightarrow F(\Phi(x), x)}\ \forall_e^2(F(\Phi)) \qquad \dfrac{\mathsf{Mon}(F) \qquad d}{F(F(\Phi)) \subseteq F(\Phi)}}{F(\Phi(x), x)}\ \Rightarrow_e}{\Phi(x) \Rightarrow F(\Phi(x), x)}\ \Rightarrow_i^\alpha}{\Phi \subseteq F(\Phi)}\ \forall_i^1 \qquad d}{(\Phi \subseteq F(\Phi)) \wedge (F(\Phi) \subseteq \Phi)}\ \wedge_i}{\exists Z.(Z \subseteq F\{Z/X\} \wedge F\{Z/X\} \subseteq Z)}\ \exists_i^2$$

M= (D, f^M, R^M)

M, v |= F

M, v |= X(t1, ... tn)   v(X) C D^n

M, v |= forall X. F   X n-ary

iff for any E subset of D^n

if v' is defined as v'(X)=E and v'(Y) = v(Y),

M, v' |= F

M, v |= exists X. F   X n-ary

iff exists E subset of D^n

if v' is defined as v'(X)=E and v'(Y) = v(Y),

M, v' |= F

$$A_{WFI} \triangleq \forall X.(\forall x.(\forall y.(y < x \Rightarrow X(y)) \Rightarrow X(x)) \Rightarrow \forall x.X(x))$$

$$A_i \triangleq c_{i+1} < c_i \qquad i \geq 0$$

$$\Theta = \{A_{WFI}\} \cup \{A_i, i \geq 0\}$$

Every finite subset of Θ has a standard model but Θ has no standard model.

Consider the language of PA_2 extended with a constant c and

$$B_i \triangleq c \neq S^i(0) \qquad i \geq 0$$

$\{B_i, i \geq 0\}$ is finitely satisfiable but not satisfiable in standard models.

forall x. forall X. [(X(0) /\ forall y. X(y) => X(S(y)) => X(x)]