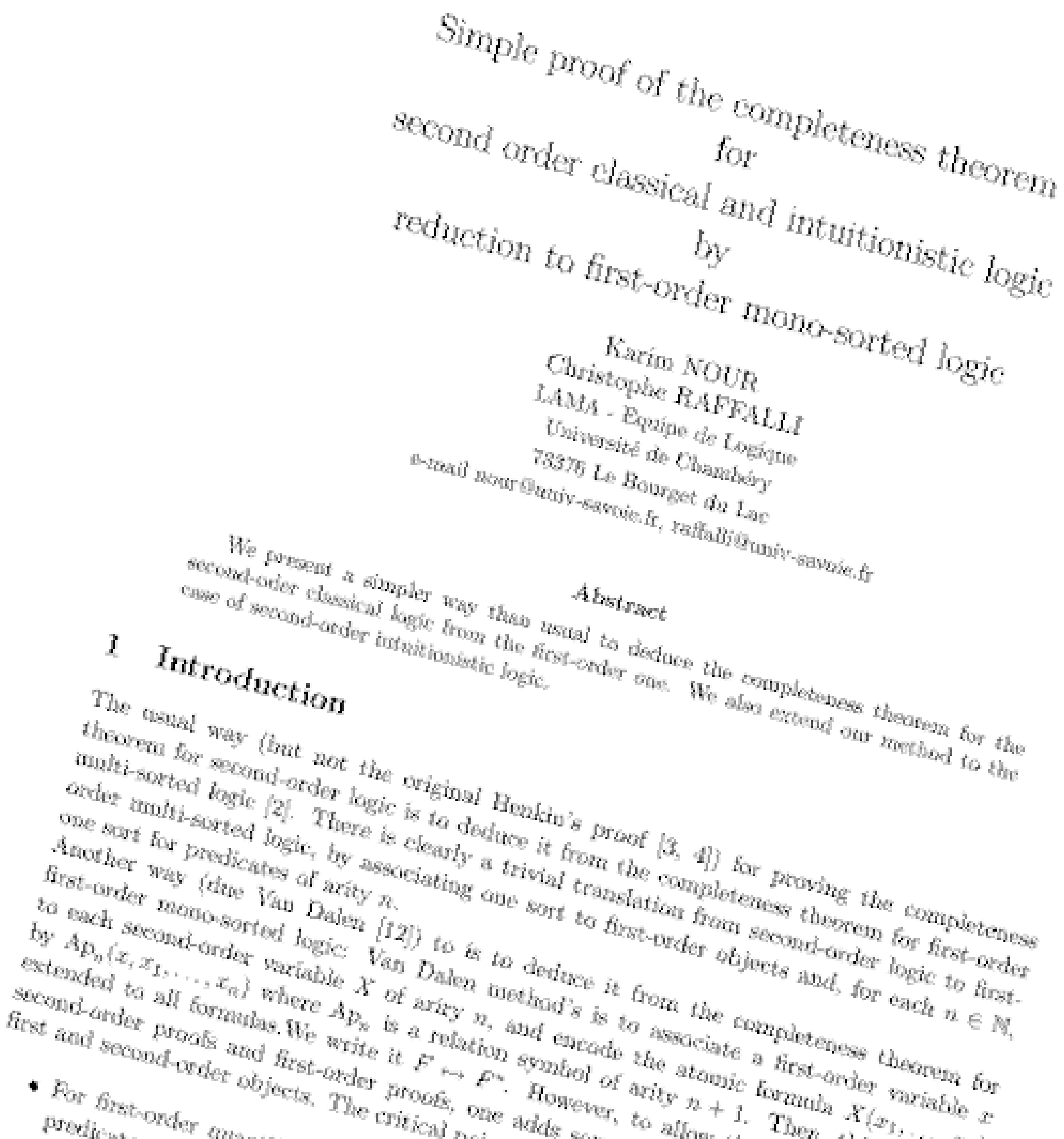
Completeness results for second-order logic



Simple proof of the completeness theorem
for
second order classical and intuitionistic logic
by
reduction to first-order mono-sorted logic

Karim NOUR
Christophe RAFFALLI
LAMA - Equipe de Logique
Université de Chambéry
73376 Le Bourget du Lac
e-mail nour@univ-savoie.fr, raffalli@univ-savoie.fr

Abstract

We present a simpler way than usual to deduce the completeness theorem for the second-order classical logic from the first-order one. We also extend our method to the case of second-order intuitionistic logic.

1 Introduction

The usual way (but not the original Henkin's proof [3, 4]) for proving the completeness theorem for second-order logic is to deduce it from the completeness theorem for first-order multi-sorted logic [2]. There is clearly a trivial translation from second-order logic to first-order multi-sorted logic, by associating one sort to first-order objects and, for each $n \in \mathbb{N}$, one sort for predicates of arity n.
Another way (due Van Dalen [12]) to is to deduce it from the completeness theorem for first-order mono-sorted logic: Van Dalen method's is to associate a first-order variable x to each second-order variable X of arity n, and encode the atomic formula $X(x_1, \ldots$ by $\mathrm{Ap}_n(x, x_1, \ldots, x_n)$ where Ap_n is a relation symbol of arity $n+1$. Then this extended to all formulas. We write it $F \mapsto F^*$. However, to allow second-order proofs and first-order proofs, one adds first and second-order objects. The critical

- For first-order

$$A_{WFI} \triangleq \forall X.(\forall x.(\forall y.(y < x \Rightarrow X(y)) \Rightarrow X(x)) \Rightarrow \forall x.X(x))$$

$$A_i \triangleq c_{i+1} < c_i \qquad i \geq 0$$

$$\Theta = \{A_{WFI}\} \cup \{A_i, i \geq 0\}$$

Every finite subset of Θ has a standard model but Θ has no standard model.

Θ has no standard model: Indeed, Let $\mathcal{M} = (\mathcal{D}, c_i^{\mathcal{M}}, <^{\mathcal{M}})$ be a model satisfying $\{A_i, i \geq 0\}$. We prove that $\mathcal{M} \not\models \mathcal{A}_{\mathcal{WFI}}$.

Let $C = \{c_i^{\mathcal{M}}, i \in \mathbb{N}\}$ and $B = \mathcal{D} \setminus C$.

Consider $A(X) = \forall y.(\forall z.(z < y \Rightarrow X(z)) \Rightarrow X(y)) \Rightarrow \forall y.X(y)$

We prove that $\mathcal{M}, [X := B] \not\models A(X)$. Clearly, $\mathcal{M}, [X := B] \not\models \forall y.X(y)$ as $c_0^{\mathcal{M}} \notin B$: it is therefore sufficient to prove that $\mathcal{M}, [X := B] \models \forall y.(\forall z.(z < y \Rightarrow X(z)) \Rightarrow X(y))$.

Let $d \in B$, then $\mathcal{M}, [X := B, y := d] \models X(y)$ so that $\mathcal{M}, [X := B, y := d] \models (\forall z.(z < y \Rightarrow X(z)) \Rightarrow X(y))$.

Otherwise, $d \in C$: let n be such that $d = c_n^{\mathcal{M}}$ In particular, we have $c_{n+1}^{\mathcal{M}} <^{\mathcal{M}} d$ and $c_{n+1}^{\mathcal{M}} \notin B$ so that $\mathcal{M}, [X := B, y := d] \not\models \forall z.(z < y \Rightarrow X(z))$. We conclude that $\mathcal{M}, [X := B, y := d] \models \forall z.(z < y \Rightarrow X(z)) \Rightarrow X(y)$.

From the above, one deduce that $\mathcal{M}, [X := B] \models \forall y.(\forall z.(z < y \Rightarrow X(z)) \Rightarrow X(y))$ and finally that $\mathcal{M} \not\models A_{WFI}$.

Θ is finitely satisfiable: We show that any $\Theta_k = \{A_{WFI}\} \cup \{A_i, 0 \leq i \leq k\}$ is satisfiable.

Indeed, consider $\mathcal{M} = (\mathbb{N}, c_i^{\mathcal{M}}, <^{\mathcal{M}})$ with $<^{\mathcal{M}}$ the strict ordering over $\mathbb{N}$ and $c_i^{\mathcal{M}} = max(k + 1 - i, 0)$.

A_{WFI} holds in $\mathbb{N}$ and each $A_i, i \leq k$ holds as well.

Consider the language of PA_2 extended with a constant c and

$$B_i \triangleq c \neq S^i(0) \qquad i \geq 0$$

$\{B_i, i \geq 0\}$ is finitely satisfiable but not satisfiable in standard models.

$$A_{WFI} \triangleq \forall X.(\forall x.(\forall y.(y < x \Rightarrow X(y)) \Rightarrow X(x)) \Rightarrow \forall x.X(x))$$

$$A_i \triangleq c_{i+1} < c_i \qquad i \geq 0$$

$$\Theta = \{A_{WFI}\} \cup \{A_i, i \geq 0\}$$

Every finite subset of Θ has a standard model but Θ has no standard model.

A strong form of incompleteness...

Consider PA_2: (only with the axioms for $+, \times$).

- It has a finite axiomatisation.
- The set of first-order closed formulas that can be derived in PA_2 is recursively enumerable since we have a notion of deduction (which is recursive).
- it contains first-order arirthmetic

Gödel's first incompleteness theorem entails that there is a Π^0_1 formula F which is true in $\mathbb{N}$ but cannot be derived in PA_2. In particular, $\mathsf{PA}_2 \cup \{\neg F\}$ does not derive a contradiction but has no standard model.

In particular, this shows that the incompleteness phenomenon is strong: there cannot be a second-order proof system with a completeness properties for standard models.

Definition 2.1 (second-order language) *Let $\mathcal{L}_2$, the language of second-order logic, be the following:*

- *The logical symbols* $\bot$, $\to$, $\wedge$, $\vee$, $\forall$ *and* $\exists$.
- *A countable set* $\mathcal{V}$ *of first-order variables :* $x_0, x_1, x_2, \ldots$
- *A countable set* Σ *of constants and functions symbols (of various arity) :* $a, b, f, g, h, \ldots$.
- *Using* $\mathcal{V}$ *and* Σ *we construct the set of first-order terms* $\mathcal{T}$ *:* $t_1, t_2, \ldots$
- *For each* $n \in \mathbb{N}$, *a countable set* $\mathcal{V}_n$ *of second-order variables of arity* n *:* $X_0^n, X_1^n, X_2^n, \ldots$.

To simplify, we omit second-order constants (they can be replaced by free variables).

Definition 2.8 (comprehension schemas) *The second-order comprehension schema* SC_2 *is the set of all closed formulas* $SC_2(G; x_1, \ldots, x_n; \chi_1, \ldots, \chi_m)$ *where* $\{x_1, \ldots, x_n\} \subset \mathcal{V}$ *and* $\mathcal{F}_v(G) \subseteq \{x_1, \ldots, x_n, \chi_1, \ldots, \chi_m\}$ *and*

$$SC_2(G; x_1, \ldots, x_n; \chi_1, \ldots, \chi_m) = \forall \chi_1 \ldots \forall \chi_m \exists X^n \forall x_1 \ldots \forall x_n \, (G \leftrightarrow X^n(x_1, \ldots, x_n)) \in SC_2$$

where $X^n \notin F_v(G)$.

Comprehension schema is provable in second-order logic

Definition 2.8 (comprehension schemas) *The second-order comprehension schema* SC_2 *is the set of all closed formulas* $SC_2(G; x_1, \ldots, x_n; \chi_1, \ldots, \chi_m)$ *where* $\{x_1, \ldots, x_n\} \subset \mathcal{V}$ *and* $\mathcal{F}_v(G) \subseteq \{x_1, \ldots, x_n, \chi_1, \ldots, \chi_m\}$ *and*

$$SC_2(G; x_1, \ldots, x_n; \chi_1, \ldots, \chi_m) = \forall \chi_1 \ldots \forall \chi_m \exists X^n \forall x_1 \ldots \forall x_n \, (G \leftrightarrow X^n(x_1, \ldots, x_n)) \in SC_2$$

where $X^n \notin F_v(G)$.

$$\dfrac{\forall X^n . F}{F[G/X^n(y_1 \ldots y_n)]}\ \forall^2_e(G)$$

weak universal elimination

$$\dfrac{\forall X^n . F}{F[Y^n/X^n]}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[G]}{G \to G}\to_i \quad \dfrac{[G]}{G \to G}\to_i}{G \leftrightarrow G}\wedge_i}{\forall x_1 \ldots \forall x_n \, G \leftrightarrow G}\forall^1_i *}{\exists X^n \forall x_1 \ldots \forall x_n (G \leftrightarrow X^n(x_1 \ldots x_n))}\exists^2_i(G)}{SC_2(G; x_1 \ldots x_n; \chi_1 \ldots \chi_m)}\forall^2_i$$

$$\dfrac{F[G/X^n(y_1 \ldots y_n)]}{\exists X^n . F}\ \exists^2_i(G)$$

$$G \leftrightarrow X^n(x_1 \ldots x_n).$$

Definition 3.1 (second-order classical model) *A second-order model for $\mathcal{L}_2$ is given by a tuple $\mathcal{M}_2 = (\mathcal{D}, \overline{\Sigma}, \{\mathcal{P}_n\}_{n\in\mathbb{N}})$ where*

$$\mathcal{M}_2^F = (\mathcal{D}, \overline{\Sigma}, \{\mathcal{P}(\mathcal{D}^n)\}_{n\in\mathbb{N}}).$$

- *$\mathcal{D}$ is a non empty set.*
- *$\overline{\Sigma}$ contains a function $\overline{f}$ from $\mathcal{D}^n$ to $\mathcal{D}$ for each function f of arity n in Σ.*
- *$\mathcal{P}_n \subseteq \mathcal{P}(\mathcal{D}^n)$ for each $n \in \mathbb{N}$. The set $\mathcal{P}_n$ of subsets of $\mathcal{D}^n$ will be used as the range for the second-order quantification of arity n. For $n = 0$, we assume that $\mathcal{P}_0 = \mathcal{P}(\mathcal{D}^0) = \{0, 1\}$ because $\mathcal{P}(\mathcal{D}^0) = \mathcal{P}(\emptyset) = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$.*

An $\mathcal{M}_2$-interpretation σ is a function on $\mathcal{V} \cup \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ such that $\sigma(x) \in \mathcal{D}$ for $x \in \mathcal{V}$ and $\sigma(X^n) \in \mathcal{P}_n$ for $X^n \in \mathcal{V}_n$.
If σ is a $\mathcal{M}_2$-interpretation, we define $\sigma(t)$ the interpretation of a first-order term by induction with $\sigma(f(t_1, \ldots, t_n)) = \overline{f}(\sigma(t_1), \ldots, \sigma(t_n))$.
Then if σ is a $\mathcal{M}_2$-interpretation we define $\mathcal{M}_2, \sigma \models A$ for a formula A by induction as follows:

- $\mathcal{M}_2, \sigma \models X^n(t_1, \ldots, t_n)$ *iff* $(\sigma(t_1), \ldots, \sigma(t_n)) \in \sigma(X^n)$
- $\mathcal{M}_2, \sigma \models A \to B$ *iff* $\mathcal{M}_2, \sigma \models A$ *implies* $\mathcal{M}_2, \sigma \models B$
- $\mathcal{M}_2, \sigma \models A \wedge B$ *iff* $\mathcal{M}_2, \sigma \models A$ *and* $\mathcal{M}_2, \sigma \models B$
- $\mathcal{M}_2, \sigma \models A \vee B$ *iff* $\mathcal{M}_2, \sigma \models A$ *or* $\mathcal{M}_2, \sigma \models B$
- $\mathcal{M}_2, \sigma \models \forall x\, A$ *iff for all* $v \in \mathcal{D}$ *we have* $\mathcal{M}_2, \sigma[x := v] \models A$
- $\mathcal{M}_2, \sigma \models \exists x\, A$ *iff there exists* $v \in \mathcal{D}$ *such that* $\mathcal{M}_2, \sigma[x := v] \models A$
- $\mathcal{M}_2, \sigma \models \forall X^n\, A$ *iff for all* $\pi \in \mathcal{P}_n$ *we have* $\mathcal{M}_2, \sigma[X^n := \pi] \models A$
- $\mathcal{M}_2, \sigma \models \exists X^n\, A$ *iff there exists* $\pi \in \mathcal{P}_n$ *such that* $\mathcal{M}_2, \sigma[X^n := \pi] \models A$

$\vdash^i_k \quad i \in \{1,2\} \quad k \in \{i, c\}$

Theorem 3.6 (Completeness of second order classical semantic) *Let A be a closed second-order formula. $\vdash^2_c A$ iff for any second-order model $\mathcal{M}_2$ such that $\mathcal{M}_2 \models SC_2$ we have $\mathcal{M}_2 \models A$.*

(1) Full second-order models

(2) Henkin models (range of relation var becomes a parameter of the model).

(3) First-order (many sorted) models of SOL. $X^n(t_1 \dots t_n)$.

$(t_1, \dots t_n) \in X^n$

$\mathcal{M} = (\mathcal{D}^0, \Sigma, (\mathcal{D}^i)_{i \geq 1})$

$\mathcal{D} : \mathcal{D} \times \dots \times \mathcal{D} \times \mathcal{D}^n$

$\in_n (t_1, \dots t_n, X^n)$.

$$\mathcal{L}_2 \xrightarrow{\ *\ } \mathcal{L}_1 . \qquad \Sigma$$

$$\in_m$$

1. If $\Gamma \vdash^2_R A$ then $\Gamma^*, SC_2^* \vdash^1_R A^*$

2. A converse.

3. A semantical translation.

4. A Bonus: a notion of SO intuitionistic models for which NJ_2 is complete

Definition 2.2 (first-order language) *Let $\mathcal{L}_1$, a particular language of first-order logic, be the following:*

- *A countable set $\mathcal{V}$ of first-order variables : $x_0, x_1, x_2, \ldots$ (it is simpler to use the same set of first-order variables in $\mathcal{L}_1$ and $\mathcal{L}_2$).*
- *A countable set Σ of constants and functions symbols (of various arity) : $a, b, f, g, h, \ldots$. Here again we use the same set as for $\mathcal{L}_2$.*
- *For each $n \in \mathbb{N}$, a relation symbol Ap_n of arity $n+1$.*

Given $\mathcal{L}_2$ a SO language.

$(X(t_1, \ldots t_n))^* \to Ap_n(\varphi_n(X), t_1, \ldots t_n)$.

Definition 2.4 (coding) *We choose for each $n \in \mathbb{N}$ a bijection ϕ_n from $\mathcal{V}_n$ to $\mathcal{V}$. The fact that it is a bijection for each n is the main point in our method.*
Let F be a second-order formula, we define a first-order formula F^* by induction as follows:

- $\perp^* = \perp$
- $(X^n(t_1, \ldots, t_n))^* = Ap_n(\phi_n(X^n), t_1, \ldots, t_n)$
- $(A \Diamond B)^* = A^* \Diamond B^*$ *where* $\Diamond \in \{\to, \wedge, \vee\}$
- $(Qx\, A)^* = Qy(A[x := y])^*$ *where* $y \notin \mathcal{F}_v(A^*)$ *and* $Q \in \{\forall, \exists\}$
- $(QX^n\, A)^* = Qy(A[X^n := Y^n])^*$ *where* $\Phi_n(Y^n) = y$, $y \notin \mathcal{F}_v(A^*)$ *and* $Q \in \{\forall, \exists\}$

$(Qx.A)^*$

$\boxed{Qy.(A[y/x])^*}$

$\forall y. X(y).$

$\forall y. Ap_1(\varphi_1(X), y)$

Example 2.6 $(\forall X(X(x) \to X(y)))^* = \forall z(Ap_1(z, x) \to Ap_1(z, y))$. *This example illustrates why we need renaming. For instance, if $\Phi_1(X)$ were equal to x or y in $(X(x) \to X(y))^*$.*

$\forall x Ap_1(x, y)$

$\forall X. X(y).$

Remark 2.7 *The mapping $F \mapsto F^*$ is not surjective, for instance there is no antecedent for $\forall x\, Ap_1(x, x)$ or $Ap_1(f(a), a)$.*

$F^* = \forall x\, Ap_1(x, x)$

Definition *The first-order comprehension schema* SC_1 *is defined simply as* $SC_2^* = \{F^*, F \in SC_2\}$

Remark 2.9 *Let* $F = X(x)$ *where* $\Phi_1(X) = x$. *We have:*

- $SC_2(F;x;X) = \forall X \exists Y \forall x (F \leftrightarrow Y(x)) \in SC_2$.
- $SC_2(F;x;X)^* = (\forall X \exists Y \forall x (F \leftrightarrow Y(x)))^* = \forall z \exists y \forall x (Ap_1(z,x) \leftrightarrow Ap_1(y,x)) \in SC_1$.

It is easy to see that $(\forall X \exists Y \forall x (F \leftrightarrow Y(x)))^* = \forall z \exists y \forall x (F[X := Z]^* \leftrightarrow Ap_1(y,x))$ *where* $\phi_1(Z) = z \neq x$.
In general we have the following result : for each second-order formula G *there is a variable substitution* σ *such that*

$$\begin{aligned} SC_2(G; x_1, \ldots, x_n; \chi_1, \ldots, \chi_m)^* &= (\forall \chi_1 \ldots \forall \chi_m \exists X^n \forall x_1 \ldots \forall x_n (G \leftrightarrow X^n(x_1, \ldots, x_n)))^* \\ &= \underline{\forall y_1 \ldots \forall y_m \exists x} \forall x_1 \ldots \forall x_n (G[\sigma]^* \leftrightarrow Ap_n(x, x_1, \ldots, x_n)). \end{aligned}$$

Theorem 2.10 *Let* Γ *be a second-order context and* A *a second-order formula. If* $\Gamma \vdash_k^2 A$ *then* $\Gamma^*, SC_1 \vdash_k^1 A^*$ $(k \in \{i, c\})$.

$\Downarrow$

$\Gamma, SC_2 \vdash^2_{kw} A$

proof: By induction on the derivation of $\Gamma \vdash_k^2 A$

Definition 2.11 (reverse coding) *Let F be a first-order formula, we define a second-order formula $F^\diamond$ by induction as follows:*

$A \longrightarrow A^* \longrightarrow A^{*\diamond}$

$A^\diamond$

$\mathcal{J}_v(A^\diamond) \cap \bigcup_{n \in \mathbb{N}} \{\phi_n^{-1}(x)\} = \{X^{i_1} \dots X^{i_p}\}$

- $\perp^\diamond = \perp$
- $Ap_n(x, t_1, \dots, t_n)^\diamond = X^n(t_1, \dots, t_n)$ *where* $X^n = \phi_n^{-1}(x)$
- $Ap_n(t, t_1, \dots, t_n)^\diamond = \perp$ *if t is not a variable.*
- $(A \Diamond B)^\diamond = A^\diamond \Diamond B^\diamond$ *where* $\Diamond \in \{\rightarrow, \wedge, \vee\}$
- $(Qx\, A)^\diamond = QxQX^{i_1} \dots QX^{i_p} A^\diamond$ *where* $Q \in \{\forall, \exists\}$, $X^n = \phi_n^{-1}(x)$ *for all* $n \in \mathbb{N}$, $i_1 < i_2 < \dots < i_p$

$\forall x\, Ap_1(x, x)^\diamond =$

$\forall x\, \forall \phi_1^{-1}(x).\ \underbrace{\phi_1^{-1}(x)}(x)$

Lemma 2.17 *Let Γ be a first-order context and A a first-order formula. If $\Gamma \vdash^1_k A$ then $\Gamma^\diamond \vdash^2_k A^\diamond$ $(k \in \{i, c\})$.*

2.17

$SC_1, \Gamma^* \vdash A^* \Longrightarrow$

$SC_{(2)}, \Gamma \vdash A$

$\Gamma \vdash A$

proof: By induction on the derivation of $\Gamma \vdash^1_k A$. The only difficult cases are the case of the elimination of $\forall$ and the introduction of $\exists$ which are treated in the same way as the examples

Let Γ be a first-order context, $F = Ap_1(x, y) \to Ap_2(x, y, y) \vee Ap_1(y, x)$ and t a term.
We have :

- $(\forall x\, F)^\diamond = \forall x \forall X^1 \forall X^2 (X^1(y) \to X^2(y, y) \vee Y^1(x))$ *and* $(\exists x\, F)^\diamond = \exists x \exists X^1 \exists X^2 (X^1(y) \to X^2(y, y) \vee Y^1(x))$ *(where $\phi_1(Y^1) = y$).*
- *If $t = z$, then $(F[x := t])^\diamond = Z^1(y) \to Z^2(y, y) \vee Y^1(z)$ (where $\phi_1(Z^1) = \phi_2(Z^2) = z$) and if t is not a variable, then $(F[x := t])^\diamond = \perp \to \perp \vee Y^1(t)$*

We remark that :

- $(F[x := z])^\diamond = Z^1(y) \to Z^2(y, y) \vee Y^1(z) = F^\diamond[X^1 := Z^1][x := z]$ *if z is a variable such that $\phi_1(Z^1) = \phi_2(Z^2) = z$.*
- $(F[x := t])^\diamond = \perp \to \perp \vee Y^1(t) = F^\diamond[X^1 := \lambda x_1 \perp][x := t]$ *if t is not a variable.*

and then :

- *If $\Gamma^\diamond \vdash^2_k (\forall x\, F)^\diamond$, then (by using some $\forall$-elimination rules) $\Gamma^\diamond \vdash^2_k (F[x := t])^\diamond$.*
- *If $\Gamma^\diamond \vdash^2_k (F[x := t])^\diamond$, then (by using some $\exists$-introduction rules) $\Gamma^\diamond \vdash^2_k (\exists x\, F)^\diamond$.*

Theorem 2.18 *Let Γ be a second-order context and A a second-order formula. If $\Gamma^*, SC_1 \vdash^1_k A^*$ then $\Gamma \vdash^2_k A$ $(k \in \{i, c\})$.*

Lemma 2.13 *If A is a second order formula then $\vdash^2_i A^{*\diamond} \leftrightarrow A$.*

Corollary 2.15 $\vdash^2_i (SC_1)^\diamond \leftrightarrow SC_2$

proof:

Lemma 2.17 *Let Γ be a first-order context and A a first-order formula. If $\Gamma \vdash^1_k A$ then $\Gamma^\diamond \vdash^2_k A^\diamond$ $(k \in \{i, c\})$.*

Now, we will use the translation between L2 and L1 to obtain completeness:

1- we know that a first-order provability of F* under assumptions Gamma*, SC1* entails the second-order probability of F;
2- if one can relate the fact that F is a second-order semantical consequence of Gamma to a similar semantical relation between first order statements Gamma* and F*, we can rely on Gödel completeness theorem for predicate calculus.
3- that is our next, and final task: relate semantical consequences by turning a second-order model of L2 into a first-order model of L1. We shall also ensure that the translated model satisfies SC1.

Definition 3.3 (semantical translation) *Let $\mathcal{M}_1 = (\mathcal{D}, \overline{\Sigma}, \{\alpha_n\}_{n\in\mathbb{N}})$ be a first-order model. We define a second-order model ${\mathcal{M}_1}^\diamond = (\mathcal{D}, \overline{\Sigma}, \{\mathcal{P}_n\}_{n\in\mathbb{N}})$ where $\mathcal{P}_0 = \{0, 1\}$ and for $n > 0$, $\mathcal{P}_n = \{|a|_n; a \in \mathcal{D}\}$ where $|a|_n = \{\underline{(a_1, \ldots, a_n)} \in \mathcal{D}^n; (a\ \boxed{a_1, \ldots, a_n}) \in \alpha_n\}$.*
Let σ be an $\mathcal{M}_1$-interpretation, we define $\sigma^\diamond$ an ${\mathcal{M}_1}^\diamond -$ interpretation by $\sigma^\diamond(x) = \sigma(x)$ if $x \in \mathcal{V}$ and $\sigma^\diamond(X^n) = |\sigma(\phi(X^n))|_n$.

$\mathcal{M}_1$

$\alpha_n = A_{P_n}$, $X^n(t_1 - t_n)$,

$A_{P_n}(\varphi_n(X^n)) \boxed{t_1 - t_n}$

Lemma 3.4 *For any first-order model $\mathcal{M}_1$, any $\mathcal{M}_1$-interpretation σ and any second order formula A, $\mathcal{M}_1, \sigma \models A^*$ if and only if ${\mathcal{M}_1}^\diamond, \sigma^\diamond \models A$.*

proof: By induction on the formula A, this is an immediate consequence of the definition of semantical translation. □

Corollary 3.5 *For any first-order model $\mathcal{M}_1$, $\mathcal{M}_1 \models SC_1$ if and only if ${\mathcal{M}_1}^\diamond \models SC_2$.*

proof: Immediate consequence of lemma 3.4 using the fact that formulas in SC_1 and SC_2 are closed. □

Theorem 3.6 (Completeness of second order classical semantic) *Let A be a closed second-order formula. $\vdash_c^2 A$ iff for any second-order model $\mathcal{M}_2$ such that $\mathcal{M}_2 \models SC_2$ we have $\mathcal{M}_2 \models A$.*

proof: $\Longrightarrow$ Usual direct proof by induction on the proof of $\vdash_c^2 A$.
$\Longleftarrow$ Let $\mathcal{M}_1$ be a first-order model such that $\mathcal{M}_1 \models SC_1$. Using corollary 3.5 we have ${\mathcal{M}_1}^\diamond \models SC_2$ and by hypothesis, we get ${\mathcal{M}_1}^\diamond \models A$. Then using lemma 3.4 we have $\mathcal{M}_1 \models A^*$. As this is true for any first-order model satisfying SC_1, the first-order completeness theorem gives $SC_1 \vdash_c^1 A^*$ and this leads to the wanted result $\vdash_c^2 A$ using theorem 2.18. $\square$

$M_1^\diamond \models A$ then $M_1 \models A^*$, $\underbrace{(M_1 \models SC_1 \Rightarrow M_1 \models A^*.}_{SC_1 \vdash A^*.}$

$\Longrightarrow \vdash^2 A$

Intuitionistic completeness

Definition 4.5 (first-order intuitionistic model) *A first-order Kripke model is given by a tuple* $\mathcal{K}_1 = (\mathcal{K}, 0, \leq, \{\mathcal{D}_p\}_{p\in\mathcal{K}}, \{\overline{\Sigma}_p\}_{p\in\mathcal{K}}, \{\alpha_{n,p}\}_{n\in\mathbb{N},p\in\mathcal{K}}, \Vdash)$ *where*

- $(\mathcal{K}, \leq, 0)$ *is a partially ordered set with* 0 *as bottom element.*
- $\mathcal{D}_p$ *are non empty sets such that for all* $p, q \in \mathcal{K}$, $p \leq q$ *implies* $\mathcal{D}_p \subseteq \mathcal{D}_q$.
- $\overline{\Sigma}_p$ *contains a function* $\overline{f}_p$ *from* $\mathcal{D}_p^n$ *to* $\mathcal{D}_p$ *for each function* f *of arity* n *in* Σ. *Moreover, for all* $p, q \in \mathcal{K}$, $p \leq q$ *implies that for all* $(a_1, \ldots, a_n) \in \mathcal{D}_p^n \subseteq \mathcal{D}_q^n$ *we have* $\overline{f}_p(a_1, \ldots, a_n) = \overline{f}_q(a_1, \ldots, a_n)$.
- $\alpha_{n,p}$ *are subsets of* $\mathcal{D}_p^{n+1}$ *such that for all* $p, q \in \mathcal{K}$, *for all* $n \in \mathbb{N}$, $p \leq q$ *implies* $\alpha_{n,p} \subseteq \alpha_{n,q}$.
- $\Vdash$ *is the relation defined by* $p \Vdash Ap_n(a, a_1, \ldots, a_n)$ *if and only if* $p \in \mathcal{K}$ *and* $(a, a_1, \ldots, a_n) \in \alpha_{n,p}$.

A $\mathcal{K}_1$*-interpretation* σ *at level* p *is a function from* $\mathcal{V}$ *to* $\mathcal{D}_p$.

Definition 4.1 (second-order intuitionistic model) *A second-order Kripke model for* $\mathcal{L}_2$ *is given by a tuple* $\mathcal{K}_2 = (\mathcal{K}, 0, \leq, \{\mathcal{D}_p\}_{p\in\mathcal{K}}, \{\Sigma_p\}_{p\in\mathcal{K}}, \{\Pi_{n,p}\}_{n\in\mathbb{N},p\in\mathcal{K}})$ *where*

- $(\mathcal{K}, \leq, 0)$ *is a partially ordered set with* 0 *as bottom element.*

- $\mathcal{D}_p$ *are non empty sets such that for all* $p, q \in \mathcal{K}$, $p \leq q$ *implies* $\mathcal{D}_p \subseteq \mathcal{D}_q$.

- $\overline{\Sigma}_p$ *contains a function* $\overline{f}_p$ *from* $\mathcal{D}_p^n$ *to* $\mathcal{D}_p$ *for each function* f *of arity* n *in* Σ. *Moreover, for all* $p, q \in \mathcal{K}$, $p \leq q$ *implies that for all* $(a_1, \ldots, a_n) \in \mathcal{D}_p^n \subseteq \mathcal{D}_q^n$ *we have* $\overline{f}_p(a_1, \ldots, a_n) = \overline{f}_q(a_1, \ldots, a_n)$.

- $\Pi_{n,p}$ *are non empty sets of increasing functions* $(P_q)_{q\geq p}$ *such that for all* $q \geq p, P_q \in \mathcal{P}(\mathcal{D}_q^n)$ *(increasing means for all* $q, q' \geq p$, $q \leq q'$ *implies* $P_q \subseteq P_{q'}$*). Moreover, if* $q \geq p$ *and* $\pi \in \Pi_{n,p}$ *then* π *restricted to all* $q' \geq q$ *belongs to* $\Pi_{n,q}$.

  In particular, an element of $\Pi_{0,p}$ *is a particular increasing function in* {*0*,*1*} *with* $0 = \emptyset$ *and* $1 = \{\emptyset\}$.

A $\mathcal{K}_2$*-interpretation* σ *at level* p *is a function* σ *such that* $\sigma(x) \in \mathcal{D}_p$ *for* $x \in \mathcal{V}$ *and* $\sigma(X^n) \in \Pi_{n,p}$ *for* $X^n \in \mathcal{V}_n$.

Definition 4.6 (semantical translation) *Let*

$$\mathcal{K}_1 = (\mathcal{K}, 0, \leq, \{\mathcal{D}_p\}_{p\in\mathcal{K}}, \{\overline{\Sigma}_p\}_{p\in\mathcal{K}}, \{\alpha_{n,p}\}_{n\in\mathbb{N},p\in\mathcal{K}}, \Vdash)$$

be a first-order Kripke model. We define a second-order Kripke model

$$\mathcal{K}_1{}^\diamond = (\mathcal{K}, 0, \leq, \{\mathcal{D}_p\}_{p\in\mathcal{K}}, \{\overline{\Sigma}_p\}_{p\in\mathcal{K}}, \{\Pi_{n,p}\}_{n\in\mathbb{N},p\in\mathcal{K}})$$

where $\Pi_{n,p} = \{|a|_n; a \in \mathcal{D}_p\}$ *with for all* $q \geq p$, $|a|_n(q) = \{(a_1, \ldots, a_n) \in \mathcal{D}_q^n; (a, a_1, \ldots, a_n) \in \alpha_{n,q}\}$.
Let σ *be a* $\mathcal{K}_1$*-interpretation at level* p*, we define* $\sigma^\diamond$ *a* $\mathcal{K}_1{}^\diamond$*-interpretation at level* p *by* $\sigma^\diamond(x) = \sigma(x)$ *and* $\sigma^\diamond(X^n) = |\sigma(\phi(X^n))|_n$.

Lemma 4.7 *For any first-order Kripke model* $\mathcal{K}_1$*, any* $\mathcal{K}_1$*-interpretation* σ *at level* p *and any second order formula* A*,* $\mathcal{K}_1, \sigma, p \Vdash A^*$ *if and only if* $\mathcal{K}_1{}^\diamond, \sigma^\diamond, p \Vdash A$.

Corollary 4.8 *For any first-order Kripke model* $\mathcal{K}_1$, $\mathcal{K}_1 \Vdash SC_1$ *if and only if* $\mathcal{K}_1{}^\diamond \Vdash SC_2$.

Theorem 4.9 (Completeness of second order intuitionistic semantic) *Let* A *be a closed second-order.* $\vdash_i^2 A$ *iff for all second-order Kripke model* $\mathcal{K}_2$ *such that* $\mathcal{K}_2 \Vdash SC_2$ *we have* $\mathcal{K}_2 \Vdash A$.