Maximum Weight $b$-Matchings in Random-Order Streams

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Definition of a $b$-Matching

$\leq 3$

$\leq 1$

$v_2$

$v_4$

$v_3$

$v_1$

$\leq 2$

$\leq 1$

$\leq 1$

Figure: An input graph with capacities $b_{v_1} = 2$, $b_{v_2} = 3$, and $b_{v_3} = b_{v_4} = 1$
Definition of a $b$-Matching

Figure: An input graph with capacities $b_{v_1} = 2$, $b_{v_2} = 3$, and $b_{v_3} = b_{v_4} = 1$
An output $b$-matching $M$, in red
The Maximum Weight $b$-Matching Problem

Figure: Finding a maximum weight $b$-matching
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The weight of the highlighted matching $M$ is $2 + 3 + 4 = 9$. 
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In this work, we assume the weights are small integers in a range $\{1, \ldots, W\}$.
Degree and Weighted Degree

Figure: Defining the degree and weighted degree of vertex $v_2$ in some subgraph $H$
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- $\deg_H(v_2) = 2$ (and $\deg_G(v_2) = 3$)
Degree and Weighted Degree

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- $\deg_H(v_2) = 2$ (and $\deg_G(v_2) = 3$)
- $\text{wdeg}_H(v_2) = 3 + 4 = 7$ (and $\text{wdeg}_G(v_2) = 3 + 4 + 2 = 9$)
Some Definitions

In the *semi-streaming* model:

- edges of $E$ arrive over time in a stream $S = \langle e_1, \ldots, e_m \rangle$
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- we have access only to a *limited memory*
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- edges of $E$ arrive over time in a *stream* $S = \langle e_1, \ldots, e_m \rangle$
- we have access only to a **limited memory**

In the *random-order semi-streaming* model:
- the permutation of the edges given in the stream is assumed to be chosen **uniformly at random** among all the possible permutations
In the adversarial (worst-case order) semi-streaming model:

- For unweighted maximum $(b)$-matching approximation:
  - 2 approximation using the trivial greedy algorithm
  - $1 + \ln 2 \approx 1.69$ inapproximability [Kapralov, 2021]

- For maximum weight $(b)$-matching approximation:
  - $5$ [Feigenbaum, Kannan, McGregor, Suri, and Zhang, 2005]
  - $5.83$ [McGregor, 2005]
  - $5.58$ [Zelke, 2010]
  - $4.91 + \varepsilon$ [Epstein, Levin, Mestre, and Segev, 2010]
  - $2 + \varepsilon$ [Crouch and Stubbs, 2014]

- For maximum weight $b$-matching approximation:
  - $3 + \varepsilon$ [Levin and Wajc, 2021]
  - $2 + \varepsilon$ [Huang and Sellier, 2021]
Previous Results

In the *adversarial* (worst-case order) semi-streaming model:

- For unweighted maximum \((b-)\)matching approximation:
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- For maximum weight matching approximation:
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In the *random-order* semi-steaming model:

- For maximum unweighted matching approximation:
  - 1.98 (bipartite), 1.99 (general) [Konrad, Magniez, and Mathieu, 2012]
  - 1.86 (bipartite) [Konrad, 2018]
  - $5/3 \approx 1.67$ (bipartite), $11/6 \approx 1.84$ (general) [Farhadi, Hajiaghayi, Mai, Rao, and Rossi, 2020]
  - $3/2 + \varepsilon$ (general) [Bernstein, 2020]
  - $3/2 - \delta$ (general, $\delta \sim 10^{-14}$) [Assadi and Behnezhad, 2021]
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- For maximum (weight) $b$-matching approximation:
  - no previous result in the random-order setting
### Previous Results and Our Contribution

<table>
<thead>
<tr>
<th>Maximization problem</th>
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**Table:** Approximation ratios for adversarial and random-order streams (we assume weights are small integers in \{1, \ldots, W\}).
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Table: Approximation ratios for adversarial and random-order streams (we assume weights are small integers in \(\{1, \ldots, W\}\)). Comparison with our results.
Our Contribution

We generalize the result of [Bernstein, 2020] (\(3/2 + \varepsilon\) approximation for simple matchings in random-order streams) to integer-weighted \(b\)-matchings.

**Theorem**

*We can extract with high probability from a randomly-ordered stream of edges having integer weights in \(\{1, \ldots, W\}\) a weighted \(b\)-matching with an approximation ratio of*

\[
2 - \frac{1}{2W} + \varepsilon,
\]

*using \(O(\max(|M_G|, n) \cdot \text{poly}(\log(m), W, 1/\varepsilon))\) memory.*
To obtain algorithms for random-order streams:

- for unweighted simple matching [Bernstein, 2020]: use of *Edge-Degree Constrained Subgraph*
To obtain algorithms for random-order streams:

- for unweighted simple matching [Bernstein, 2020]: use of \textit{Edge-Degree Constrained Subgraph}
- for integer-weighted $b$-matching [our paper]: introduce a generalization of the \textit{Edge-Degree Constrained Subgraph}
Definition (EDCS [Bernstein and Stein, 2015])

Let $G = (V, E)$ be a graph, and $H$ a subgraph of $G$. Given any integer parameters $\beta > \beta^-$, we say that $H$ is a $(\beta, \beta^-)$-EDCS of $G$ if $H$ satisfies the following properties:

1. For any $(u, v) \in H$, $\deg_H(u) + \deg_H(v) \leq \beta$
2. For any $(u, v) \in G \setminus H$, $\deg_H(u) + \deg_H(v) \geq \beta^-$.

Figure: Example with $\beta = 5$ and $\beta^- = 4$
Simple facts on about EDCSes:

- they always exist (can be built in polynomial time)
- they are of size $O(n \cdot \beta)$
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- they always exist (can be built in polynomial time)
- they are of size $O(n \cdot \beta)$

Main result [Bernstein and Stein, 2015]:

**Theorem (Main theorem about EDCSes)**

For $\beta / \beta^- \leq 1$, a $(\beta, \beta^-)$-EDCS always contain a $3/2 + \varepsilon$ approximation of the maximum cardinality matching.

This bound is tight.
Generalizing EDCS

Definition (EDCS)

Let $\beta > \beta^-$, then $H$ is a $(\beta, \beta^-)$-EDCS of $G$ if $H$ satisfies the following properties:

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Some properties we want to have for weighted $b$-matchings:
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Some properties we want to have for weighted $b$-matchings:

- favoring edges having large weights
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Some properties we want to have for weighted $b$-matchings:

- favoring edges having large weights
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Some properties we want to have for weighted $b$-matchings:

- favoring edges having large weights
- homogeneity
- taking into account the capacities $b_v$
Generalizing EDCS

Definition \((w-b\text{-EDCS})\)

Let \(\beta > \beta^-\), then \(H\) is a \((\beta, \beta^-)\)-\(w-b\)-EDCS of \(G\) if \(H\) satisfies the following properties:

1. For any \((u, v) \in H\), \(\deg_H(u) + \deg_H(v) \leq \beta \cdot w(u, v)\)
2. For any \((u, v) \in G \setminus H\), \(\deg_H(u) + \deg_H(v) \geq \beta^- \cdot w(u, v)\)

Figure: Favoring larger edges
Generalizing EDCS

Definition ($w$-$b$-EDCS)

Let $\beta > \beta^-$, then $H$ is a $(\beta, \beta^-)$-$w$-$b$-EDCS of $G$ if $H$ satisfies the following properties:

1. For any $(u, v) \in H$, $w\deg_H(u) + w\deg_H(v) \leq \beta \cdot w(u, v)$
2. For any $(u, v) \in G \setminus H$, $w\deg_H(u) + w\deg_H(v) \geq \beta^- \cdot w(u, v)$

We want to have an homogeneous definition, so that multiplying all the weights of the graph $G$ by a factor $\lambda$ does not change the EDCS structure in the graph:

$$w'(u, v) = \lambda \cdot w(u, v)$$
Generalizing EDCS

Definition (weighted $b$-EDCS)

Let $\beta > \beta^-$, then $H$ is a $(\beta, \beta^-)$-$w$-$b$-EDCS of $G$ if $H$ satisfies the following properties:

- For any $(u, v) \in H$, \[ \frac{\text{wdeg}_H(u)}{b_u} + \frac{\text{wdeg}_H(v)}{b_v} \leq \beta \cdot w(u, v) \]
- For any $(u, v) \in G \setminus H$, \[ \frac{\text{wdeg}_H(u)}{b_u} + \frac{\text{wdeg}_H(v)}{b_v} \geq \beta^- \cdot w(u, v) \]

We take into account the capacity of a vertex:
- a vertex with a large capacity should be able to have more edges in the $w$-$b$-EDCS
Introduction
EDCS Technique
Conclusion

Generalizing EDCS

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Let $\beta > \beta^-$, then $H$ is a $(\beta, \beta^-)$-w-$b$-EDCS of $G$ if $H$ satisfies the following properties:

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2. For any $(u, v) \in G \setminus H$, $\frac{w\deg_H(u)}{b_u} + \frac{w\deg_H(v)}{b_v} \geq \beta^- \cdot w(u, v)$
Similarly, for weighted $b$-EDCSes:

- they always exist (can be built in pseudo-polynomial time)
- they are of size $O(|M_G| \cdot \beta)$
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Main result:

**Theorem (Main theorem about weighted $b$-EDCSes)**

For $\beta/\beta^- \text{ close enough to 1}$, a $(\beta, \beta^-)$-w-$b$-EDCS always contain a $2 - \frac{1}{2W} + \varepsilon$ approximation of the maximum weight matching.

This bound is tight.
Proof for Weighted EDCS

In the following:

- we prove the result for bipartite weighted matchings, as the generalization to weighted \(b\)-matchings will be a reduction to this case
- we will explain what adaptations of the proof of [Assadi and Bernstein, 2018] is required in that case

**Lemma**

For \(\frac{\beta}{\beta^-} \) close enough to 1, a \((\beta, \beta^-)\)-w-b-EDCS always contain a \(2 - \frac{1}{2W} + \varepsilon\) approximation of the maximum weight matching in a bipartite graph.
Existing Proof for Unweighted Simple Matchings

Figure: Witness set $\overline{A} \cup B$ from Hall’s marriage theorem — can also be seen as a vertex cover (picture from [Assadi and Bernstein, 2018])
Existing Proof for Unweighted Simple Matchings

**Figure:** Witness set $\overline{A} \cup B$ from Hall’s marriage theorem — can also be seen as a vertex cover (picture from [Assadi and Bernstein, 2018])

We want to bound $|S| \geq 2(|M_G| - |M_H|)$. 

Definition (EDCS)

Let $\beta \geq \beta^- + 1$, then $H$ is a $(\beta, \beta^-)$-EDCS of $G$ if $H$ satisfies the following properties:

1. For any $(u, v) \in H$, $\deg_H(u) + \deg_H(v) \leq \beta$
2. For any $(u, v) \in G \setminus H$, $\deg_H(u) + \deg_H(v) \geq \beta^-$. 

Consider the edges $\tilde{E}$ incident to $S$ in $H$. 

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Existing Proof for Unweighted Simple Matchings

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Consider the edges $\tilde{E}$ incident to $S$ in $H$. By Property ii:

$$|S| \cdot \beta^- / 2 \leq |\tilde{E}|$$
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Consider the edges $\tilde{E}$ incident to $S$ in $H$. By Property ii:

$$|S| \cdot \beta^- / 2 \leq |\tilde{E}|$$

and using convexity and averaging arguments,

$$|\tilde{E}| \leq (\beta - \beta^- / 2) \cdot |M_H|$$
Existing Proof for Unweighted Simple Matchings

**Definition (EDCS)**

Let $\beta \geq \beta^- + 1$, then $H$ is a $(\beta, \beta^-)$-EDCS of $G$ if $H$ satisfies the following properties:

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Consider the edges $\tilde{E}$ incident to $S$ in $H$. By Property ii:

$$|S| \cdot \frac{\beta^-}{2} \leq |\tilde{E}|$$

and using convexity and averaging arguments,

$$|\tilde{E}| \leq (\beta - \beta^-/2) \cdot |M_H|$$

hence,

$$|S| \leq (2\beta/\beta^- - 1) \cdot |M_H| \leq (1 + \varepsilon) \cdot |M_H|$$
Theorem (Kőnig-Egerváry duality theorem)

In any edge-weighted bipartite subgraph $H$ of $G$, the maximum weight of a matching equals the smallest weight of a $w$-vertex-cover.

In other words, there exist integers $(\alpha_v)_{v \in V}$ such that:

- $\sum_{v \in V} \alpha_v = w(M_H)$
- for all $(u, v) \in H$, $w(u, v) \leq \alpha_u + \alpha_v$
Proof for Weighted EDCS

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- for all $(u, v) \in H$, $w(u, v) \leq \alpha_u + \alpha_v$

We now consider the optimal matching $M_G$ in $G$. The first idea is to use this duality theorem to relate $w(M_G)$ to $w(M_H)$, with a leftover term that will be analyzed in the second part of the proof (just like the set $S$ previously).
We introduce the notion of *good* and *bad* edges:

- Let \((u, v) \in M_G\) such that \(\beta^- \cdot w(u, v) \leq \beta \cdot (\alpha_u + \alpha_v)\) is called a *good edge*; the set of good edges is denoted as \(M_{good}\);
- Let \((u, v) \in M_G\) such that \(\beta^- \cdot w(u, v) > \beta \cdot (\alpha_u + \alpha_v)\) is called a *bad edge*; the set of bad edges is denoted as \(M_{bad}\).
Proof for Weighted EDCS

Now, denoting by $V_{bad}$ the set of vertices which are the endpoints of a bad edge and such that $w\text{deg}_H(u) - \beta \cdot \alpha_u > 0$, we can get

$$\beta^\cdot w(M_G) \leq \beta \cdot w(M_H) + \sum_{v \in V_{bad}} (w\text{deg}_H(v) - \beta \cdot \alpha_v)$$
Proof for Weighted EDCS

Now, denoting by $V_{bad}$ the set of vertices which are the endpoints of a bad edge and such that $w\deg_H(u) - \beta \cdot \alpha_u > 0$, we can get

$$\beta^- \cdot w(M_G) \leq \beta \cdot w(M_H) + \sum_{v \in V_{bad}} (w\deg_H(v) - \beta \cdot \alpha_v)$$

Claim

We have the inequality

$$\sum_{v \in V_{bad}} (w\deg_H(v) - \beta \cdot \alpha_v) \leq \left(\beta + W - \frac{\beta^-}{2W \cdot (1 + \varepsilon/4)}\right) \cdot w(M_H)$$
**Proof for Weighted EDCS**

**Claim**

*We have the inequality*

$$\sum_{v \in V_{bad}} (w_{\text{deg}}_H(v) - \beta \cdot \alpha_v) \leq \left( \beta + W - \frac{\beta^-}{2W \cdot (1 + \epsilon/4)} \right) \cdot w(M_H)$$

- $V_{bad}$ plays somehow here the role of $S$
Proof for Weighted EDCS

Claim

We have the inequality

\[
\sum_{v \in V_{bad}} (w \deg_H(v) - \beta \cdot \alpha_v) \leq \left( \beta + W - \frac{\beta}{2W \cdot (1 + \varepsilon/4)} \right) \cdot w(M_H)
\]

- \( V_{bad} \) plays somehow here the role of \( S \)
- however we cannot just use the set of edges in \( H \) incident to \( V_{bad} \) — some vertices are partially covered
Proof for Weighted EDCS

Claim

We have the inequality

$$\sum_{v \in V_{bad}} (w \deg_H(v) - \beta \cdot \alpha_v) \leq \left( \beta + W - \frac{\beta^-}{2W \cdot (1 + \varepsilon/4)} \right) \cdot w(M_H)$$

- $V_{bad}$ plays somehow here the role of $S$
- however we cannot just use the set of edges in $H$ incident to $V_{bad}$ — some vertices are partially covered
- we need to build an auxiliary graph $H_{bad}$ made of a well-chosen set of edges $\tilde{E}$
Proof for Weighted EDCS

Claim

We have the inequality

$$\sum_{v \in V_{bad}} (w_{\text{deg}_H(v)} - \beta \cdot \alpha_v) \leq \left( \beta + W - \frac{\beta^{-}}{2W \cdot (1 + \epsilon/4)} \right) \cdot w(M_H)$$

- $V_{bad}$ plays somehow here the role of $S$
- however we cannot just use the set of edges in $H$ incident to $V_{bad}$ — some vertices are partially covered
- we need to build an auxiliary graph $H_{bad}$ made of a well-chosen set of edges $\tilde{E}$
- then a similar averaging/convexity argument can be used
Proof for Weighted EDCS

Lemma

For $\beta/\beta^{-}$ close enough to 1, a $(\beta, \beta^{-})$-w-b-EDCS always contain a $2 - \frac{1}{2W} + \varepsilon$ approximation of the maximum weight matching in a bipartite graph.

Theorem

For $\beta/\beta^{-}$ close enough to 1, a $(\beta, \beta^{-})$-w-b-EDCS always contain a $2 - \frac{1}{2W} + \varepsilon$ approximation of the maximum weight matching.

Using the probabilistic method and Lovasz Local Lemma, as in [Assadi and Bernstein, 2018].
Definition (weighted $b$-EDCS)

Let $\beta > \beta^-$, then $H$ is a $(\beta, \beta^-)$-$w$-$b$-EDCS of $G$ if $H$ satisfies the following properties:

i. For any $(u, v) \in H$, \[ \frac{w\deg_H(u)}{b_u} + \frac{w\deg_H(v)}{b_v} \leq \beta \cdot w(u, v) \]

ii. For any $(u, v) \in G \setminus H$, \[ \frac{w\deg_H(u)}{b_u} + \frac{w\deg_H(v)}{b_v} \geq \beta^- \cdot w(u, v) \]

Theorem (Main theorem about weighted $b$-EDCSes)

*For $\beta/\beta^-$ close enough to 1, a $(\beta, \beta^-)$-$w$-$b$-EDCS always contain a $2 - \frac{1}{2W} + \varepsilon$ approximation of the maximum weight matching.*

Can be used to design an algorithm for random-order streams.