Maximum Weight b-Matchings in Random-Order Streams

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Definition of a b-Matching

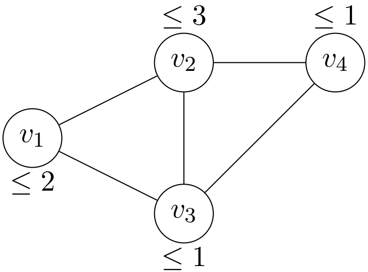

Figure: An input graph with capacities $b_{v_1} = 2$, $b_{v_2} = 3$, and $b_{v_3} = b_{v_4} = 1$

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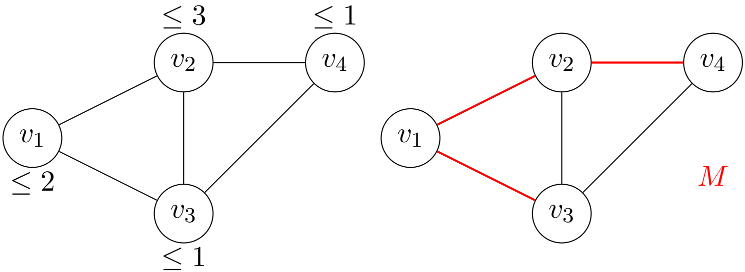

Figure: An input graph with capacities $b_{v_1} = 2$, $b_{v_2} = 3$, and $b_{v_3} = b_{v_4} = 1$
An output b-matching M, in red

The Maximum Weight b-Matching Problem

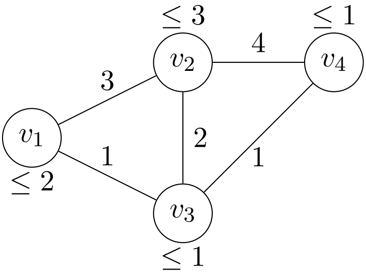

Figure: Finding a maximum weight b-matching

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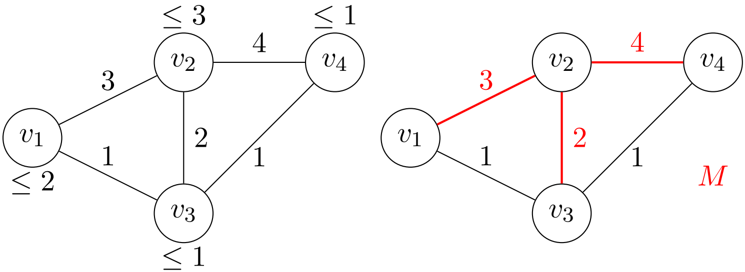

Figure: Finding a maximum weight b-matching
The weight of the highlighted matching M is $2 + 3 + 4 = 9$.

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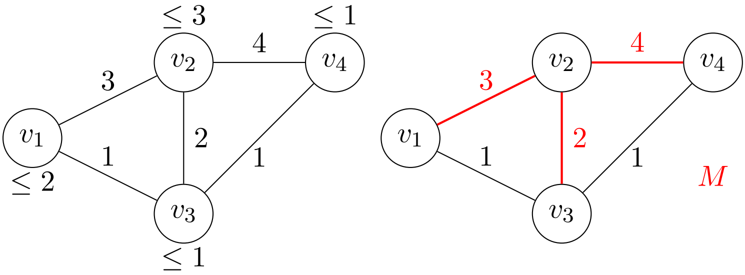

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In this work, we assume the weights are small integers in a range $\{1, \ldots, W\}$.

Degree and Weighted Degree

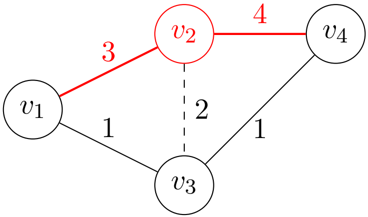

Figure: Defining the degree and weighted degree of vertex v_2 in some subgraph H

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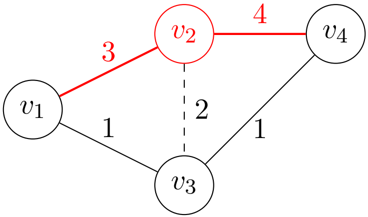

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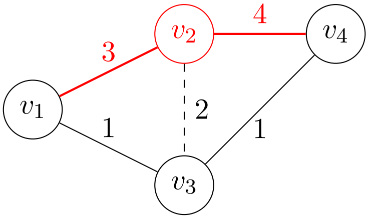

Figure: Defining the degree and weighted degree of vertex v_2 in some subgraph H

- $\deg_H(v_2) = 2$ (and $\deg_G(v_2) = 3$)
- $\mathbf{w}\deg_H(v_2) = 3 + 4 = 7$ (and $\mathbf{w}\deg_G(v_2) = 3 + 4 + 2 = 9$)

Some Definitions

In the *semi-streaming* model:

- edges of E arrive over time in a **stream** $S = \langle e_1, \ldots, e_m \rangle$

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In the *random-order semi-streaming* model:

- the permutation of the edges given in the stream is assumed to be chosen **uniformly at random** among all the possible permutations

Previous Results

In the *adversarial* (worst-case order) semi-streaming model:

- For unweighted maximum (b-)matching approximation:
  - 2 approximation using the trivial greedy algorithm
  - $1 + \ln 2 \approx 1.69$ inapproximability [Kapralov, 2021]

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  - 6 [Feigenbaum, Kannan, McGregor, Suri, and Zhang, 2005]
  - 5.83 [McGregor, 2005]
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- For maximum weight b-matching approximation:
  - $4 + \varepsilon$ [Crouch and Stubbs, 2014]
  - $3 + \varepsilon$ [Levin and Wajc, 2021]
  - $2 + \varepsilon$ [Huang and Sellier, 2021]

Previous Results

In the *random-order* semi-steaming model:

- For maximum unweighted matching approximation:
  - 1.98 (bipartite), 1.99 (general) [Konrad, Magniez, and Mathieu, 2012]
  - 1.86 (bipartite) [Konrad, 2018]
  - $5/3 \approx 1.67$ (bipartite), $11/6 \approx 1.84$ (general) [Farhadi, Hajiaghayi, Mai, Rao, and Rossi, 2020]
  - $3/2 + \varepsilon$ (general) [Bernstein, 2020]
  - $3/2 - \delta$ (general, $\delta \sim 10^{-14}$) [Assadi and Behnezhad, 2021]

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- For maximum weight matching approximation:
  - $2 - \delta$ (general, $\delta \sim 10^{-17}$) [Gamlath, Kale, Mitrovic, and Svensson, 2019]
- For maximum (weight) b-matching approximation:
  - no previous result in the random-order setting

Previous Results and Our Contribution

Maximization problem	Adversarial	Random-Order
Unweighted matching	2	$3/2 - \delta$
Weighted matching	2	$2 - \delta$
Unweighted b-matching	2	2
Weighted b-matching	2	2

Table: Approximation ratios for adversarial and random-order streams (we assume weights are small integers in $\{1, \ldots, W\}$).

Previous Results and Our Contribution

Maximization problem	Adversarial	Random-Order
Unweighted matching	2	$3/2-\delta$
Weighted matching	2	$2-\delta$ $(2-1/(2W)+\varepsilon)$
Unweighted b-matching	2	2 $(3/2+\varepsilon)$
Weighted b-matching	2	2 $(2-1/(2W)+\varepsilon)$

Table: Approximation ratios for adversarial and random-order streams (we assume weights are small integers in $\{1,\ldots,W\}$). Comparison with our results.

Our Contribution

We generalize the result of [Bernstein, 2020] ($3/2 + \varepsilon$ approximation for simple matchings in random-order streams) to integer-weighted b-matchings.

Theorem

We can extract with high probability from a randomly-ordered stream of edges having integer weights in $\{1, \ldots, W\}$ a weighted b-matching with an approximation ratio of

$$2 - \frac{1}{2W} + \varepsilon,$$

using $O(\max(|M_G|, n) \cdot poly(\log(m), W, 1/\varepsilon))$ memory.

Technique

To obtain algorithms for random-order streams:

- for unweighted simple matching [Bernstein, 2020]: use of *Edge-Degree Constrained Subgraph*

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To obtain algorithms for random-order streams:

- for unweighted simple matching [Bernstein, 2020]: use of *Edge-Degree Constrained Subgraph*
- for integer-weighted b-matching [our paper]: introduce a generalization of the *Edge-Degree Constrained Subgraph*

Edge-Degree Constrained Subgraphs

Definition (EDCS [Bernstein and Stein, 2015])

Let $G = (V, E)$ be a graph, and H a subgraph of G. Given any integer parameters $\beta > \beta^-$, we say that H is a (β, β^-)-EDCS of G if H satisfies the following properties:

1. For any $(u, v) \in H$, $\deg_H(u) + \deg_H(v) \leq \beta$
2. For any $(u, v) \in G \backslash H$, $\deg_H(u) + \deg_H(v) \geq \beta^-$.

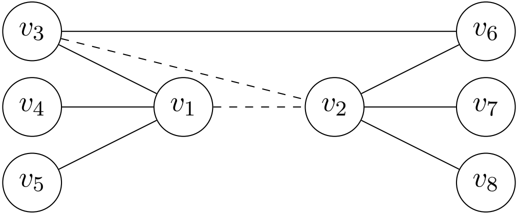

Figure: Example with $\beta = 5$ and $\beta^- = 4$

Edge-Degree Constrained Subgraphs

Simple facts on about EDCSes:

- they always exist (can be built in polynomial time)
- they are of size $O(n \cdot \beta)$

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Main result [Bernstein and Stein, 2015]:

Theorem (Main theorem about EDCSes)

For β/β^- close enough to 1, a (β, β^-)-EDCS always contain a $3/2 + \varepsilon$ approximation of the maximum cardinality matching.

This bound is tight.

Generalizing EDCS

Definition (EDCS)

Let $\beta > \beta^-$, then H is a (β, β^-)-EDCS of G if H satisfies the following properties:

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Some properties we want to have for weighted b-matchings:

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Some properties we want to have for weighted b-matchings:

- favoring edges having large weights
- homogeneity
- taking into account the capacities b_v

Generalizing EDCS

Definition (w-b-EDCS)

Let $\beta > \beta^-$, then H is a (β, β^-)-w-b-EDCS of G if H satisfies the following properties:

1. For any $(u,v) \in H$, $\deg_H(u) + \deg_H(v) \leq \beta \cdot w(u,v)$
2. For any $(u,v) \in G \backslash H$, $\deg_H(u) + \deg_H(v) \geq \beta^- \cdot w(u,v)$

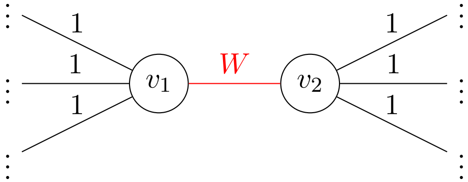

Figure: Favoring larger edges

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2. For any $(u, v) \in G \backslash H$, $\mathbf{w}\text{deg}_H(u) + \mathbf{w}\text{deg}_H(v) \geq \beta^- \cdot w(u, v)$

We want to have an homogeneous definition, so that multiplying all the weights of the graph G by a factor λ does not change the EDCS structure in the graph:

$$w'(u, v) = \lambda \cdot w(u, v)$$

Generalizing EDCS

Definition (weighted b-EDCS)

Let $\beta > \beta^-$, then H is a (β, β^-)-w-b-EDCS of G if H satisfies the following properties:

1. For any $(u,v) \in H$, $\frac{\mathbf{w}\text{deg}_H(u)}{b_u} + \frac{\mathbf{w}\text{deg}_H(v)}{b_v} \leq \beta \cdot w(u,v)$
2. For any $(u,v) \in G \backslash H$, $\frac{\mathbf{w}\text{deg}_H(u)}{b_u} + \frac{\mathbf{w}\text{deg}_H(v)}{b_v} \geq \beta^- \cdot w(u,v)$

We take into account the capacity of a vertex:

- a vertex with a large capacity should be able to have more edges in the w-b-EDCS

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Similarly, for weighted b-EDCSes:

- they always exist (can be built in pseudo-polynomial time)
- they are of size $O(|M_G| \cdot \beta)$

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Main result:

Theorem (Main theorem about weighted b-EDCSes)

For β/β^- close enough to 1, a (β, β^-)-w-b-EDCS always contain a $2 - \frac{1}{2W} + \varepsilon$ approximation of the maximum weight matching.

This bound is tight.

Proof for Weighted EDCS

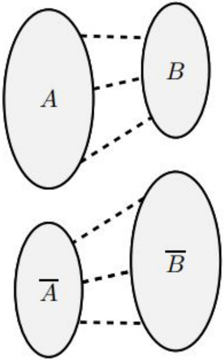
In the following:

- we prove the result for bipartite weighted matchings, as the generalization to weighted b-matchings will be a reduction to this case
- we will explain what adaptations of the proof of [Assadi and Bernstein, 2018] is required in that case

Lemma

For β/β^- close enough to 1, a (β, β^-)-w-b-EDCS always contain a $2 - \frac{1}{2W} + \varepsilon$ approximation of the maximum weight matching in a bipartite graph.

Existing Proof for Unweighted Simple Matchings



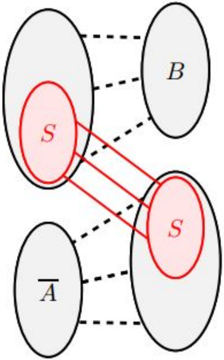


Figure: Witness set $\overline{A} \cup B$ from Hall's marriage theorem — can also be seen as a vertex cover (picture from [Assadi and Bernstein, 2018])

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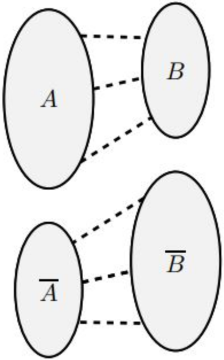

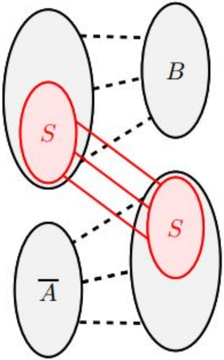

Figure: Witness set $\overline{A} \cup B$ from Hall's marriage theorem — can also be seen as a vertex cover (picture from [Assadi and Bernstein, 2018])

We want to bound $|S| \geq 2(|M_G| - |M_H|)$.

Existing Proof for Unweighted Simple Matchings

Definition (EDCS)

Let $\beta \geq \beta^- + 1$, then H is a (β, β^-)-EDCS of G if H satisfies the following properties:

- (i) For any $(u, v) \in H$, $\deg_H(u) + \deg_H(v) \leq \beta$
- (ii) For any $(u, v) \in G \backslash H$, $\deg_H(u) + \deg_H(v) \geq \beta^-$.

Consider the edges $\tilde{E}$ incident to S in H.

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$$|S| \cdot \beta^- / 2 \leq |\tilde{E}|$$

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Consider the edges $\tilde{E}$ incident to S in H. By Property ii:

$$|S| \cdot \beta^-/2 \leq |\tilde{E}|$$

and using convexity and averaging arguments,

$$|\tilde{E}| \leq (\beta - \beta^-/2) \cdot |M_H|$$

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hence,

$$|S| \leq (2\beta/\beta^- - 1) \cdot |M_H| \leq (1 + \varepsilon) \cdot |M_H|$$

.

Proof for Weighted EDCS

Theorem (Kőnig-Egerváry duality theorem)

In any edge-weighted bipartite subgraph H of G, the maximum weight of a matching equals the smallest weight of a w-vertex-cover.

In other words, there exist integers $(\alpha_v)_{v \in V}$ such that:

- $\sum_{v \in V} \alpha_v = w(M_H)$
- for all $(u, v) \in H$, $w(u, v) \leq \alpha_u + \alpha_v$

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We now consider the optimal matching M_G in G. The first idea is to use this duality theorem to relate $w(M_G)$ to $w(M_H)$, with a leftover term that will be analyzed in the second part of the proof (just like the set S previously).

Proof for Weighted EDCS

We introduce the notion of *good* and *bad* edges:

- $(u, v) \in M_G$ such that $\beta^- \cdot w(u, v) \leq \beta \cdot (\alpha_u + \alpha_v)$ is called a *good edge*; the set of good edges is denoted as M_{good};
- $(u, v) \in M_G$ such that $\beta^- \cdot w(u, v) > \beta \cdot (\alpha_u + \alpha_v)$ is called a *bad edge*; the set of bad edges is denoted as M_{bad}.

Proof for Weighted EDCS

Now, denoting by V_{bad} the set of vertices which are the endpoints of a bad edge and such that $\mathbf{w}\deg_H(u) - \beta \cdot \alpha_u > 0$, we can get

$$\beta^- \cdot w(M_G) \leq \beta \cdot w(M_H) + \sum_{v \in V_{bad}} (\mathbf{w}\deg_H(v) - \beta \cdot \alpha_v)$$

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Claim

We have the inequality

$$\sum_{v \in V_{bad}} (\mathbf{w}\text{deg}_H(v) - \beta \cdot \alpha_v) \leq \left(\beta + W - \frac{\beta^-}{2W \cdot (1 + \varepsilon/4)} \right) \cdot w(M_H)$$

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- V_{bad} plays somehow here the role of S
- however we cannot just use the set of edges in H incident to V_{bad} — some vertices are partially covered
- we need to build an auxiliary graph H_{bad} made of a well-chosen set of edges $\tilde{E}$

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Claim

We have the inequality

$$\sum_{v \in V_{bad}} (\mathbf{w}\mathrm{deg}_H(v) - \beta \cdot \alpha_v) \leq \left(\beta + W - \frac{\beta^-}{2W \cdot (1 + \varepsilon/4)} \right) \cdot w(M_H)$$

- V_{bad} plays somehow here the role of S
- however we cannot just use the set of edges in H incident to V_{bad} — some vertices are partially covered
- we need to build an auxiliary graph H_{bad} made of a well-chosen set of edges $\tilde{E}$
- then a similar averaging/convexity argument can be used

Proof for Weighted EDCS

Lemma

For β/β^- close enough to 1, a (β, β^-)-w-b-EDCS always contain a $2 - \frac{1}{2W} + \varepsilon$ approximation of the maximum weight matching in a bipartite graph.

Theorem

For β/β^- close enough to 1, a (β, β^-)-w-b-EDCS always contain a $2 - \frac{1}{2W} + \varepsilon$ approximation of the maximum weight matching.

Using the probabilistic method and Lovasz Local Lemma, as in [Assadi and Bernstein, 2018].

Conclusion

Definition (weighted b-EDCS)

Let $\beta > \beta^-$, then H is a (β, β^-)-w-b-EDCS of G if H satisfies the following properties:

1. For any $(u, v) \in H$, $\frac{\mathbf{w}\mathrm{deg}_H(u)}{b_u} + \frac{\mathbf{w}\mathrm{deg}_H(v)}{b_v} \leq \beta \cdot w(u, v)$
2. For any $(u, v) \in G \backslash H$, $\frac{\mathbf{w}\mathrm{deg}_H(u)}{b_u} + \frac{\mathbf{w}\mathrm{deg}_H(v)}{b_v} \geq \beta^- \cdot w(u, v)$

Theorem (Main theorem about weighted b-EDCSes)

For β/β^- close enough to 1, a (β, β^-)-w-b-EDCS always contain a $2 - \frac{1}{2W} + \varepsilon$ approximation of the maximum weight matching.

Can be used to design an algorithm for random-order streams.