Parameterized Matroid-Constrained Maximum Coverage

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Matroid-Constrained Maximum Coverage
Figure: A maximum $k$-vertex cover problem with $k = 2$
Figure: A maximum $k$-vertex cover problem with $k = 2$, the value of the cover is $12 + 2 + 1 + 4 + 3 + 2 + 5$
Maximum $k$ Vertex Cover

Let $G = (V, E)$ be a graph.

- a non-negative weight $w(e)$ is associated to each edge $e \in E$
- an edge $e = (u, v)$ is called covered by a set $S \subseteq V$ if at least one of its endpoints is in $S$, i.e. $u \in S$ or $v \in S$
- we will denote $E_G(S)$ the sum of the weights of the edges covered the set $S \subseteq V$, i.e.

$$E_G(S) = \sum_{e=(u,v) \in E, \{u,v\} \cap S \neq \emptyset} w(e)$$

- in the maximum $k$-vertex cover problem, we want a set containing at most $k$ elements maximizing $E_G$, i.e.

$$\underset{S \subseteq V, |S| \leq k}{\text{argmax}} E_G(S)$$
Figure: A maximum matroid-constrained vertex cover problem (one vertex per color can be taken)
Figure: A maximum matroid-constrained vertex cover problem (one vertex per color can be taken), the value of the cover is $12 + 4 + 3 + 2 + 5$
\[ M = (V, \mathcal{I}) \]

is a matroid on the ground set \( V \) if these conditions hold for \( \mathcal{I} \subseteq \mathcal{P}(V) \):

1. \( \emptyset \in \mathcal{I} \),
2. if \( X \subseteq Y \in \mathcal{I} \), then \( X \in \mathcal{I} \),
3. if \( X, Y \in \mathcal{I}, |Y| > |X| \), there exists an element \( e \in Y \setminus X \) so that \( X \cup \{e\} \in \mathcal{I} \),

the sets in \( \mathcal{I} \) are the independent sets and the rank \( r_M \) of the matroid \( M \) is defined as \( \max_{X \in \mathcal{I}} |X| \).
Let $G = (V, E)$ be a graph and $\mathcal{M} = (V, \mathcal{I})$ a matroid over $V$:

- in the matroid-constrained maximum vertex cover problem, we want a set independent in $\mathcal{M}$ maximizing $E_G$, i.e.

$$\text{argmax}_{S \subseteq V, S \in \mathcal{I}} E_G(S)$$
this work generalizes a kernelization method developed for maximum coverage under \textit{cardinality constraint} to the more general \textit{matroid constraint}

to simplify this presentation, instead of considering a frequency-bounded coverage function we only consider a vertex-cover function
The problem of maximizing a submodular function under matroid constraint has been studied extensively
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Interest of the Problem

The problem of maximizing a submodular function under matroid constraint has been studied extensively:

- the problem is easy when: the submodular function is linear
- in our work, we study a slightly more complicated case: the function is a cover function of bounded frequency

Figure: Vertex cover function
For the maximum $k$-vertex cover problem:

- greedy provides a $1 - 1/e$ approximation [Hochbaum and Pathria, 1998]
- a LP-based approach and a technique of pipage rounding give a ratio of $3/4$ [Ageev and Sviridenko, 2000]
- the current best ratio is 0.92, attained using a kernelization method [Manurangsi, 2018]
- it is not possible to have a Polynomial Time Approximation Scheme [Guo, Niedermeier, and Wernicke, 2005]
Previous Results

Here we recall the definition of an FPT-AS [Marx, 2008]:

\textbf{Definition}

Given a parameter function $\kappa$ associating a natural number to each instance $x \in I$ of a given problem, a \textit{Fixed-Parameter Tractable Approximation Scheme} (FPT-AS) is an algorithm that can provide a $(1 - \varepsilon)$ approximation in $f(\varepsilon, \kappa(x)) \cdot |x|^{O(1)}$ time.
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- in our case $\kappa$ is the rank $k$ of the matroid
- FTP-AS for $k$-vertex cover [Marx, 2008; Manurangsi, 2018]
- approximate kernel of size $k/\varepsilon$ for $k$-vertex-cover [Manurangsi, 2018]
- approximate kernels of size $O(k/\varepsilon)$ for partition and laminar matroids [Huang and Sellier, 2022] and for transversal matroids [Kamiyama, 2022]
Let \( \mathcal{M} = (V, \mathcal{I}) \) be a matroid and let \( G = (V, E) \) be a weighted graph. We can compute in polynomial time an approximate kernel \( V' \) of size \( k \cdot \rho \) containing a \( 1 - 1/\rho \) approximate solution of the matroid-constrained maximum vertex cover problem for any integer \( \rho \).

Using a brute force enumeration, we can find a \( 1 - \varepsilon \) approximation in \( (1/\varepsilon)^{O(k)} n^{O(1)} \) time, i.e., an FPT-AS.
The idea is to:

- build an *approximate kernel*, i.e., a smaller graph containing a \((1 - \varepsilon)\)-approximation of the optimal solution
- find the optimal solution in that smaller graph using brute force
Previous Kernelization Technique

- $V'$ contains the $k/\varepsilon$ vertices having the largest weighted degrees
Previous Kernelization Technique

Figure: Kernelization technique developed in [Manurangsi, 2018] for the maximum $k$-vertex cover problem

- $V'$ contains the $k/\varepsilon$ vertices having the largest weighted degrees
- the elements in $O^{out}$ are replaced by random elements drawn from $V'$
Previous Kernelization Technique

Figure: Kernelization technique developed in [Manurangsi, 2018] for the maximum $k$-vertex cover problem

- A vertex in $O^{out}$ is replaced by a vertex in $V'$ having a larger weighted degree, i.e. for instance,

$$\deg_w(u_1) \leq \deg_w(v_1), \ldots, \deg_w(u_r) \leq \deg_w(v_{12})$$
Previous Kernelization Technique

Figure: Kernelization technique developed in [Manurangsi, 2018] for the maximum $k$-vertex cover problem

- there is two types of double counting: (i) edges between $O^{in}$ and sampled vertices, and (ii) edges between sampled vertices
To go from $k$-vertex-cover to matroid constrained vertex cover:
- the sampling cannot be done the same way, as we have to sample independent sets
- hence the kernel $V'$ also has to be built differently (example: we should not take only vertices of one color in our kernel $V'$)

Figure: A maximum matroid-constrained vertex cover problem
The sampling procedure has to:

- select **independent sets**
- the elements of the kernel must have the **same probability** to be sampled
- these events have to be pairwise **negatively correlated**
Definition

Suppose that $\mathcal{M} = (V, \mathcal{I})$ is a matroid. Then we can define $\tau \mathcal{M} = (V, \mathcal{I}_\tau)$ as the union of $\tau$ matroids $\mathcal{M}$, as follows: $S \in \mathcal{I}_\tau$ if $S$ can be partitioned into $S_1 \cup \cdots \cup S_\tau$ so that each $S_i \in \mathcal{I}$.

Theorem

Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid and let $G = (V, E)$ be a weighted graph. Let $V'$ be a maximum weight independent set in $\rho \mathcal{M}$, with respect to the weighted degrees $\text{deg}_w(v)$. Then $V'$ contains a $1 - 1/\rho$ approximate solution of the matroid-constrained maximum vertex cover problem.

For uniform matroids, we get exactly the same kernel as in [Manurangsi, 2018].
Definition

The \textit{density} of a subset $U \subseteq V$ in the matroid $\mathcal{M}$ is defined as

$$\rho_{\mathcal{M}}(U) = \frac{|U|}{r_{\mathcal{M}}(U)}.$$
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Definition

Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid, and $\rho$ be a positive integer. A subset $V' \subseteq V$ is called a $\rho$-DBS in $\mathcal{M}$ if $\rho_{\mathcal{M}}(V') = \rho$ and for all $U \subseteq V'$, $\rho_{\mathcal{M}}(U) \leq \rho$.

These Density-Balanced Subset have nice sampling and contraction properties that allow to adapt the previous sampling technique. These subsets appear naturally when building matroid unions.
Definition

Let \( \mathcal{M} = (V, \mathcal{I}) \) be a matroid, and \( \rho \) be a positive integer. A subset \( V' \subseteq V \) is called a \( \rho \)-DBS in \( \mathcal{M} \) if \( \rho_{\mathcal{M}}(V') = \rho \) and for all \( U \subseteq V' \), \( \rho_{\mathcal{M}}(U) \leq \rho \).

Figure: The subset spanned in \( V' \) by \( \{v_1, v_2\} \) is of size bounded by \( 2 \cdot \rho \), meaning that there are at least \( (r_{\mathcal{M}}(V') - 2) \cdot \rho \) elements still usable for the sampling.
Conclusion

Results:

- **natural generalization** of the kernelization process used in the $k$-vertex-cover problem
- closes the gap between the **cardinality constraint** and the more general **matroid constraint**

**Theorem**

Let $V'$ be a maximum weight independent set in $\rho M$, with respect to the weighted degrees. Then $V'$ contains a $1 - 1/\rho$ approximate solution of the matroid-constrained maximum vertex cover problem.