Parameterized Matroid-Constrained Maximum Coverage

François Sellier

INSTITUT DE RECHERCHE EN INFORMATIQUE FONDAMENTALE

MINES ParisTech | PSL

Maximum k Vertex Cover

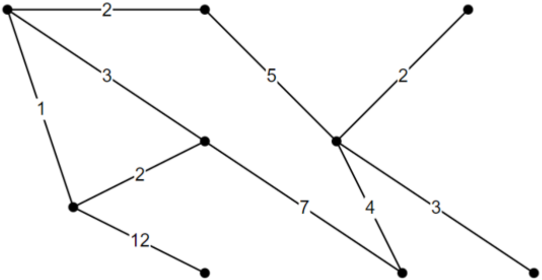

Figure: A maximum k-vertex cover problem with $k = 2$

Maximum k Vertex Cover

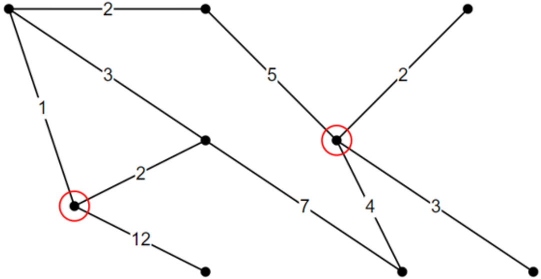

Figure: A maximum k-vertex cover problem with $k = 2$, the value of the cover is $12 + 2 + 1 + 4 + 3 + 2 + 5$

Maximum k Vertex Cover

Let $G = (V, E)$ be a graph.

- a non-negative *weight* $w(e)$ is associated to each edge $e \in E$
- an edge $e = (u, v)$ is called *covered* by a set $S \subseteq V$ if at least one of its endpoints is in S, *i.e* $u \in S$ or $v \in S$
- we will denote $E_G(S)$ the sum of the weights of the edges covered the set $S \subseteq V$, *i.e.*

$$E_G(S) = \sum_{e=(u,v)\in E, \{u,v\}\cap S \neq \emptyset} w(e)$$

- in the *maximum k-vertex cover* problem, we want a set containing at most k elements maximizing E_G, *i.e.*

$$\underset{S\subseteq V, |S|\leq k}{\operatorname{argmax}} E_G(S)$$

Matroid-Constrained Maximum Vertex Cover

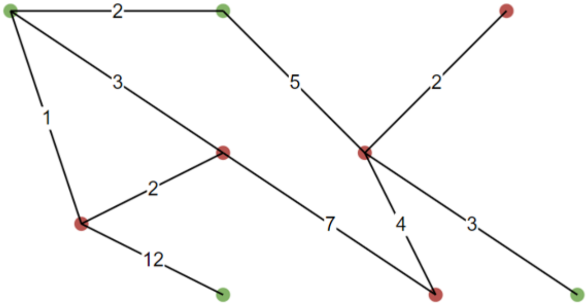

Figure: A maximum matroid-constrained vertex cover problem (one vertex per color can be taken)

Matroid-Constrained Maximum Vertex Cover

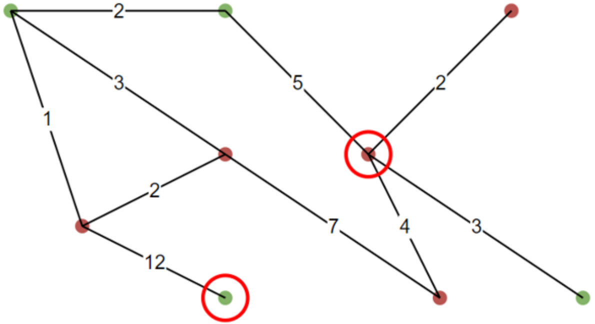

Figure: A maximum matroid-constrained vertex cover problem (one vertex per color can be taken), the value of the cover is $12 + 4 + 3 + 2 + 5$

Matroid-Constrained Maximum Vertex Cover

$\mathcal{M} = (V, \mathcal{I})$ is a *matroid* on the ground set V if these conditions hold for $\mathcal{I} \subseteq \mathcal{P}(V)$:

1. $\emptyset \in \mathcal{I}$,
2. if $X \subseteq Y \in \mathcal{I}$, then $X \in \mathcal{I}$,
3. if $X, Y \in \mathcal{I}, |Y| > |X|$, there exists an element $e \in Y \setminus X$ so that $X \cup \{e\} \in \mathcal{I}$,

the sets in $\mathcal{I}$ are the *independent sets* and the *rank* $r_{\mathcal{M}}$ of the matroid $\mathcal{M}$ is defined as $\max_{X \in \mathcal{I}} |X|$.

Matroid-Constrained Maximum Vertex Cover

Let $G = (V, E)$ be a graph and $\mathcal{M} = (V, \mathcal{I})$ a matroid over V:

- in the *matroid-constrained maximum vertex cover* problem, we want a set independent in $\mathcal{M}$ maximizing E_G, *i.e.*

$$\operatorname*{argmax}_{S \subseteq V, S \in \mathcal{I}} E_G(S)$$

General Outline

- this work generalizes a kernelization method developed for maximum coverage under **cardinality constraint** to the more general **matroid constraint**
- to simplify this presentation, instead of considering a frequency-bounded coverage function we only consider a vertex-cover function

Interest of the Problem

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- the problem is easy when:
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- in our work, we study a slightly more complicated case:
  - the function is a cover function of bounded frequency

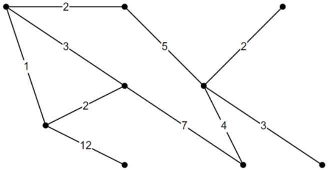

Figure: Vertex cover function

Previous Results

For the maximum k-vertex cover problem:

- greedy provides a $1 - 1/e$ approximation [Hochbaum and Pathria, 1998]
- a LP-based approach and a technique of pipage rounding give a ratio of $3/4$ [Ageev and Sviridenko, 2000]
- the current best ratio is 0.92, attained using a kernelization method [Manurangsi, 2018]
- it is not possible to have a Polynomial Time Approximation Scheme [Guo, Niedermeier, and Wernicke, 2005]

Previous Results

Here we recall the definition of an FPT-AS [Marx, 2008]:

Definition

Given a parameter function κ associating a natural number to each instance $x \in I$ of a given problem, a *Fixed-Parameter Tractable Approximation Scheme* (FPT-AS) is an algorithm that can provide a $(1-\varepsilon)$ approximation in $f(\varepsilon, \kappa(x)) \cdot |x|^{O(1)}$ time.

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- in our case κ is the rank k of the matroid
- FTP-AS for k-vertex cover [Marx, 2008; Manurangsi, 2018]
- approximate kernel of size k/ε for k-vertex-cover [Manurangsi, 2018]
- approximate kernels of size $O(k/\varepsilon)$ for partition and laminar matroids [Huang and Sellier, 2022] and for transversal matroids [Kamiyama, 2022]

Our Contribution

Theorem

Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid and let $G = (V, E)$ be a weighted graph. We can compute in polynomial time an approximate kernel V' of size $k \cdot \rho$ containing a $1 - 1/\rho$ approximate solution of the matroid-constrained maximum vertex cover problem for any integer ρ.

Using a brute force enumeration, we can find a $1 - \varepsilon$ approximation in $(1/\varepsilon)^{O(k)}\, n^{O(1)}$ time, i.e., an FPT-AS.

Kernelization Framework

The idea is to:

- build an *approximate kernel*, *i.e.*, a smaller graph containing a $(1-\varepsilon)$-approximation of the optimal solution
- find the optimal solution in that smaller graph using bruteforce

Previous Kernelization Technique

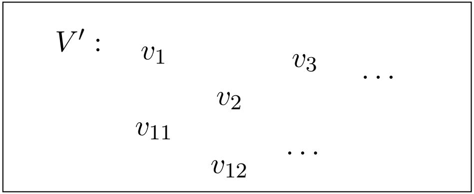

Figure: Kernelization technique developed in [Manurangsi, 2018] for the maximum k-vertex cover problem

- V' contains the k/ε vertices having the largest weighted degrees

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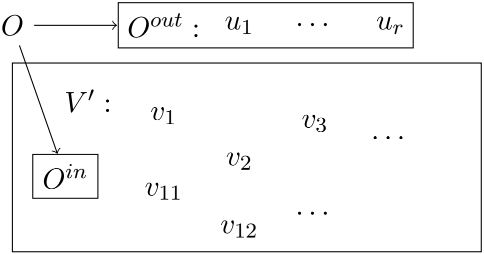

Figure: Kernelization technique developed in [Manurangsi, 2018] for the maximum k-vertex cover problem

- V' contains the k/ε vertices having the largest weighted degrees
- the elements in O^{out} are replaced by random elements drawn from V'

Previous Kernelization Technique

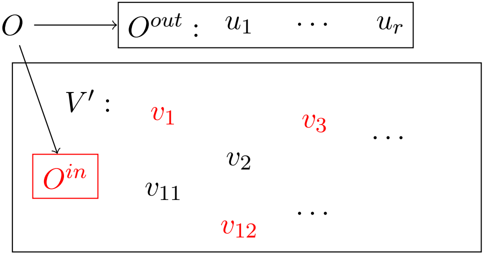

Figure: Kernelization technique developed in [Manurangsi, 2018] for the maximum k-vertex cover problem

- a vertex in O^{out} is replaced by a vertex in V' having a larger weighted degree, *i.e.* for instance,

$$\deg_w(u_1) \leq \deg_w(v_1), \ldots, \deg_w(u_r) \leq \deg_w(v_{12})$$

Previous Kernelization Technique

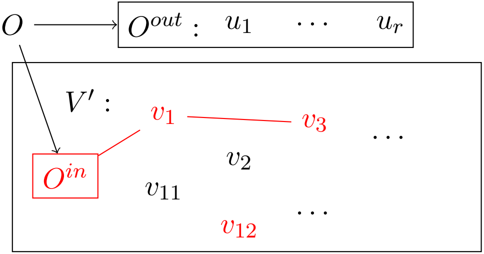

Figure: Kernelization technique developed in [Manurangsi, 2018] for the maximum k-vertex cover problem

- there is two types of double counting: (i) edges between O^{in} and sampled vertices, and (ii) edges between sampled vertices

Kernelization Framework

To go from k-vertex-cover to matroid constrained vertex cover:

- the sampling cannot be done the same way, as we have to sample independent sets
- hence the kernel V' also has to be built differently (example: we should not take only vertices of one color in our kernel V')

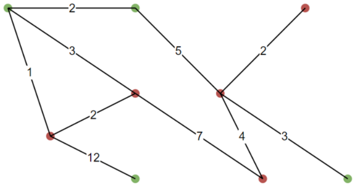

Figure: A maximum matroid-constrained vertex cover problem

Kernelization Framework

The sampling procedure has to:

- select **independent sets**
- the elements of the kernel must have the **same probability** to be sampled
- these events have to be pairwise **negatively correlated**

Kernelization Framework

Definition

Suppose that $\mathcal{M} = (V, \mathcal{I})$ is a matroid. Then we can define $\tau\mathcal{M} = (V, \mathcal{I}_\tau)$ as the union of τ matroids $\mathcal{M}$, as follows: $S \in \mathcal{I}_\tau$ if S can be partitioned into $S_1 \cup \cdots \cup S_\tau$ so that each $S_i \in \mathcal{I}$.

Theorem

Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid and let $G = (V, E)$ be a weighted graph. Let V' be a maximum weight independent set in $\rho\mathcal{M}$, with respect to the weighted degrees $\deg_w(v)$. Then V' contains a $1 - 1/\rho$ approximate solution of the matroid-constrained maximum vertex cover problem.

For uniform matroids, we get exactly the same kernel as in [Manurangsi, 2018].

Kernelization Framework

Definition

The *density* of a subset $U \subseteq V$ in the matroid $\mathcal{M}$ is defined as

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Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid, and ρ be a positive integer. A subset $V' \subseteq V$ is called a *ρ-DBS* in $\mathcal{M}$ if $\rho_{\mathcal{M}}(V') = \rho$ and for all $U \subseteq V'$, $\rho_{\mathcal{M}}(U) \leq \rho$.

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These Density-Balanced Subset have nice sampling and contraction properties that allow to adapt the previous sampling technique. These subsets appear naturally when building matroid unions.

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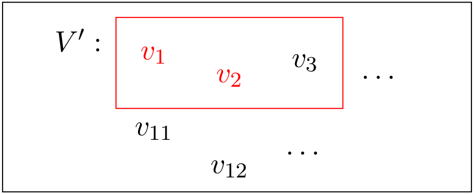

Figure: The subset spanned in V' by $\{v_1, v_2\}$ is of size bounded by $2 \cdot \rho$, meaning that there are at least $(r_{\mathcal{M}}(V') - 2) \cdot \rho$ elements still usable for the sampling.

Conclusion

Results:

- **natural generalization** of the kernelization process used in the k-vertex-cover problem
- closes the gap between the **cardinality constraint** and the more general **matroid constraint**

Theorem

Let V' be a maximum weight independent set in $\rho\mathcal{M}$, with respect to the weighted degrees. Then V' contains a $1 - 1/\rho$ approximate solution of the matroid-constrained maximum vertex cover problem.