Cartesian Coherent Differential Categories

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IRIF

June 15, 2023

Plan

The category **Vect** is not a CCC

Category Vect

Objects: $\mathbb{R}$ vector spaces
Morphisms: $\mathbf{Vect}(E, F) := \{\text{linear maps } E \to F\}$

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&\simeq \mathbf{Vect}(E \otimes F, G)
\end{aligned}
$$

Closure with regard to a tensor product $\otimes$

Refresher: linear logic

Logic of resources [**Girard_1987**]

- $A \multimap B$: Consume exactly one resource A to produce one B (Linearity)
- $A \otimes B$: A and B at the same time (Bilinearity)
- $A \,\&\, B$: Can choose between A or B (but not both) (Projections)
- $!A$: resource A duplicable and erasable.

Recovers the usual logic $A \Rightarrow B := !A \multimap B$.

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Models of linear logic

- Symmetric monoidal category $(\mathcal{L}, \otimes)$ closed with regard to $\otimes$
- Cartesian product &.
- Comonad ! such that $!(A \,\&\, B) \simeq !A \otimes !B$
- Kleisli category: $\mathcal{L}_!$: $\mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$ is a CCC

Finiteness spaces

Model of linear logic **Fin** [Ehrhard and Regnier 2003]:

- Objects: topological vector spaces
- Morphisms: continuous linear maps
- $\mathbf{Fin}_!(X, Y) := \mathbf{Fin}(!X, Y)$: CCC of analytic functions

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Analytic function

For example, if $f : \mathbb{R} \to \mathbb{R}$,

$$f(x+h) = \sum_{n=0}^{\infty} a_n(x) h^n$$

If $f : E \to F$

$$f(x+h) = \sum_{n=0}^{\infty} f_n(x)(\underbrace{h, \ldots, h}_{\text{n times}})$$

Taylor development

If $f : \mathbb{R} \to \mathbb{R}$ is analytic then $f' : \mathbb{R} \to \mathbb{R}$ is analytic, so $f^{(n)} : \mathbb{R} \to \mathbb{R}$ is analytic and

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If $f : E \to F$ is analytic then the derivative is analytic

$$f' : E \quad \to \quad (E \multimap F)$$

$$f(x+h) \simeq f(x) + f'(x) \cdot h + o(h)$$

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So $f^{(n)} : E \to (\underbrace{E \multimap \cdots \multimap E}_{\text{n times}} \multimap F)$ is analytic and

$$f(x+h) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) \cdot (\underbrace{h, \ldots, h}_{\text{n times}})$$

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Syntax side

Derivative of a Λ-term

$$\frac{\partial M}{\partial x} \cdot N$$

Substitute exactly one occurrence of x by N in M.

Taylor Development

$$(\lambda x.M)N \to \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n M}{\partial x^n} \cdot (N, \ldots, N) \right) [0/x]$$

Resource calculus: performs Taylor expansion in everything at the same time.

Plan

Differential Category [Blute, Cockett, and Seely 2006]

Differential : in a model $\mathcal{L}$ of LL

For $f \in \mathcal{L}(!X, Y)$

$$f' \in \mathcal{L}(!X \otimes X,\ Y) \simeq \mathcal{L}(!X, X \multimap Y)$$

$+$ axioms of differential calculus

Example : $(f \otimes g)' = f' \otimes g + f \otimes g'$

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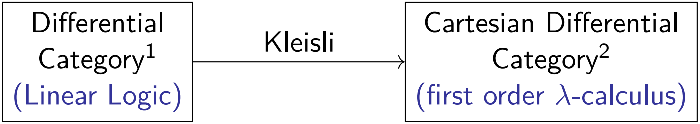
$\mathcal{L}$ must be additive

- $\mathcal{L}(X, Y)$ is a commutative monoïd
- $0 \circ g = 0$ and $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$ (left additive)
- $h \circ 0 = 0$ and $h \circ (f_1 + f_2) = h \circ f_1 + h \circ f_2$ (additive)

Cartesian Differential Categories

Recall: $\mathcal{L}_!(X, Y) := \mathcal{L}(!X, Y)$ is a CCC



[1] Blute, Cockett, and Seely 2006.
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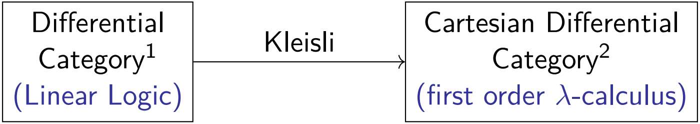

- Adding compatibility with closure: models of resource calculus Bucciarelli, Ehrhard, and Manzonetto 2010
- Adding axioms for Taylor property (in a qualitative setting) Manzonetto 2012

Example: relation model

[1] Blute, Cockett, and Seely 2006.
[2] Blute, Cockett, and Seely 2009.

Additive is too much

Non-deterministic: $\mathtt{true}, \mathtt{false} \in \mathcal{L}(1, 1 \oplus 1)$.
What is $\mathtt{true} + \mathtt{false}$?

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In models

Interesting models $\mathcal{L}$ of LL in which $\mathcal{L}_!$ is a category with analytic morphisms, with restricted addition

- Coherence spaces, non-uniform coherence spaces
- Probabilistic coherence spaces

In probabilistic coherence spaces

Probabilistic coherent spaces **Pcoh**: not additive

- Object N of natural numbers
- $\mu, \nu \in \mathbf{Pcoh}(1, N)$ are sub-probability distributions on $\mathbb{N}$
- $\mu + \nu$: measure on $\mathbb{N}$ of mass ≤ 2
- $\mu + \nu \in \mathbf{Pcoh}(1, N)$ if their total mass is ≤ 1

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Yet: ! in **Pcoh** is deeply analytic.

Example: generating function. A (sub)probability distribution $\mathbb{P}$ on $\mathbb{N}$ is characterized by

$$t \in [0, 1] \mapsto \sum_{n=0}^{\infty} \mathbb{P}\{n\} t^n$$

Coherent differentiation

Coherent differentiation (models)

Introduced in Ehrhard 2023, in models of LL.
Differentiation is a distributive law between ! and a functor S that encodes summability.

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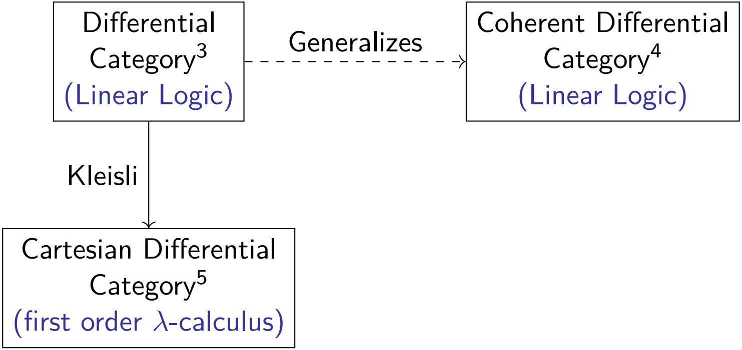
Coherent differentiation (syntax)

Coherent differential PCF Ehrhard 2022

- PCF with differential operator
- Coherent differentiation induces a model + subject reduction
- The relational model is adequate
- Reduction is deterministic (proof using adequacy and **Pcoh**)

Plan

Diagram of generality



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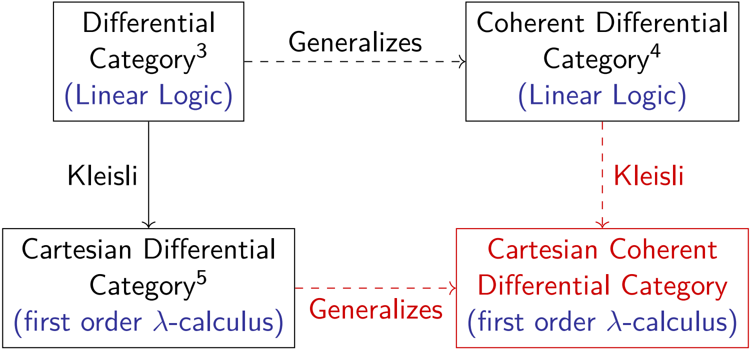

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Left summability structure (candidate)

Action on objects $\widetilde{\mathrm{D}}X$: "object of summable elements"

- $\pi_0, \pi_1 \in \mathcal{C}(\widetilde{\mathrm{D}}X, X)$ jointly monic
- $\sigma \in \mathcal{C}(\widetilde{\mathrm{D}}X, X)$ sum
- A morphism $0 \in \mathcal{C}(X, Y)$

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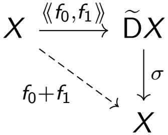
$f_0, f_1 \in \mathcal{C}(X, Y)$ are summable if there is $\langle\!\langle f_0, f_1 \rangle\!\rangle \in \mathcal{C}(X, \widetilde{\mathrm{D}}Y)$ such that $\pi_i \circ \langle\!\langle f_0, f_1 \rangle\!\rangle = f_i$. Define



Note: π_0, π_1, σ are not natural

Additivity

Compatibility with composition

If g_0 and g_1 are summable.

- $0 \circ f = 0$ and $(g_0 + g_1) \circ f = g_0 \circ f + g_1 \circ f$ (left additive)
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Additivity

h additive: $h \circ 0 = 0$ and $h \circ (f_0 + f_1) = h \circ f_0 + h \circ f_1$

Left summability structure:

- π_0, π_1, σ are additive
- Axioms that endows $\mathcal{C}(X, Y)$ with the structure of a partially additive monoid, see Arbib and Manes 1980

Differentiation

An operator for differentiation

Given $f \in \mathcal{C}(X, Y)$, there is

$$\begin{array}{rccc} \widetilde{\mathrm{D}}f : & \widetilde{\mathrm{D}}X & \to & \widetilde{\mathrm{D}}Y \\ & \langle\!\langle x, u \rangle\!\rangle & \mapsto & \langle\!\langle f(x), f'(x).u \rangle\!\rangle \end{array}$$

Aka: $\pi_0 \circ \widetilde{\mathrm{D}}f = f \circ \pi_0$.

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Axioms of differentiation: very structural properties

- π_0, π_1 are linear (h linear if h additive and $\pi_1 \circ \widetilde{\mathrm{D}} h = h \circ \pi_1$)
- σ, 0 are linear ($0' = 0$ and $(f+g)' = f' + g'$)
- $\widetilde{\mathrm{D}}$ is a functor (Chain rule)

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- $\widetilde{\mathrm{D}}$ is a functor (Chain rule)
- $\widetilde{\mathrm{D}}$ is a monad with unit ι_0 and sum θ (The differential is additive = Leibniz)
- c and l are natural (Schwarz + the differential is linear)

What are ι_0, θ, c and l ?

$$\iota_0 \circ x = \langle\!\langle x, 0 \rangle\!\rangle$$

$$\theta \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle = \langle\!\langle x, u + v \rangle\!\rangle$$

$$\mathsf{c} \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle = \langle\!\langle \langle\!\langle x, v \rangle\!\rangle, \langle\!\langle u, w \rangle\!\rangle \rangle\!\rangle$$

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One can see $\widetilde{\mathrm{D}}^2 f \in \mathcal{C}(\widetilde{\mathrm{D}}^2 X, \widetilde{\mathrm{D}}^2 Y)$ as

$$\widetilde{\mathrm{D}}^2 f \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle$$
$$=$$
$$\langle\!\langle \langle\!\langle f(x) \ , \ f'(x).u \rangle\!\rangle \ , \ \langle\!\langle f'(x) \cdot v \ , \ f^{(2)}(x) \cdot (u, v) + f'(x) \cdot w \rangle\!\rangle \rangle\!\rangle$$

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Formal version of the reasoning

The axioms rewrite as equational properties on the differential. Those equations on $\widetilde{\mathsf{D}} X = X \,\&\, X$ give Cartesian Differential Categories.

Plan

Compatibility with Cartesian Product

Compatibility with the product

- Compatibility with sum: "the sum on pairs is the coordinate wise sum"
- Compatibility with differential: "the projections are linear"

$$\mathsf{c}_{\&} : \widetilde{\mathsf{D}}(X \mathbin{\&} Y) \simeq \widetilde{\mathsf{D}}X \mathbin{\&} \widetilde{\mathsf{D}}Y$$

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$$\mathsf{c}_{\&} : \widetilde{\mathrm{D}}(X \mathbin{\&} Y) \simeq \widetilde{\mathrm{D}}X \mathbin{\&} \widetilde{\mathrm{D}}Y$$

Induce a strength

$$\Phi^0 = \widetilde{\mathrm{D}}X_0 \mathbin{\&} X_1 \xrightarrow{\mathrm{id}\&\iota_0} \widetilde{\mathrm{D}}X_0 \mathbin{\&} \widetilde{\mathrm{D}}X_1 \xrightarrow{\mathsf{c}_{\&}^{-1}} \widetilde{\mathrm{D}}(X_0 \mathbin{\&} X_1)$$

Partial derivative of $f \in \mathcal{C}(X_0 \mathbin{\&} X_1, Y)$:

$$\widetilde{\mathrm{D}}_0 f = \widetilde{\mathrm{D}}f \circ \Phi^0 \in \mathcal{C}(\widetilde{\mathrm{D}}X_0 \mathbin{\&} X_1, \widetilde{\mathrm{D}}Y)$$

$$\langle \langle\!\langle x, u \rangle\!\rangle, y \rangle \mapsto (f\langle x, y \rangle, f'\langle x, y \rangle \cdot \langle\!\langle u, 0 \rangle\!\rangle)$$

Leibniz and Schwarz

Leibniz

In analysis :

$$f'(x, y) \cdot (u, v) = \partial_0 f(x, y) \cdot u + \partial_1 f(x, y) \cdot v$$

In Cartesian Coherent Differential Categories

$$\widetilde{\mathrm{D}} f \circ \mathrm{c}_{\&}^{-1} = \theta \circ \widetilde{\mathrm{D}}_0 \widetilde{\mathrm{D}}_1 f = \theta \circ \widetilde{\mathrm{D}}_1 \widetilde{\mathrm{D}}_0 f$$

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Schwarz

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$$\partial_0 \partial_1 f = \partial_1 \partial_0 f$$

In Cartesian Coherent Differential Categories

$$\widetilde{\mathsf{D}}_0 \widetilde{\mathsf{D}}_1 f = \mathsf{c} \circ \widetilde{\mathsf{D}}_1 \widetilde{\mathsf{D}}_0 f$$

Multilinear map

$n+1$-additive map

$\varphi \in \mathcal{C}(X_0 \mathbin{\&} \cdots \mathbin{\&} X_n, Y)$ such that for any i,

$$
\begin{aligned}
\varphi \circ (f_0 \mathbin{\&} \cdots \mathbin{\&} 0 \mathbin{\&} \cdots \mathbin{\&} f_n) &= 0 \\
\varphi \circ (f_0 \mathbin{\&} \cdots \mathbin{\&} h_0 + h_1 \mathbin{\&} \cdots \mathbin{\&} f_n) &= \varphi \circ (f_0 \mathbin{\&} \cdots \mathbin{\&} h_0 \mathbin{\&} \cdots \mathbin{\&} f_n) + \\
&\quad\ \varphi \circ (f_0 \mathbin{\&} \cdots \mathbin{\&} h_1 \mathbin{\&} \cdots \mathbin{\&} f_n)
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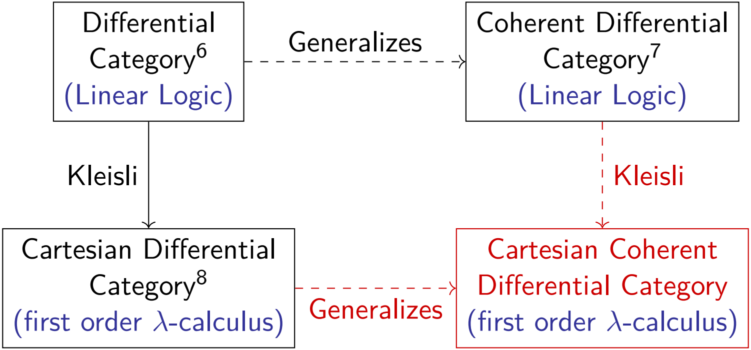
$n+1$-linear map

$\varphi \in \mathcal{C}(X_0 \,\&\, \cdots \,\&\, X_n, Y)$ that is $n+1$-additive and such that

$$\pi_1 \circ \widetilde{\mathrm{D}}_i \varphi = \varphi \circ (\mathrm{id} \,\&\, \cdots \,\&\, \pi_1 \,\&\, \cdots \,\&\, \mathrm{id})$$

Important notion for PCF semantics: `succ`, `pred` are linear, `if`, `let` are bilinear.

What we did so far



We introduced a first order coherent differential calculus, defined a semantic, and proved subject reduction.

[6] Blute, Cockett, and Seely 2006.

[7] Ehrhard 2023.

[8] Blute, Cockett, and Seely 2009.

Plan

Short term

Cartesian Closed Coherent Differential Categories

- Compatibility of differential with regard to closure
- Fixpoints

It should be a model of the coherent differential PCF.

Compatibility with closure

Compatibility with the closure:

- Compatibility with sum: "the sum of two functions is the point wise sum"
- Compatibility with the differential: "the evaluation is linear in its fuctionnal coordinate"

$$\widetilde{\mathrm{D}}(X \Rightarrow Y) \simeq X \Rightarrow \widetilde{\mathrm{D}}Y$$

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Internal derivative

$$\widetilde{\mathsf{D}}^{\mathsf{int}} \in \mathcal{C}(X \Rightarrow Y, \widetilde{\mathsf{D}}X \Rightarrow \widetilde{\mathsf{D}}Y)$$

$$X \Rightarrow Y \,\&\, \widetilde{\mathsf{D}}X \xrightarrow{\Phi^1} \widetilde{\mathsf{D}}(X \Rightarrow Y \,\&\, X) \xrightarrow{\widetilde{\mathsf{D}}\mathbf{ev}} \widetilde{\mathsf{D}}Y$$

Interpreting coherent differential PCF

Adding fixpoint:

- Define $f \leq g$ if there is h such that $f + h = g$
- This is an order (subtleties with antisymmetry). Assume that it is an ω-cpo
- This should give back the partially additive monoids of Arbib and Manes 1980
- Ask the differential to be continuous.

The differential commutes with the fixpoint.

Long term: model of resource calculus

$$(\lambda x.M)N \to \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n M}{\partial x^n} \cdot (N, \ldots, N) \right) [0/x]$$

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In coherence spaces:

- Terms of type `nat` are interpreted by singletons or empty sets.
- Two sets are summable if and only if one is empty
- So only one of the approximant has a non-empty semantic!

Ongoing work: Taylor functor

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