Coherent Taylor expansion as a bimonad

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Making differentiation functorial
In cartesian differential categories, differentiation is axiomatized as an operator
\[ D : X \times Y \to Y \times Y \]
\[ D(f, a) = f'(x) \cdot a \]
The chain rule of differentiation corresponds to an equation on \( D \).
\[ D(g \circ f) = Dg \circ (f \circ p_1, Df) \]
\[ (g \circ f)'(x) \cdot a = g'(f(x)) \cdot f'(x) \cdot a \]
One of the main ideas those last years: making this a functorial equation. Let
\[ f : X \to Y \]
\[ g : X \to Y \times Y \]
The chain rule becomes equivalent to the equation
\[ T(f(x)) = f(x) \cdot g(f(x)) \]
\[ T : (S, +, 0) \to (S, S', S') \]
Remarkable observation: other properties of differentiation become naturality equations. For example, let \( \theta : (X \times X) \times (X \times X) \to X \times X \)
\[ \theta(x, u, v) = (x, x + v) \]
Then: \( \theta \circ T^2 f = Tf \circ \theta \iff f'(x) \cdot (u + v) = f'(x) \cdot u + f'(x) \cdot v \]

Coherent differentiation

Issue with addition
In (cartesian) differential categories: unrestricted sum operation \( \sigma : X \times X \to X \).
- Math pov: objects are vector spaces/modules
- Computer science pov: morphisms are non deterministic computation

Main idea: take \( TX \neq X \) and restrain sum.
- If \( TX = X \times X \): cartesian differential categories
- If \( TX \) is the tangent bundle of \( X \): tangent categories
- If \( TX \) is a left summability structure: coherent differentiation

Coherent differentiation extends differentiation to deterministic models of \( LL \): coherence spaces, probabilistic coherence spaces, etc.

A partial notion of summation
Let \( L \) be a model of linear logic with \( 0 \)-morphisms.

Summability structure in models of \( LL \)
Let \( S : \mathbb{L} \to \mathbb{L} \) be a functor. Intuitively, \( SX = \{ (x_i)_{i=0}^n \mid x_i \text{ is defined} \} \)
Assume that there exists the following natural transformations:
- \( p_0, p_1 \in L(SX, X) \) jointly monic
- \( \pi : \{ x_{0}, x_i \} \to x_i \)
- \( \sum \sigma \in L(SX, X) \)
\[ \sigma : \{ x_{0}, x_i \} \to x_0 + x_i \]
\( \delta, \eta \in \mathcal{L}(X, Y) \) are summable if :
\[ \exists \delta, \eta \in L(X, SY) \text{ s.t. } \pi_0 \circ \delta, \eta = \delta, \eta : X \to \{ \delta(x), \eta(x) \} \]
+ some axioms on \( S \): the hom-sets are finite partially additive monoids

Additivity and left additivity
- Left additivity: \( (g + g) \circ f = g \circ f + g \circ f \)
- Additivity: \( g \circ (\delta + \eta) = g \circ \delta + g \circ \eta \)
This summability structure on \( L \) induces a left summability structure on \( L \), (similar, but there is no additivity).

Differentiation in \( L \)
- Define \( t_0, \theta, c \) and \( l \) natural transformations in \( L \).
\[ t_0 \in L(X, SX) \]
\[ t_0(x) = \{ x, 0 \} \]
\[ \theta \in L(SX, SX) \]
\[ \theta(x, u, v) = (x, u + v) \]
\[ c \in L(S(SX), S') \]
\[ c(x, u) = \{ x, u \} \]
\[ l \in L(SX, S') \]
\[ l(x) = \{ x, 0 \} \]
Then \( (S, +, 0) \) is a monad, \( (S, \cdot, 1) \) is a comonad, and \( (S, +, 0, \cdot, 1, c) \) is a bimonad.

Differentiation: functor \( T : L \to L \) that extends \( S \) to \( L \) (chain rule).
\[ T : \text{ISX} \to SY \]
\[ (x, u) \mapsto f'(x) \cdot u \]

Other properties of differentiation: naturality of the families above in \( L \).

Important observation
\[ (T, \text{Der} (\cdot), \text{Der} (\cdot)) \text{ is a monad, but } (T, \text{Der} (\cdot), \text{Der} (\cdot)) \text{ is not a comonad because } \text{Der} (\cdot) \text{ is not natural.} \]

Coherent Taylor expansion

The functor \( T \) performs a first order Taylor expansion. What if it could perform the whole Taylor expansion? Remarkably, this can be done with a very similar theory.

\( \omega \)-summability structures
We introduce an infinitary counterpart of summability structures.
\[ SX = \{ (x_i)_{i=0}^\infty \mid x_i \text{ is defined} \} \]
+ some axioms on \( S \): the hom-sets are partially additive monoids

It has a similar bimonad structure as before, and Taylor expansion is again a functor \( T : \mathbb{L} \to \mathbb{L} \) that extends \( S \). Intuitively, if \( f : \mathbb{R} \to \mathbb{R} \) is analytic we can write
\[ f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i \]
where \( f^i \) can be computed by the Faà Di Bruno formula. Then \( Tf \) can be seen as
\[ Tf(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i \]
The bimonad structure of \( T \)
We ask the same axioms as coherent differentiation, with an additional one that enforces the analyticity of maps: \( \text{Der} (\cdot) \) is natural in \( L \). \( T \) is now a bimonad!

Only one axiom is not about the bimonad \( T \): the compatibility between \( \partial \) and \( m^2 \).

A syntax for coherent Taylor expansion

Using coherent differentiation, Ehrhard introduced a deterministic PCF with both fixpoints and differentiation, and a straightforward probabilistic extension.

This axiomatization of Taylor expansion suggests that this calculus can feature Taylor expansion.

This calculus should be related to the computation of higher order derivatives in automated differentiation.