An introduction to Coherent differentiation

Aymeric Walch

IRIF

January 12, 2023

Introduction

The third leg of Curry Howard

Syntaxe (Programs)	Semantic (Category)
Types	Objets
Programs $A \Rightarrow B$	Morphisms $\mathcal{L}(A, B)$

A philosophy of semantic:

- Scout for model with interesting properties not captured by syntax
- Refine those properties into syntax
- Rinse and repeat

Linear Logic

Semantic: Morphisms are linear
Syntax : Ressource consumption

Linear Logic

Semantic: Morphisms are linear
Syntax : Ressource consumption

Differential logic/calculus

Semantic: Morphisms are analytic
Syntax: Ressource sensitive substitution (differentiation)

Linear Logic

Semantic: Morphisms are linear
Syntax : Ressource consumption

Differential logic/calculus

Semantic: Morphisms are analytic
Syntax: Ressource sensitive substitution (differentiation)

Coherent differentiation

Semantic: Sum is only partially defined
Syntax: ???

Plan

Plan

The category **Vect** is not a CCC

Category **Vect**

Objects: $\mathbb{R}$ vectorial spaces
Morphisms: $\mathbf{Vect}(E, F) = \{\text{linear maps } E \to F\}$

The category **Vect** is not a CCC

Category Vect

Objects: $\mathbb{R}$ vectorial spaces
Morphisms: $\mathbf{Vect}(E, F) = \{\text{linear maps } E \to F\}$

- **Vect** is Cartesian : $E \,\&\, F := E \times F$
- $\mathbf{Vect}(F, G)$ is a $\mathbb{R}$ vectorial space

The category **Vect** is not a CCC

Category **Vect**

Objects: $\mathbb{R}$ vectorial spaces
Morphisms: $\mathbf{Vect}(E, F) = \{\text{linear maps } E \to F\}$

- **Vect** is Cartesian : $E \,\&\, F := E \times F$
- $\mathbf{Vect}(F, G)$ is a $\mathbb{R}$ vectorial space
- But $\mathbf{Vect}(E, \mathbf{Vect}(F, G)) \not\simeq \mathbf{Vect}(E \times F, G)$

$$\begin{aligned}\mathbf{Vect}(E, \mathbf{Vect}(F, G)) &\simeq \{\text{bilinear maps } E \times F \to G\} \\ &\simeq \mathbf{Vect}(E \otimes F, G)\end{aligned}$$

Closure with regard to a tensor product $\otimes$

Linear Logic

Logic of ressources [Girard 1987]

- $A \multimap B$: Consume exactly one ressource A to produce one B (Linearity)
- $A \otimes B$: A and B at the same time (Bilinearity)
- $A \& B$: Can chose between A ou B (but not both) (Projections)
- $!A$: ressource A duplicable and erasable.

Recovers the usual logic $A \Rightarrow B := !A \multimap B$.

Linear Logic

Logic of ressources [Girard 1987]

- $A \multimap B$: Consume exactly one ressource A to produce one B (Linearity)
- $A \otimes B$: A and B at the same time (Bilinearity)
- $A \,\&\, B$: Can chose between A ou B (but not both) (Projections)
- $!A$: ressource A duplicable and erasable.

Recovers the usual logic $A \Rightarrow B :=!A \multimap B$.

Models of linear logic

Symetric monoidal category $(\mathcal{L}, \otimes)$ closed with regard to $\otimes$
Cartesian product $\&$.
Comonad ! such that $!(A \,\&\, B) \simeq !A \otimes !B$

Kleisli Category of the exponential

Comonad

der $\in \mathcal{L}(!X, X)$, dig $\in \mathcal{L}(!X, !!X)$ Allows to define:

- Dereliction : for any $f \in \mathcal{L}(X, Y)$, $f \circ \mathrm{der} \in \mathcal{L}(!X, Y)$
- Promotion : for any $f \in \mathcal{L}(!X, Y)$, $f^! := !f \circ \mathrm{dig} \in \mathcal{L}(!X, !Y)$

Kleisli category $\mathcal{L}_!$

- Objects: the objects of $\mathcal{L}$
- Morphisms: $\mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$
- Identity from dereliction: der $\in \mathcal{L}_!(X, X)$
- Composition from promotion: if $f \in \mathcal{L}_!(X, Y)$, $g \in \mathcal{L}_!(Y, Z)$ $g \circ_! f = g \circ f^!$
- Dereliction: induces a functor $\mathcal{K}_! : \mathcal{L} \to \mathcal{L}_!$

Finitness spaces

Model of linear logic **Fin** [Ehrhard 2005]:

- Objects: topological vector spaces
- Morphisms: continuous linear maps

! on **Fin**

Fin$(!X, Y)$: analytic functions

$$f(x+h) = \sum_{n=0}^{\infty} f_n(h, \ldots, h)$$

With f_n n-linear symetric. For example, if $f : \mathbb{R} \to \mathbb{R}$,

$$f(x+h) = \sum_{n=0}^{\infty} a_n h^n$$

Plan

Differential in **Fin**

Differential in analysis

$f : X \to Y$ differentiable in $x \in X$ if

$$f(x+u) \approx f(x) + f'(x).u$$

$f'(x) : X \to Y$ is <u>linear</u>

Differential in **Fin**

Differential in analysis

$f : X \to Y$ differentiable in $x \in X$ if

$$f(x+u) \approx f(x) + f'(x).u$$

$f'(x) : X \to Y$ is <u>linear</u>

Differential in **Fin**

Analytic $\Rightarrow$ Diffferentiable. For $f \in \mathbf{Fin}(!X, Y)$

$$f' \in \mathbf{Fin}(!X \otimes X, Y) \quad (1)$$
$$(x\ ,\ u) \mapsto f'(x).u \quad (2)$$

Iterated derivative, analiticity

Iterated derivative

$f' \in \mathbf{Fin}(!X \otimes X, Y) \simeq \mathbf{Fin}(!X, \mathbf{Fin}(X, E))$ can be differentiated.

$$f^{(2)}(x) \in \mathbf{Fin}(X, \mathbf{Fin}(X, E)) \simeq \mathbf{Fin}(X \otimes X, E)$$

$f^{(2)}(x) \in \mathbf{Fin}(X \otimes X, Y)$: bilinear morphism
$f^{(n)}(x) \in \mathbf{Fin}(X \otimes \ldots \otimes X, Y)$: n-linear morphism

Link with analyticity

$f \in \mathbf{Fin}(!X, Y)$ is analytic

$$f(x+h) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) \cdot (h, \ldots, h)$$

Syntax side

Derivative of a Λ-term

$$\frac{\partial M}{\partial x} \cdot N$$

Substitute exactly one occurence of x by N in M.

Taylor Development

$$(\lambda x.M)N \to \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n M}{\partial x^n} \cdot (N, \ldots, N) \right) [0/x]$$

Differential Category [BLUTE, Cockett, and SEELY 2006]

Differential : in a model $\mathcal{L}$ of LL

If $f \in \mathcal{L}(!X, Y)$:

$$f' \in \mathcal{L}(!X \otimes X, Y)$$

$+$ axioms of differential calculus

Example : $(f \otimes g)' = f' \otimes g + f \otimes g'$

Differential Category [BLUTE, Cockett, and SEELY 2006]

Differential : in a model $\mathcal{L}$ of LL

If $f \in \mathcal{L}(!X, Y)$:

$$f' \in \mathcal{L}(!X \otimes X, Y)$$

$+$ axioms of differential calculus

Example : $(f \otimes g)' = f' \otimes g + f \otimes g'$

$\mathcal{L}$ must be additive

$\mathcal{L}$: enriched over commutative monoid

- $\mathcal{L}(X, Y)$ is a monoïd
- $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$
- $h \circ (f_1 + f_2) = h \circ f_1 + h \circ f_2$

Additive is too much

Non deterministic: $\mathtt{true}, \mathtt{false} \in \mathcal{L}(1, 1 \oplus 1)$.
What is $\mathtt{true} + \mathtt{false}$?

In syntax, sum is not that wild

- In the earlier Taylor development: only one branch of $\sum_{n=0}^{\infty} \cdots$ is non zero.
- Probabilistic Taylors expansion: the coefficients of the sum has total mass ≤ 1, the sum is a probabilistic branching.

In semantic

Probabilistic coherent spaces **Pcoh**: not additive

- $X := (|X|, \mathcal{P}(|X|))$ discrete space
- $\mu, \nu \in \mathbf{Pcoh}(1, X)$ are sub-probability distributions on $|X|$
- $\mu + \nu$: measure on $|X|$ of mass ≤ 2
- $\mu + \nu \in \mathbf{Pcoh}(1, X)$ if their total mass is ≤ 1

In semantic

Probabilistic coherent spaces **Pcoh**: not additive

- $X := (|X|, \mathcal{P}(|X|))$ discrete space
- $\mu, \nu \in \mathbf{Pcoh}(1, X)$ are sub-probability distributions on $|X|$
- $\mu + \nu$: measure on $|X|$ of mass ≤ 2
- $\mu + \nu \in \mathbf{Pcoh}(1, X)$ if their total mass is ≤ 1

Yet: ! in **Pcoh** is deeply analytic.

Ex: Infinite head and tail. Coin C of parameter p.

$$\sum_{n=1}^{\infty} p^{n-1}(1-p)C^n$$

Plan

Partial commutative monoid

$(\mathcal{L}(X, Y), +, 0)$ is a partial commutative monoïd. Write $f \boxplus g$ if $f + g$ is defined

Neutral element 0

$f \boxplus 0$, $0 \boxplus f$ and $f + 0 = 0 + f = f$

Commutativity

If $f \boxplus g$ then $g \boxplus f$ and $g + f = f + g$

Partial commutative monoid

Associativity

If $f \boxplus g$ and $(f+g) \boxplus h$, then $g \boxplus h$, $f \boxplus (g+h)$ and

$$(f+g)+h = f+(g+h)$$

Partial commutative monoid

Associativity

If $f \boxplus g$ and $(f+g) \boxplus h$, then $g \boxplus h$, $f \boxplus (g+h)$ and

$$(f+g)+h = f+(g+h)$$

Strong associativity: positivity of coefficients.

Ex: In $[-1,1]$, $x \boxplus y$ if $|x|+|y| \leq 1$

- $(-0.5+0.5)+1$ defined
- $-0.5+(0.5+1)$ not defined

$\rightarrow +$ is not associative on $[-1,1]$

$\rightarrow +$ is associative on $[0,1]$

Compatibility with addition

$\mathcal{L}$ is enriched over partial commutative monoids.

Compatibility with composition

If $f_0 \boxplus f_1$ and $g_0 \boxplus g_1$

- Left: $(g_0 + g_1) \circ f = g_0 \circ f + g_1 \circ f$ and $0 \circ f = 0$
- Right: $g \circ (f_0 + f_1) = g \circ f_0 + g \circ f_1$ and $g \circ 0 = 0$

Compatibility with addition

$\mathcal{L}$ is enriched over partial commutative monoids.

Compatibility with composition

If $f_0 \boxplus f_1$ and $g_0 \boxplus g_1$

- Left: $(g_0 + g_1) \circ f = g_0 \circ f + g_1 \circ f$ and $0 \circ f = 0$
- Right: $g \circ (f_0 + f_1) = g \circ f_0 + g \circ f_1$ and $g \circ 0 = 0$

In practice, the partiality is given by the structure of the category itself through a functor

$$\mathsf{S} = (1 \,\&\, 1) \multimap \_$$

Summability structure [Ehrhard 2021]

Summability structure

- Functor S: $\mathsf{S}X$ “set of pairs of summable elements of X”
- Projections : $\pi_0, \pi_1 : \mathsf{S}X \to X$ (natural) jointly monic: $(\forall i \in \{0,1\}, \pi_i \circ f = \pi_i \circ g) \Rightarrow f = g$
- Sum : $\sigma : \mathsf{S}X \to X$ (natural)

Summability structure [Ehrhard 2021]

Summability structure

- Functor S: SX "set of pairs of summable elements of X"
- Projections : $\pi_0, \pi_1 : SX \to X$ (natural) jointly monic: $(\forall i \in \{0,1\}, \pi_i \circ f = \pi_i \circ g) \Rightarrow f = g$
- Sum : $\sigma : SX \to X$ (natural)

Induced Sum

- $f_0 \boxplus f_1$ if $\exists \langle f_0, f_1 \rangle_S : X \to SY$

$$\text{s.t. } \begin{cases} \pi_0 \circ \langle f_0, f_1 \rangle_S = f_0 \\ \pi_1 \circ \langle f_0, f_1 \rangle_S = f_1 \end{cases}$$

- $f_1 + f_2 = \sigma \circ (f_1, f_2)_S$
- 3 axioms (neutral element, commutativity, associativity)

Summability Monad

Monad S

Monad Unit $\iota_0 \in \mathcal{L}(X, \mathsf{S}X)$

$$\begin{aligned} \iota_0 = \langle \mathrm{id}, 0 \rangle_S : X &\to \mathsf{S}X \\ x &\mapsto (x, 0) \end{aligned}$$

Monad Sum $\tau \in \mathcal{L}(\mathsf{S}^2 X, \mathsf{S}X)$

$$\begin{aligned} \tau = \langle \pi_0\pi_0, \pi_0\pi_1 + \pi_1\pi_0 \rangle_S : \mathsf{S}^2 X &\to \mathsf{S}X \\ ((x, u), (v, \epsilon)) &\mapsto (x, u + v) \end{aligned}$$

Summability Monad

Monad S

Monad Unit $\iota_0 \in \mathcal{L}(X, \mathsf{S}X)$

$$\begin{aligned} \iota_0 = \langle \mathrm{id}, 0 \rangle_S : X &\to \mathsf{S}X \\ x &\mapsto (x, 0) \end{aligned}$$

Monad Sum $\tau \in \mathcal{L}(\mathsf{S}^2 X, \mathsf{S}X)$

$$\begin{aligned} \tau = \langle \pi_0\pi_0, \pi_0\pi_1 + \pi_1\pi_0 \rangle_S : \mathsf{S}^2 X &\to \mathsf{S}X \\ ((x, u), (v, \epsilon)) &\mapsto (x, u + v) \end{aligned}$$

Compatibility of $+$ with $\otimes$: The monad is strong:
$\mathsf{L} \in \mathcal{L}(\mathsf{S}X \otimes \mathsf{S}Y, \mathsf{S}(X \otimes Y))$

$$(x, u) \otimes (y, v) \mapsto (x \otimes y, x \otimes v + u \otimes y)$$

Very similar to dual numbers (real with ϵ such that $\epsilon^2 = 0$)

Plan

Coherent differentiation

Coherent differentiation

For $f : !X \to Y$

$$f(x+u) \simeq f(x) + f'(x).u$$

$f(x)$ and $f'(x).u$ must be summable

$$\begin{array}{cccc} Df : & !SX & \to & SY \\ & (x,u) & \mapsto & (f(x), f'(x).u) \end{array}$$

Coherent differentiation

Coherent differentiation

For $f : !X \to Y$

$$f(x+u) \simeq f(x) + f'(x).u$$

$f(x)$ and $f'(x).u$ must be summable

$$\begin{array}{rccc} Df : & !SX & \to & SY \\ & (x,u) & \mapsto & (f(x), f'(x).u) \end{array}$$

Advantage: the chain rule becomes a functoriality property

$$(g \circ f)'(x) \cdot u = g'(f(x)) \cdot f'(x) \cdot u$$

$$\iff (\mathrm{D}g \circ \mathrm{D}f(x,u)) = \mathrm{D}(g \circ f)(x,u)$$

Axioms of differentiation

Coherent differentiation

$$Df \in \mathcal{L}(!\mathrm{S}X, \mathrm{S}Y) = \mathcal{L}_!(\mathrm{S}X, \mathrm{S}Y)$$

Axioms: Intuitively

- (D-struct) $\mathcal{K}_!(\pi_0) \circ_! \mathrm{D}f = f \circ_! \mathcal{K}_!(\pi_0)$
- (Chain rule) D is a functor (in $\mathcal{L}_!$)
- (Leibniz) $f'(x) \cdot 0 = 0$, $f'(x) \cdot (u + v) = f'(x) \cdot u + f'(x) \cdot v$
- (Linearity) $\frac{\partial f'(x) \cdot u}{\partial u} = f'(x)$
- (Swchartz) $f^{(2)}(x)$ is symetric
- (Bilinearity) If $\phi \in \mathcal{L}(X \otimes Y, Z)$, $\phi'(x, y) \cdot (u, v) = \phi(x, v) + \phi(u, y)$

In terms of structure extension to $\mathcal{L}_!$

Axioms: As I see them

- (D-struct) $\mathcal{K}_!(\pi_0) : \mathsf{D} \Rightarrow \mathsf{Id}$ natural
- (Chain rule) D is a functor that extends S to $\mathcal{L}_!$: $\mathsf{D}(\mathcal{K}_\mathsf{M}(h)) = \mathcal{K}_\mathsf{M}(\mathsf{S}h)$.
- (Leibniz) $\mathcal{K}_!(\iota_0) : \mathsf{Id} \Rightarrow \mathsf{D}$, $\mathcal{K}_!(\tau) : \mathsf{D}^2 \Rightarrow \mathsf{D}$ natural (D is a Monad that extends S)
- (Linearity) $\mathcal{K}_!(\mathbf{l}) : \mathsf{D} \Rightarrow \mathsf{D}^2$ natural
- (Swchartz) $\mathcal{K}_!(\mathbf{c}) : \mathsf{D}^2 \Rightarrow \mathsf{D}^2$ is natural

In term of distributive Law

Distributive law

Distributive law between a functor F and the comonad !

$$\lambda_F \in \mathcal{L}(!FX, F!X)$$

Subject to 2 compatibility conditions (with dig and der)

Distributive Law, fact 1

For $F : \mathcal{L} \to \mathcal{L}$, bijection between

- Extensions $\widehat{F} : \mathcal{L}_! \to \mathcal{L}_!$
- Distributive laws $\lambda_F \in \mathcal{L}(!FX, F!X)$

Functorial bijection.

Distributive law associated to:

- *Id*: $\mathrm{id} \in \mathcal{L}(!X, !X)$
- D: $\partial_X \in \mathcal{L}(!\mathsf{S}X, \mathsf{S}!X)$
- D^2: $\partial_{\mathsf{S}X} \circ \mathsf{S}\partial_X \in \mathcal{L}(!\mathsf{S}^2X, \mathsf{S}^2X)$

Distributive law associated to:

- *Id*: $\mathrm{id} \in \mathcal{L}(!X, !X)$
- D: $\partial_X \in \mathcal{L}(!\mathrm{S}X, \mathrm{S}!X)$
- D^2: $\partial_{\mathrm{S}X} \circ \mathrm{S}\partial_X \in \mathcal{L}(!\mathrm{S}^2 X, \mathrm{S}^2 X)$

Distributive Law, fact 2

If $\alpha_X \in \mathcal{L}(FX, GX)$ natural, $\widehat{F}, \widehat{G} : \mathcal{L}_! \to \mathcal{L}_!$, extensions, λ_F, λ_G associated distributive laws. $\mathcal{K}_!(\alpha) : \widehat{F} \Rightarrow \widehat{G}$ if and only if

$$\begin{array}{ccc} !FX & \xrightarrow{\lambda_F} & F!X \\ {\scriptstyle !\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ !GX & \xrightarrow[\lambda_G]{} & G!X \end{array}$$

α is a morphism between λ_F and λ_G.

Axioms as distributive law

Axioms: as introduced intially

- (D-struct) π_0 morphism between id and ∂
- (Chain rule) ∂ distributive law
- (Leibniz) ι_0 morphism between id and ∂.
  τ morphism between $\partial \circ \mathsf{S}\partial$ and ∂
- (Linearity) **l** morphism of distributive law
- (Swchartz) **c** morphism of distributive law

The Journey so far

- enriched over Partial commutative Monoid
- Partiality: a strong Monad S, close to dual numbers
- Differentiation: extension of the Monad S to a Monad D on $\mathcal{L}_!$ that preserves naturality of key operators
- Distributive Laws: the axioms correspond to diagrams on $\partial : !S \Rightarrow S!$
- Side note: $\mathcal{L}_S$ is a model of LL

Plan

Coherent differentiation generalizes differentiation

Give an additive category model of differential LL

Summability structure

$$SX = X \mathbin{\&} X, \pi_i = \mathbf{p}_i$$

$$\langle f_0, f_1 \rangle_S = \langle f_0, f_1 \rangle, \ \sigma = \mathbf{p}_0 + \mathbf{p}_1$$

We check that $\sigma \circ \langle f_0, f_1 \rangle_S = f_1 + f_2$

Differentiation

Differentiation

- Define $f^{\mathbf{coh}} : !(X \,\&\, X) \to Y$. Intuitively: $(x, u) \mapsto f'(x) \cdot u$

$$f^{\mathbf{coh}} : !(X \,\&\, X) \xrightarrow{\sim} !X \otimes !X \xrightarrow{id_{!X} \otimes \mathsf{der}} !X \otimes X \xrightarrow{f'} Y$$

Differentiation

Differentiation

- Define $f^{\mathbf{coh}} : !(X \,\&\, X) \to Y$. Intuitively: $(x, u) \mapsto f'(x) \cdot u$

$$f^{\mathbf{coh}} : !(X \,\&\, X) \xrightarrow{\sim} !X \otimes !X \xrightarrow{id_{!X} \otimes \mathsf{der}} !X \otimes X \xrightarrow{f'} Y$$

- $Df : !(X \,\&\, X) \to SY$. Intuitively: $(x, u) \mapsto (f(x), f^{\mathbf{coh}}(x, u))$

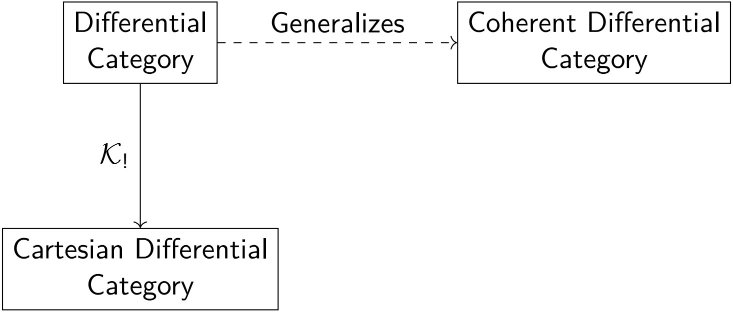
$$Df = \langle f \circ_! \mathbf{p}_0, f^{\mathbf{coh}} \rangle$$

One to one correspondance between axioms of differential categories and coherent differentiation.

Plan

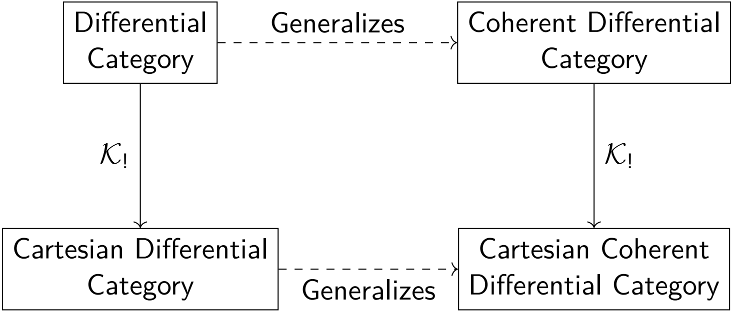
Diagram of generality

Diagram of generality



Left additive: $(f_0 + f_1) \circ g = f_0 \circ g + f_1 \circ g$

Diagram of generality



Left additive: $(f_0 + f_1) \circ g = f_0 \circ g + f_1 \circ g$

Compatibility with Cartesian Product

Compatibility with the product:

- Compatibility with sum: "the sum of two pairs is the coordinate wise sum.
- Compatibility with differential: "the projections are linear"

$$\mathrm{D}(X \mathbin{\&} Y) \simeq \mathrm{D}X \mathbin{\&} \mathrm{D}Y$$

Compatibility with Cartesian Product

Compatibility with the product:

- Compatibility with sum: "the sum of two pairs is the coordinate wise sum.
- Compatibility with differential: "the projections are linear"

$$\mathrm{D}(X \mathbin{\&} Y) \simeq \mathrm{D}X \mathbin{\&} \mathrm{D}Y$$

Strength and partial derivative

Induce a strength

$$\Phi^0 \in \mathcal{L}_!(\mathrm{D}X_0 \mathbin{\&} X_1, \mathrm{D}(X_0 \mathbin{\&} X_1))$$

Partial derivative of $f \in \mathcal{L}_!(X_0 \mathbin{\&} X_1, Y)$:

$$\mathrm{D}_0 f = \mathrm{D}f \circ \Phi^0 \in \mathcal{L}_!(\mathrm{D}X_0 \mathbin{\&} X_1, \mathrm{D}Y)$$

Comptatibility with the closure

Compatibility with the closure:

- Compatibility with sum: "the sum of two functions is the pointwise sum"
- Compatibility with the differential: "the evaluation is linear in its fuctionnal coordinate"

$$\mathrm{D}(X \Rightarrow Y) \simeq X \Rightarrow \mathrm{D}Y$$

Comptatibility with the closure

Compatibility with the closure:

- Compatibility with sum: "the sum of two functions is the pointwise sum"
- Compatibility with the differential: "the evaluation is linear in its fuctionnal coordinate"

$$\mathrm{D}(X \Rightarrow Y) \simeq X \Rightarrow \mathrm{D}Y$$

Internal derivative

For $f \in \mathcal{L}_{\mathrm{M}}(X, Y \Rightarrow Z)$

$$\mathrm{D}^{\mathrm{int}} f \in \mathcal{L}_{\mathrm{M}}(X, \mathrm{D}Y \Rightarrow \mathrm{D}Z)$$

Plan

Terms

Terms

$$M, N := (M)N \mid \lambda x.M \mid \mathrm{D}M \mid \mathrm{D}^d\pi_i(M) \mid \mathrm{D}^d\iota_i(M) \mid \mathrm{D}^d\Theta(M) \mid \mathrm{D}^d\mathbf{c}(M)$$

Types

$$A, B := \mathrm{D}^d\iota \mid A \times B \mid A \Rightarrow B$$

$\mathrm{D}(A \times B) := \mathrm{D}A \times \mathrm{D}B$
$\mathrm{D}(A \Rightarrow B) := A \Rightarrow \mathrm{D}B$

Terms

Terms

$$M, N := (M)N \mid \lambda x.M \mid \mathrm{D}M \mid \mathrm{D}^d\pi_i(M) \mid \mathrm{D}^d\iota_i(M) \mid \mathrm{D}^d\Theta(M) \mid \mathrm{D}^d\mathbf{c}(M)$$

Types

$$A, B := \mathrm{D}^d\iota \mid A \times B \mid A \Rightarrow B$$

$\mathrm{D}(A \times B) := \mathrm{D}A \times \mathrm{D}B$
$\mathrm{D}(A \Rightarrow B) := A \Rightarrow \mathrm{D}B$

$$\frac{\Gamma \vdash M : A \Rightarrow B}{\Gamma \vdash \mathrm{D}M : \mathrm{D}A \Rightarrow \mathrm{D}B} \quad \frac{\Gamma \vdash M : \mathrm{D}^{d+1}A}{\Gamma \vdash \mathrm{D}^d\pi_i(M) : \mathrm{D}^d A}$$

Reduction rules

Some rules

- $\mathrm{D}^d\pi_i(\mathrm{D}^{e+1}f(M)) \to \mathrm{D}^e f(\mathrm{D}^d\pi_i(M))$ (linearity of the constructors)
- $\mathrm{D}^d\pi_1(\mathrm{D}^d\Theta(M)) \to \mathrm{D}^d\pi_0(\mathrm{D}^d\pi_1(M)) + \mathrm{D}^d\pi_1(\mathrm{D}^d\pi_0(M))$

Last rule: reduction $\Lambda \to \mathcal{M}_{fin}(\Lambda)$. Lift the reduction to

$$\mathcal{M}_{fin}(\Lambda) \to \mathcal{M}_{fin}(\Lambda)$$

Reduction rules

Some rules

- $\mathrm{D}^d\pi_i(\mathrm{D}^{e+1}f(M)) \to \mathrm{D}^e f(\mathrm{D}^d\pi_i(M))$ (linearity of the constructors)
- $\mathrm{D}^d\pi_1(\mathrm{D}^d\Theta(M)) \to \mathrm{D}^d\pi_0(\mathrm{D}^d\pi_1(M)) + \mathrm{D}^d\pi_1(\mathrm{D}^d\pi_0(M))$

Last rule: reduction $\Lambda \to \mathcal{M}_{fin}(\Lambda)$. Lift the reduction to

$$\mathcal{M}_{fin}(\Lambda) \to \mathcal{M}_{fin}(\Lambda)$$

Probabilistic setting: multidistributions monad

Deterministic krivine machin using a memory structure on the projections

Differential

$$\mathrm{D}(\lambda x^{A}.M) \to \lambda x^{\mathrm{D}A}.\partial(x, M)$$

If $\Gamma, x : A \vdash M : B$ then

$$\Gamma, x : \mathrm{D}A \vdash \partial(x, M) : \mathrm{D}B$$

$$\partial(x, x) = x \quad \partial(x, y) = \iota_0(y)$$

$$\partial(x, (M)N) = \Theta((\mathrm{D}\partial(x, M))\partial(x, N))$$

$$\partial(x, \mathrm{D}N) = \mathbf{c}(\mathrm{D}\partial(x, N))$$

Can be extended to PCF:

- `succ` and `pred`: linear constructors
- `if` : $\iota \times (A \times A) \to A$ and `let` : $\iota \times A \to A$ bilinear constructors
- Admits fixpoint

And a lot more to come Krivine Machine, adequacy of **Rel**, deterministic Krivine machine, adequacy of **Pcoh** (?)

Thank you for your attention!