An introduction to Coherent differentiation

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Introduction

The third leg of Curry Howard

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A philosophy of semantic:

- Scout for model with interesting properties not captured by syntax
- Refine those properties into syntax
- Rinse and repeat
Linear Logic

Semantic: Morphisms are linear
Syntax: Ressource consumption
Linear Logic
Semantic: Morphisms are linear
Syntax: Resource consumption

Differential logic/calculus
Semantic: Morphisms are analytic
Syntax: Resource sensitive substitution (differentiation)
Linear Logic
Semantic: Morphisms are linear
Syntax: Ressourse consumption

Differential logic/calculus
Semantic: Morphisms are analytic
Syntax: Ressource sensitive substitution (differentiation)

Coherent differentiation
Semantic: Sum is only partially defined
Syntax: ???
Plan

1. Vector spaces as a model of Linear Logic
2. Introduction to differential calculus
3. Partial sum
4. Differentiation
5. Link with differential categories
6. Cartesian Coherent Differential Categories
7. A sketch of a syntax
Plan

1. Vector spaces as a model of Linear Logic
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The category $\textbf{Vect}$ is not a CCC

**Category $\textbf{Vect}$**

Objects: $\mathbb{R}$ vectorial spaces
Morphisms: $\text{Vect}(E, F) = \{\text{linear maps } E \rightarrow F\}$
The category **Vect** is not a CCC

**Category Vect**

- Objects: $\mathbb{R}$ vectorial spaces
- Morphisms: $\text{Vect}(E, F) = \{\text{linear maps } E \to F\}$

- **Vect** is Cartesian: $E \& F := E \times F$
- **Vect**($F, G$) is a $\mathbb{R}$ vectorial space
The category $\textbf{Vect}$ is not a CCC

**Category $\textbf{Vect}$**

Objects: $\mathbb{R}$ vectorial spaces

Morphisms: $\textbf{Vect}(E, F) = \{\text{linear maps } E \to F\}$

- $\textbf{Vect}$ is Cartesian: $E \& F := E \times F$
- $\textbf{Vect}(F, G)$ is a $\mathbb{R}$ vectorial space
- But $\textbf{Vect}(E, \textbf{Vect}(F, G)) \not\cong \textbf{Vect}(E \times F, G)$

$$\textbf{Vect}(E, \textbf{Vect}(F, G)) \cong \{\text{bilinear maps } E \times F \to G\}$$

$$\cong \textbf{Vect}(E \otimes F, G)$$

Closure with regard to a tensor product $\otimes$
Linear Logic

Logic of ressources [Girard 1987]

- $A \rightarrow B$: Consume exactly one ressource $A$ to produce one $B$ (Linearity)
- $A \otimes B$: $A$ and $B$ at the same time (Bilinearity)
- $A \& B$: Can chose between $A$ ou $B$ (but not both) (Projections)
- $!A$: ressource $A$ duplicable and erasable.

Recovers the usual logic $A \Rightarrow B := !A \rightarrow B$. 
Vector spaces as a model of Linear Logic

Linear Logic

Logic of ressources [Girard 1987]

- $A \multimap B$: Consume exactly one ressource $A$ to produce one $B$ (Linearity)
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Recovers the usual logic $A \Rightarrow B : = !A \multimap B$.

Models of linear logic

Symetric monoidal category $(\mathcal{L}, \otimes)$ closed with regard to $\otimes$
Cartesian product $\&$
Comonad $!$ such that $!(A \& B) \simeq!A \otimes!B$
Kleisli Category of the exponential

Comonad

\( \text{der} \in \mathcal{L}(!X, X), \text{dig} \in \mathcal{L}(!X, !!X) \) Allows to define:

- **Dereliction**: for any \( f \in \mathcal{L}(X, Y) \), \( f \circ \text{der} \in \mathcal{L}(!X, Y) \)
- **Promotion**: for any \( f \in \mathcal{L}(!X, Y) \), \( f^! := !f \circ \text{dig} \in \mathcal{L}(!X, !Y) \)

Kleisli category \( \mathcal{L}_! \)

- **Objects**: the objects of \( \mathcal{L} \)
- **Morphisms**: \( \mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y) \)
- **Identity from dereliction**: \( \text{der} \in \mathcal{L}_!(X, X) \)
- **Composition from promotion**: if \( f \in \mathcal{L}_!(X, Y) \), \( g \in \mathcal{L}_!(Y, Z) \)
  \[ g \circ_! f = g \circ f^! \]
- **Dereliction**: induces a functor \( \mathcal{K}_! : \mathcal{L} \to \mathcal{L}_! \)
Finitness spaces

Model of linear logic **Fin** [Ehrhard 2005]:

- Objects: topological vector spaces
- Morphisms: continuous linear maps

**Fin(!X, Y):** analytic functions

\[ f(x + h) = \sum_{n=0}^{\infty} f_n(h, \ldots, h) \]

With \( f_n \) \( n \)-linear symmetric. For example, if \( f : \mathbb{R} \rightarrow \mathbb{R} \),

\[ f(x + h) = \sum_{n=0}^{\infty} a_n h^n \]
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Differential in \textbf{Fin}

Differential in analysis

\( f : X \rightarrow Y \) differentiable in \( x \in X \) if

\[
    f(x + u) \approx f(x) + f'(x).u
\]

\( f'(x) : X \rightarrow Y \) is \underline{linear}
Differential in \textbf{Fin}

\textbf{Differential in analysis}

$f : X \to Y$ differentiable in $x \in X$ if

\[ f(x + u) \approx f(x) + f'(x).u \]

$f'(x) : X \to Y$ is \underline{linear}

\textbf{Differential in \textbf{Fin}}

Analytic $\Rightarrow$ Differentiable. For $f \in \textbf{Fin}(!X, Y)$

\[ f' \in \textbf{Fin}(!X \otimes X, Y) \]

\[ (x, u) \mapsto f'(x).u \]
Iterated derivative, analiticity

**Iterated derivative**

\[ f' \in \text{Fin}(!X \otimes X, Y) \simeq \text{Fin}(!X, \text{Fin}(X, E)) \text{ can be differentiated.} \]

\[ f^{(2)}(x) \in \text{Fin}(X, \text{Fin}(X, E)) \simeq \text{Fin}(X \otimes X, E) \]

\[ f^{(2)}(x) \in \text{Fin}(X \otimes X, Y) : \text{bilinear morphism} \]

\[ f^{(n)}(x) \in \text{Fin}(X \otimes \ldots \otimes X, Y) : n\text{-linear morphism} \]

**Link with analyticity**

\[ f \in \text{Fin}(!X, Y) \text{ is analytic} \]

\[ f(x + h) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) \cdot (h, \ldots, h) \]
Syntax side

Derivative of a $\Lambda$-term

$$\frac{\partial M}{\partial x} \cdot N$$

Substitute exactly one occurrence of $x$ by $N$ in $M$.

Taylor Development

$$(\lambda x. M) N \rightarrow \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\partial^n M}{\partial x^n} \cdot (N, \ldots, N) \right) [0/x]$$
Differential Category [BLUTE, Cockett, and SEELY 2006]

Differential: in a model $\mathcal{L}$ of LL

If $f \in \mathcal{L}(!X, Y)$:

$$f' \in \mathcal{L}(!X \otimes X, Y)$$

+ axioms of differential calculus

Example: $(f \otimes g)' = f' \otimes g + f \otimes g'$
Differential Category [BLUTE, Cockett, and SEELY 2006]

Differential: in a model $\mathcal{L}$ of LL

If $f \in \mathcal{L}(!X, Y)$:

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Example: $(f \otimes g)' = f' \otimes g + f \otimes g'$

$\mathcal{L}$ must be additive

$\mathcal{L}$: enriched over commutative monoid

- $\mathcal{L}(X, Y)$ is a monoïd
- $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$
- $h \circ (f_1 + f_2) = h \circ f_1 + h \circ f_2$
Additive is too much

**Non deterministic:** $\text{true}, \text{false} \in \mathcal{L}(1, 1 \oplus 1)$. What is $\text{true} + \text{false}$?

**In syntax, sum is not that wild**

- In the earlier Taylor development: only one branch of $\sum_{n=0}^{\infty} \cdots$ is non zero.

- Probabilistic Taylors expansion: the coefficients of the sum has total mass $\leq 1$, the sum is a probabilistic branching.
In semantic

Probabilistic coherent spaces $\mathbf{Pcoh}$: not additive

- $X := (|X|, \mathcal{P}(|X|))$ discrete space
- $\mu, \nu \in \mathbf{Pcoh}(1, X)$ are sub-probability distributions on $|X|
- \mu + \nu$: measure on $|X|$ of mass $\leq 2$
- $\mu + \nu \in \mathbf{Pcoh}(1, X)$ if their total mass is $\leq 1$
In semantic

Probabilistic coherent spaces $\mathbf{P}_{\text{coh}}$: not additive

- $X := (|X|, \mathcal{P}(|X|))$ discrete space
- $\mu, \nu \in \mathbf{P}_{\text{coh}}(1, X)$ are sub-probability distributions on $|X|$
- $\mu + \nu$: measure on $|X|$ of mass $\leq 2$
- $\mu + \nu \in \mathbf{P}_{\text{coh}}(1, X)$ if their total mass is $\leq 1$

Yet: $!$ in $\mathbf{P}_{\text{coh}}$ is deeply analytic.


$$\sum_{n=1}^{\infty} p^{n-1}(1 - p)C^n$$
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Partial commutative monoid

\((\mathcal{L}(X, Y), +, 0)\) is a partial commutative monoid. Write \(f \boxplus g\) if \(f + g\) is defined.

**Neutral element 0**

\(f \boxplus 0, 0 \boxplus f\) and \(f + 0 = 0 + f = f\)

**Commutativity**

If \(f \boxplus g\) then \(g \boxplus f\) and \(g + f = f + g\)
Partial commutative monoid

**Associativity**

If \( f \boxplus g \) and \( (f + g) \boxplus h \), then \( g \boxplus h \), \( f \boxplus (g + h) \) and

\[
(f + g) + h = f + (g + h)
\]
**Partial commutative monoid**

**Associativity**

If $f \boxplus g$ and $(f + g) \boxplus h$, then $g \boxplus h$, $f \boxplus (g + h)$ and

$$(f + g) + h = f + (g + h)$$

**Strong associativity:** positivity of coefficients.

Ex: In $[-1, 1]$, $x \boxplus y$ if $|x| + |y| \leq 1$

- $(-0.5 + 0.5) + 1$ defined
- $-0.5 + (0.5 + 1)$ not defined

→ $+$ is not associative on $[-1, 1]$
→ $+$ is associative on $[0, 1]$
Compatibility with addition

\( \mathcal{L} \) is enriched over partial commutative monoids.

**Compatibility with composition**

If \( f_0 \boxplus f_1 \) and \( g_0 \boxplus g_1 \)

- **Left:** \((g_0 + g_1) \circ f = g_0 \circ f + g_1 \circ f \) and \(0 \circ f = 0\)
- **Right:** \(g \circ (f_0 + f_1) = g \circ f_0 + g \circ f_1 \) and \(g \circ 0 = 0\)
Compatibilty with addition

\( L \) is enriched over partial commutative monoids.

**Compatibility with composition**

If \( f_0 \boxplus f_1 \) and \( g_0 \boxplus g_1 \)

- **Left:** \((g_0 + g_1) \circ f = g_0 \circ f + g_1 \circ f\) and \(0 \circ f = 0\)
- **Right:** \(g \circ (f_0 + f_1) = g \circ f_0 + g \circ f_1\) and \(g \circ 0 = 0\)

In practice, the partiality is given by the structure of the category itself through a functor

\[ S = (1 \& 1) \rightarrow \_ \]
Summability structure [Ehrhard 2021]

- **Functor** $S: \mathcal{S}X$ “set of pairs of summable elements of $X$”
- **Projections**: $\pi_0, \pi_1: \mathcal{S}X \to X$ (natural) jointly monic:
  $$(\forall i \in \{0, 1\}, \pi_i \circ f = \pi_i \circ g) \Rightarrow f = g$$
- **Sum**: $\sigma: \mathcal{S}X \to X$ (natural)
Summability structure [Ehrhard 2021]

**Summability structure**

- **Functor** $S : SX \text{ “set of pairs of summable elements of } X$”
- **Projections**: $\pi_0, \pi_1 : SX \to X$ (natural) jointly monic:
  
  \[ (\forall i \in \{0, 1\}, \pi_i \circ f = \pi_i \circ g) \Rightarrow f = g \]

- **Sum**: $\sigma : SX \to X$ (natural)

**Induced Sum**

- $f_0 \boxplus f_1$ if $\exists \langle f_0, f_1 \rangle_S : X \to SY$

  \[
  \begin{aligned}
  \pi_0 \circ \langle f_0, f_1 \rangle_S &= f_0 \\
  \pi_1 \circ \langle f_0, f_1 \rangle_S &= f_1 
  \end{aligned}
  \]

- $f_1 + f_2 = \sigma \circ (f_1, f_2)_S$

- **3 axioms** (neutral element, commutativity, associativity)
Summability Monad

Monad $S$

Monad Unit $\nu_0 \in \mathcal{L}(X, SX)$

\[ \nu_0 = \langle \text{id}, 0 \rangle_S : X \rightarrow SX \]
\[ x \mapsto (x, 0) \]

Monad Sum $\tau \in \mathcal{L}(S^2X, SX)$

\[ \tau = \langle \pi_0\pi_0, \pi_0\pi_1 + \pi_1\pi_0 \rangle_S : S^2X \rightarrow SX \]
\[ ((x, u), (v, \epsilon)) \mapsto (x, u + v) \]
Summability Monad

Monad $S$

Monad Unit $\nu_0 \in \mathcal{L}(X, SX)$

$$\nu_0 = \langle \text{id}, 0 \rangle_S : X \to SX$$

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$$((x, u), (v, \epsilon)) \mapsto (x, u + v)$$

Compatibility of $+$ with $\otimes$: The monad is strong:

$L \in \mathcal{L}(SX \otimes SY, S(X \otimes Y))$

$$(x, u) \otimes (y, v) \mapsto (x \otimes y, x \otimes v + u \otimes y)$$

Very similar to dual numbers (real with $\epsilon$ such that $\epsilon^2 = 0$)
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Coherent differentiation

For $f : !X \to Y$

$$f(x + u) \simeq f(x) + f'(x).u$$

$f(x)$ and $f'(x).u$ must be summable

$$Df : !SX \to SY$$

$$(x, u) \mapsto (f(x), f'(x).u)$$
Coherent differentiation

For $f : !X \to Y$

$$f(x + u) \simeq f(x) + f'(x) \cdot u$$

$f(x)$ and $f'(x) \cdot u$ must be summable

$$Df : !SX \to SY$$
$$(x, u) \mapsto (f(x), f'(x) \cdot u)$$

Advantage: the chain rule becomes a functoriality property

$$(g \circ f)'(x) \cdot u = g'(f(x)) \cdot f'(x) \cdot u$$

$$\iff (Dg \circ Df(x, u)) = D(g \circ f)(x, u)$$
Axioms of differentiation

Coherent differentiation

\[ Df \in \mathcal{L}(!SX, SY) = \mathcal{L}_!(SX, SY) \]

Axioms: Intuitively

- (D-struct) \( K_!(\pi_0) \circ !_Df = f \circ !_K_!(\pi_0) \)
- (Chain rule) \( D \) is a functor (in \( \mathcal{L}_! \))
- (Leibniz) \( f'(x) \cdot 0 = 0, \ f'(x) \cdot (u + v) = f'(x) \cdot u + f'(x) \cdot v \)
- (Linearity) \( \frac{\partial f'(x) \cdot u}{\partial u} = f'(x) \)
- (Swchartz) \( f^{(2)}(x) \) is symmetric
- (Bilinearity) If \( \phi \in \mathcal{L}(X \otimes Y, Z) \), \( \phi'(x, y) \cdot (u, v) = \phi(x, v) + \phi(u, y) \)
In terms of structure extension to $\mathcal{L}_!$

Axioms: As I see them

- **(D-struct)** $\mathcal{K}_!(\pi_0) : D \Rightarrow \text{Id}$ natural
- **(Chain rule)** $D$ is a functor that extends $S$ to $\mathcal{L}_!$:
  \[ D(\mathcal{K}_M(h)) = \mathcal{K}_M(Sh). \]
- **(Leibniz)** $\mathcal{K}_!(\iota_0) : \text{Id} \Rightarrow D$, $\mathcal{K}_!(\tau) : D^2 \Rightarrow D$ natural ($D$ is a **Monad** that extends $S$)
- **(Linearity)** $\mathcal{K}_!(\mathbf{1}) : D \Rightarrow D^2$ natural
- **(Swchartz)** $\mathcal{K}_!(\mathbf{c}) : D^2 \Rightarrow D^2$ is natural
In term of distributive Law

Distributive law
Distributive law between a functor $F$ and the comonad $\lambda$

$$\lambda_F \in \mathcal{L}(!FX, F!X)$$

Subject to 2 compatibility conditions (with dig and der)

Distributive Law, fact 1
For $F : \mathcal{L} \to \mathcal{L}$, bijection between

- Extensions $\hat{F} : \mathcal{L}_! \to \mathcal{L}_!$
- Distributive laws $\lambda_F \in \mathcal{L}(!FX, F!X)$

Functorial bijection.
Distributive law associated to:

- $Id$: $id \in \mathcal{L}(!X, !X)$
- $D$: $\partial_X \in \mathcal{L}(!SX, S!X)$
- $D^2$: $\partial_{SX} \circ S\partial_X \in \mathcal{L}(!S^2X, S^2X)$
Distributive law associated to:

- \(\text{Id}: \ id \in \mathcal{L}(!X, !X)\)
- \(\text{D}: \partial_X \in \mathcal{L}(!SX, S!X)\)
- \(\text{D}^2: \partial_{SX} \circ S\partial_X \in \mathcal{L}(!S^2X, S^2X)\)

**Distributive Law, fact 2**

If \(\alpha_X \in \mathcal{L}(FX, GX)\) natural, \(\hat{F}, \hat{G} : \mathcal{L}_! \rightarrow \mathcal{L}_!\), extensions, \(\lambda_F, \lambda_G\) associated distributive laws. \(\mathcal{K}_!(\alpha) : \hat{F} \Rightarrow \hat{G}\) if and only if

\[
\begin{array}{ccc}
!FX & \xrightarrow{\lambda_F} & F!X \\
\downarrow & & \downarrow \\
!GX & \xrightarrow{\lambda_G} & G!X
\end{array}
\]

\(\alpha\) is a morphism between \(\lambda_F\) and \(\lambda_G\).
Axioms: as introduced initially

- **(D-struct)** $\pi_0$ morphism between id and $\partial$
- **(Chain rule)** $\partial$ distributive law
- **(Leibniz)** $\iota_0$ morphism between id and $\partial$. $\tau$ morphism between $\partial \circ S \partial$ and $\partial$
- **(Linearity)** $\mathbf{1}$ morphism of distributive law
- **(Swchartz)** $\mathbf{c}$ morphism of distributive law
The Journey so far

- enriched over Partial commutative Monoid
- Partiality: a strong Monad $S$, close to dual numbers
- Differentiation: extension of the Monad $S$ to a Monad $D$ on $\mathcal{L}_!$ that preserves naturality of key operators
- Distributive Laws: the axioms correspond to diagrams on $\mathcal{P} : !S \Rightarrow S!$
- Side note: $\mathcal{L}_S$ is a model of LL
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Coherent differentiation generalizes differentiation

Give an additive category model of differential LL

Summability structure

\[ SX = X & X, \pi_i = p_i \]

\[ \langle f_0, f_1 \rangle_s = \langle f_0, f_1 \rangle, \sigma = p_0 + p_1 \]

We check that \( \sigma \circ \langle f_0, f_1 \rangle_s = f_1 + f_2 \)
Differentiation

Define $f^{\text{coh}} : ! (X \& X) \to Y$. Intuitively: $(x, u) \mapsto f'(x) \cdot u$

$$f^{\text{coh}} : ! (X \& X) \xrightarrow{\sim} ! X \otimes ! X \xrightarrow{id_! X \otimes \text{der}} ! X \otimes X \xrightarrow{f'} Y$$
Define $f^{\text{coh}} : !(X \& X) \to Y$. Intuitively: $(x, u) \mapsto f'(x) \cdot u$

$$f^{\text{coh}} : !(X \& X) \sim !X \otimes !X \xrightarrow{id_!X \otimes \text{der}} !X \otimes X \xrightarrow{f'} Y$$

$Df : !(X \& X) \to SY$. Intuitively: $(x, u) \mapsto (f(x), f^{\text{coh}}(x, u))$

$$Df = \langle f \circ !_p, f^{\text{coh}} \rangle$$

One to one correspondance between axioms of differential categories and coherent differentiation.
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Diagram of generality

Differential Category $\rightarrow$ Generalizes $\rightarrow$ Coherent Differential Category

$\text{Left additive: } (f_0 + f_1) \circ g = f_0 \circ g + f_1 \circ g$
Diagram of generality

Differential Category \quad \overset{\mathcal{K}_!}{\longrightarrow} \quad \text{Generalizes} \quad \overset{\longrightarrow}{\text{}} \quad \text{Coherent Differential Category}

Cartesian Differential Category

Left additive: \((f_0 + f_1) \circ g = f_0 \circ g + f_1 \circ g\)
Diagram of generality

Differential Category \( \rightarrow \) Generalizes \( \rightarrow \) Coherent Differential Category

\[ \mathcal{K}_1 \]

Cartesian Differential Category \( \rightarrow \) Generalizes \( \rightarrow \) Cartesian Coherent Differential Category

Left additive: \((f_0 + f_1) \circ g = f_0 \circ g + f_1 \circ g\)
Compatibility with Cartesian Product

Compatibility with the product:

- Compatibility with sum: "the sum of two pairs is the coordinate wise sum."
- Compatibility with differential: "the projections are linear"

\[ D(X \& Y) \simeq DX \& DY \]
Compatibility with Cartesian Product

Compatibility with the product:

- Compatibility with sum: "the sum of two pairs is the coordinate wise sum."
- Compatibility with differential: "the projections are linear"

\[ D(X \& Y) \simeq DX \& DY \]

Strength and partial derivative

Induce a strength

\[ \Phi^0 \in \mathcal{L}!(DX_0 \& X_1, D(X_0 \& X_1)) \]

Partial derivative of \( f \in \mathcal{L}!(X_0 \& X_1, Y) \):

\[ D_0f = Df \circ \Phi^0 \in \mathcal{L}!(DX_0 \& X_1, DY) \]
Compatibility with the closure:

- Compatibility with sum: "the sum of two functions is the pointwise sum"

- Compatibility with the differential: "the evaluation is linear in its fuctionnal coordinate"

\[ D(X \Rightarrow Y) \sim X \Rightarrow DY \]
Compatibility with the closure:

- Compatibility with sum: "the sum of two functions is the pointwise sum"
- Compatibility with the differential: "the evaluation is linear in its fonctionnal coordinate"

\[ D(X \Rightarrow Y) \simeq X \Rightarrow D Y \]

Internal derivative

For \( f \in \mathcal{L}_M(X, Y \Rightarrow Z) \)

\[ D^{\text{int}} f \in \mathcal{L}_M(X, DY \Rightarrow DZ) \]
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Terms

\[ M, N := (M)N \mid \lambda x. M \mid DM \mid D^d\pi_i(M) \mid D^d\Gamma_i(M) \mid D^d\Theta(M) \mid D^d\Theta(M) \]

Types

\[ A, B := D^d\Gamma \mid A \times B \mid A \Rightarrow B \]

\[ D(A \times B) := DA \times DB \]
\[ D(A \Rightarrow B) := A \Rightarrow DB \]
A sketch of a syntax

Terms

\[ M, N ::= (M)N \mid \lambda x.M \mid DM \mid D^d\pi_i(M) \mid D^d\iota_i(M) \mid D^d\Theta(M) \mid D^dc(M) \]

Types

\[ A, B ::= D^d\iota \mid A \times B \mid A \Rightarrow B \]

\[ D(A \times B) ::= DA \times DB \]

\[ D(A \Rightarrow B) ::= A \Rightarrow DB \]

\[ \Gamma \vdash M : A \Rightarrow B \implies \Gamma \vdash DM : DA \Rightarrow DB \]

\[ \Gamma \vdash M : D^{d+1}A \implies \Gamma \vdash D^d\pi_i(M) : D^dA \]
Reduction rules

Some rules

- \( D^d \pi_i (D^{e+1} f(M)) \rightarrow D^e f(D^d \pi_i (M)) \) (linearity of the constructors)
- \( D^d \pi_1 (D^d \Theta(M)) \rightarrow D^d \pi_0 (D^d \pi_1 (M)) + D^d \pi_1 (D^d \pi_0 (M)) \)

Last rule: reduction \( \Lambda \rightarrow M_{fin}(\Lambda) \). Lift the reduction to

\[
M_{fin}(\Lambda) \rightarrow M_{fin}(\Lambda)
\]
Reduction rules

Some rules

- \( D^d \pi_i(D^{e+1}f(M)) \to D^e f(D^d \pi_i(M)) \) (linearity of the constructors)
- \( D^d \pi_1(D^d \Theta(M)) \to D^d \pi_0(D^d \pi_1(M)) + D^d \pi_1(D^d \pi_0(M)) \)

Last rule: reduction \( \Lambda \to \mathcal{M}_{\text{fin}}(\Lambda) \). Lift the reduction to

\[
\mathcal{M}_{\text{fin}}(\Lambda) \to \mathcal{M}_{\text{fin}}(\Lambda)
\]

Probabilistic setting: multidistributions monad
Deterministic krivine machine using a memory structure on the projections
Differential

\[ D(\lambda x^A. M) \rightarrow \lambda x^{DA}. \partial(x, M) \]

If \( \Gamma, x : A \vdash M : B \) then

\[ \Gamma, x : DA \vdash \partial(x, M) : DB \]

\[ \partial(x, x) = x \quad \partial(x, y) = \iota_0(y) \]
\[ \partial(x, (M)N) = \Theta((D\partial(x, M))\partial(x, N)) \]
\[ \partial(x, DN) = c(D\partial(x, N)) \]
Can be extended to PCF:

▶ succ and pred: linear constructors
▶ if : \(\tau \times (A \times A) \to A\) and let : \(\tau \times A \to A\) bilinear constructors
▶ Admits fixpoint

**And a lot more to come** Krivine Machine, adequacy of Rel, deterministic Krivine machine, adequacy of Pcoh (?)
Thank you for your attention!