Coherent Differentiation and Taylor expansion

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Plan

Differentiation

The distributive law $\partial \in \mathcal{L}(!\mathsf{S}X, \mathsf{S}!X)$ induces a functor T on $\mathcal{L}_!$.

The functor T

If $f \in \mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$

$$\mathsf{T}f = \ !\mathsf{S}X \xrightarrow{\partial} \mathsf{S}!X \xrightarrow{\mathsf{S}f} \mathsf{S}Y$$

Intuitively

$$\begin{array}{cccc} \mathsf{T}f : & \mathsf{S}X & \rightarrow & \mathsf{S}Y \\ & \langle\!\langle x, u \rangle\!\rangle & \mapsto & \langle\!\langle f(x), f'(x).u \rangle\!\rangle \end{array}$$

- Functoriality of T is the chain rule
- Leibniz + Schwarz + Linearity of the derivatives: naturality assumptions.

$$\begin{aligned}
&\iota_0 \in \mathcal{L}(X, \mathsf{S}X) && \iota_0 \circ x = \langle\!\langle x, 0 \rangle\!\rangle \\
&\theta \in \mathcal{L}(\mathsf{S}^2X, \mathsf{S}X) && \theta \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle = \langle\!\langle x, u + v \rangle\!\rangle \\
&\mathsf{c} \in \mathcal{L}(\mathsf{S}^2X, \mathsf{S}^2X) && \mathsf{c} \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle = \langle\!\langle \langle\!\langle x, v \rangle\!\rangle, \langle\!\langle u, w \rangle\!\rangle \rangle\!\rangle \\
&\mathsf{l} \in \mathcal{L}(\mathsf{S}X, \mathsf{S}^2X) && \mathsf{l} \circ \langle\!\langle x, u \rangle\!\rangle = \langle\!\langle \langle\!\langle x, 0 \rangle\!\rangle, \langle\!\langle 0, u \rangle\!\rangle \rangle\!\rangle
\end{aligned}$$

All natural in $\mathcal{L}$, $(\mathsf{S}, \iota_0, \theta, \sigma, \mathsf{l}, \mathsf{c})$ is a c-bimonad.

$$
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\mathsf{l} \in \mathcal{L}(\mathsf{S}X, \mathsf{S}^2 X) & \mathsf{l} \circ \langle\!\langle x, u \rangle\!\rangle = \langle\!\langle \langle\!\langle x, 0 \rangle\!\rangle , \langle\!\langle 0, u \rangle\!\rangle \rangle\!\rangle
\end{array}
$$

All natural in $\mathcal{L}$, $(\mathsf{S}, \iota_0, \theta, \sigma, \mathsf{l}, \mathsf{c})$ is a c-bimonad.

Axioms of differentiation

- ι_0 and sum θ are natural in $\mathcal{L}_!$ (Leibniz)
- c is natural in $\mathcal{L}_!$ (Schwarz)
- l is natural in $\mathcal{L}_!$ (The differential is linear)

σ is not natural in $\mathcal{L}_!$: $f(x+u) \neq f(x) + f'(x) \cdot u$

Plan

An order 2 coherent Taylor expansion

Coherent differentiation is a first order Taylor expansion.

$$\begin{array}{rccc} \mathsf{T}f : & \mathsf{S}X & \to & \mathsf{S}Y \\ & \langle\!\langle x, u \rangle\!\rangle & \mapsto & \langle\!\langle f(x), f'(x) \cdot u \rangle\!\rangle \end{array}$$

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But we can do the same for second order

$$\begin{array}{rccc} \mathsf{T}f : & \mathsf{S}X & \to & \mathsf{S}Y \\ & \langle\!\langle x, u, v \rangle\!\rangle & \mapsto & \langle\!\langle f(x),\ f'(x) \cdot u,\ \frac{1}{2} f''(x) \cdot (u, u) + f'(x) \cdot v \rangle\!\rangle \end{array}$$

We recover order 2 Taylor expansion

$$\mathsf{T}f\ \langle\!\langle x, u, 0 \rangle\!\rangle = \left\langle\!\!\left\langle f(x),\ f'(x) \cdot u,\ \frac{1}{2} f''(x) \cdot (u, u) \right\rangle\!\!\right\rangle$$

but the red term is necessary for compositionality.

Going to infinity

Introduce infinitary summability structures:

$$\mathsf{S}X = \{\langle\!\langle x_i \rangle\!\rangle_{i=0}^{\infty} \mid \sum_{i=0}^{\infty} x_i \text{ is defined}\}$$

Taylor expansion is still a functor

$$\mathsf{T}f \ \ \langle\!\langle x_i \rangle\!\rangle_{i=0}^{\infty} = \left\langle\!\!\left\langle \sum_{m \in \mathcal{M}(n)} \frac{1}{m!} \frac{\mathrm{d}^{|m|} f}{\mathrm{d}^{|m|} x}(x_0) \cdot \vec{x}_m \right\rangle\!\!\right\rangle_{n=0}^{\infty}$$

- $\mathcal{M}(n)$ is the set of multisets $m \in \mathcal{M}_{\mathrm{fin}}(\mathbb{N}^*)$ s.t. $\sum_{i \in \mathbb{N}^*} i \ m(i) = n$
- $m! = \prod_{i \in \mathbb{N}^*} m(i)!$
- $\vec{x}_m = (\underbrace{x_1, \ldots, x_1}_{m(1) \text{ times}}, \ldots, \underbrace{x_i, \ldots, x_i}_{m(i) \text{ times}}, \ldots, \underbrace{x_n, \ldots, x_n}_{m(n) \text{ times}})$

Going to infinity

We recover the usual Taylor expansion. Let $\vec{x} = \langle\!\langle x, u, 0, \dots \rangle\!\rangle$. Then

$$\vec{x}_m = (\underbrace{u, \dots, u}_{m(1) \text{ times}}, 0, \dots, 0)$$

Thus

$$\frac{1}{m!}\frac{\mathrm{d}^{|m|}f}{\mathrm{d}^{|m|}x}(x) \cdot \vec{x}_m = \begin{cases} \frac{1}{n!}\frac{\mathrm{d}^n f}{\mathrm{d}^n x}(x) \text{ if } m = [1, \dots, 1] \\ 0 \text{ otherwise} \end{cases}$$

So

$$\mathsf{T}f \;\; \langle\!\langle x, u, 0, \dots \rangle\!\rangle = \left\langle\!\!\left\langle \frac{1}{n!}\frac{\mathrm{d}^n f}{\mathrm{d}^n x}(x) \right\rangle\!\!\right\rangle_{i=0}^{\infty}$$

Same axioms

- T is a functor (Chain rule)
- T is a monad with unit ι_0 and sum θ (Leibniz)
- c is natural (Schwarz)
- l is natural (The differential is linear)
- σ is natural (Morphisms are analytic)

$$(\mathsf{T}, \iota_0, \theta, \sigma, \mathsf{l}, \mathsf{c}) \text{ is a c-bimonad}$$

Plan

Elementary summability structure

Very similar theory.

$$\mathbb{D} = 1 \mathbin{\&} 1 \mathbin{\&} \cdots \qquad \mathsf{S}X = \mathbb{D} \multimap X$$

In **Pcoh**.

$$|\mathbb{D}| = \mathbb{N} \quad \mathsf{P}\mathbb{D} = [0,1]^{\mathbb{N}}$$

and

$$|\mathsf{S}X| = \mathbb{N} \times |X| \quad \mathsf{PS}X = \{\langle\!\langle x_i \rangle\!\rangle_{i=0}^{\infty} \in \mathbb{R}_{\geq 0}^{\mathbb{N} \times |X|} \mid \sum_{i \in \mathbb{N}} x_i \in \mathsf{P}X\}$$

Bimonoid structure

Co-unit	$\mathsf{p}_0 \in \mathbf{Pcoh}(\mathbb{D}, 1)$	
Co-multitiplication	$\widetilde{\theta} \in \mathbf{Pcoh}(\mathbb{D}, \mathbb{D} \otimes \mathbb{D})$	$\widetilde{\theta}_{n,(i,j)} = \delta_{n,i+j}$
Unit	$\Delta \in \mathbf{Pcoh}(1, \mathbb{D})$	$\Delta_{*,n} = 1$
Multiplication	$\widetilde{\mathsf{I}} \in \mathbf{Pcoh}(\mathbb{D} \otimes \mathbb{D}, \mathbb{D})$	$\widetilde{\mathsf{I}}_{(n,m),k} = \delta_{n,m}\delta_{m,k}$

Taylor structure

Coalgebra $\widetilde{\partial} \in \mathbf{Pcoh}(\mathbb{D}, !\mathbb{D})$

$$\widetilde{\partial}_{n,[i_1,\dots,i_k]} = \begin{cases} 1 \text{ if } n = i_1 + \cdots + i_k \\ 0 \text{ otherwise} \end{cases}$$

Induce $\partial \in \mathbf{Pcoh}(!\mathsf{S}X, \mathsf{S}!X)$. If $m = [a_1, \dots, a_k]$

$$\partial_{p,(n,m)} = \begin{cases} \frac{m!}{p!} \text{ if } p = [(i_1, a_1), \dots, (i_k, a_k)] \text{ and } \sum_{l=1}^{k} i_l = n \\ 0 \text{ otherwise} \end{cases}$$

If $s \in \mathbf{Pcoh}(!X, Y)$, $\mathsf{T}s \in \mathbf{Pcoh}(!\mathsf{S}X, \mathsf{S}Y)$. If $p = [(i_1, a_1), \dots, (i_k, a_k)]$

$$\mathsf{T}(s)_{p,(n,b)} = \begin{cases} \frac{m!}{p!} s_{m,b} \text{ if } \sum_{l=1}^{k} i_l = n \\ 0 \text{ otherwise} \end{cases}$$

Plan

Takeaway on coherent differentiation

- Axiomatization of differentiation with partial sums
- Axioms of differentiation: functoriality and naturality
- Nice theory of partial derivatives
- Coherent differential PCF

Takeaway on Taylor expansion

- Same theory as differentiation
- Same theory of partial derivative
- The coherent differential PCF should be adapted to this setting

Deterministic (or probabilistic) calculus with a Krivine machine that counts the number of time an input is used during run time.