Plan

1. Partial sums and differentiation in a CCC
2. The whole Taylor expansion
3. Taylor expansion in the elementary case
4. Conclusion and perspectives
Differentiation

The distributive law $\partial \in \mathcal{L}(!SX, S!X)$ induces a functor $T$ on $\mathcal{L}_!$.

The functor $T$

If $f \in \mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$

$$Tf = \xrightarrow{\partial} S!X \xrightarrow{Sf} SY$$

Intuitively

$$Tf : SX \rightarrow SY \quad \langle x, u \rangle \mapsto \langle f(x), f'(x).u \rangle$$

- Functoriality of $T$ is the chain rule
- **Leibniz + Schwarz + Linearity of the derivatives**: naturality assumptions.
\[ \nu_0 \in \mathcal{L}(X, SX) \quad \nu_0 \circ x = \langle x, 0 \rangle \]

\[ \theta \in \mathcal{L}(S^2X, SX) \quad \theta \circ \langle \langle x, u \rangle, \langle v, w \rangle \rangle = \langle x, u + v \rangle \]

\[ c \in \mathcal{L}(S^2X, S^2X) \quad c \circ \langle \langle x, u \rangle, \langle v, w \rangle \rangle = \langle \langle x, v \rangle, \langle u, w \rangle \rangle \]

\[ l \in \mathcal{L}(SX, S^2X) \quad l \circ \langle x, u \rangle = \langle \langle x, 0 \rangle, \langle 0, u \rangle \rangle \]

All natural in \( \mathcal{L} \), \((S, \nu_0, \theta, \sigma, l, c)\) is a c-bimonad.
\[ \nu_0 \in \mathcal{L}(X, SX) \quad \text{and} \quad \nu_0 \circ x = \langle x, 0 \rangle \]
\[ \theta \in \mathcal{L}(S^2X, SX) \quad \text{and} \quad \theta \circ \langle \langle x, u \rangle, \langle v, w \rangle \rangle = \langle x, u + v \rangle \]
\[ c \in \mathcal{L}(S^2X, S^2X) \quad \text{and} \quad c \circ \langle \langle x, u \rangle, \langle v, w \rangle \rangle = \langle \langle x, v \rangle, \langle u, w \rangle \rangle \]
\[ l \in \mathcal{L}(SX, S^2X) \quad \text{and} \quad l \circ \langle x, u \rangle = \langle \langle x, 0 \rangle, \langle 0, u \rangle \rangle \]

All natural in \( \mathcal{L} \), \( (S, \nu_0, \theta, \sigma, l, c) \) is a c-bimonad.

**Axioms of differentiation**

▶ \( \nu_0 \) and sum \( \theta \) are natural in \( \mathcal{L}_! \) (Leibniz)

▶ \( c \) is natural in \( \mathcal{L}_! \) (Schwarz)

▶ \( l \) is natural in \( \mathcal{L}_! \) (The differential is linear)

\( \sigma \) is **not** natural in \( \mathcal{L}_! \): \( f(x + u) \neq f(x) + f'(x) \cdot u \)
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An order 2 coherent Taylor expansion

Coherent differentiation is a first order Taylor expansion.

\[
Tf : \quad SX \rightarrow SY \\
\langle x, u \rangle \mapsto \langle f(x), f'(x) \cdot u \rangle
\]

but the red term is necessary for compositionality.
An order 2 coherent Taylor expansion

Coherent differentiation is a first order Taylor expansion.

\[
Tf : \quad SX \rightarrow SY \\
\langle x, u \rangle \mapsto \langle f(x), f'(x) \cdot u \rangle
\]

But we can do the same for second order

\[
Tf : \quad SX \rightarrow SY \\
\langle x, u, v \rangle \mapsto \langle f(x), f'(x) \cdot u, \frac{1}{2}f''(x) \cdot (u, u) + f'(x) \cdot v \rangle
\]

We recover order 2 Taylor expansion

\[
Tf \langle x, u, 0 \rangle = \langle f(x), f'(x) \cdot u, \frac{1}{2}f''(x) \cdot (u, u) \rangle
\]

but the red term is necessary for compositionality.
Going to infinity

Introduce infinitary summability structures:

\[ SX = \{ \langle x_i \rangle_{i=0}^\infty \mid \sum_{i=0}^\infty x_i \text{ is defined} \} \]

Taylor expansion is still a functor

\[ Tf \langle x_i \rangle_{i=0}^\infty = \left\{ \sum_{m \in \mathcal{M}(n)} \frac{1}{m!} \frac{d^{|m|}f}{d^{|m|}x}(x_0) \cdot \tilde{x}_m \right\}_{n=0}^\infty \]

- \( \mathcal{M}(n) \) is the set of multisets \( m \in \mathcal{M}_\text{fin}(\mathbb{N}^*) \) s.t. \( \sum_{i \in \mathbb{N}^*} i \cdot m(i) = n \)
- \( m! = \prod_{i \in \mathbb{N}^*} m(i)! \)
- \( \tilde{x}_m = \left( x_1, \ldots, x_1, \ldots, x_i, \ldots, x_i, \ldots, x_n, \ldots, x_n \right) \)
  \[ m(1) \text{ times} \quad m(i) \text{ times} \quad m(n) \text{ times} \]
The whole Taylor expansion

Going to infinity

We recover the usual Taylor expansion. Let $\vec{x} = \langle x, u, 0, \ldots \rangle$. Then

$$\vec{x}_m = (u, \ldots, u, 0, \ldots, 0)_{m(1) \text{ times}}$$

Thus

$$\frac{1}{m!} \frac{d^{|m|} f}{d^{|m|} x}(x) \cdot \vec{x}_m = \begin{cases} 
\frac{1}{n!} \frac{d^n f}{d^n x}(x) & \text{if } m = [1, \ldots, 1] \\
0 & \text{otherwise}
\end{cases}$$

So

$$Tf \langle x, u, 0, \ldots \rangle = \left\langle \frac{1}{n!} \frac{d^n f}{d^n x}(x) \right\rangle_{i=0}^{\infty}$$
Same axioms

- $T$ is a functor (Chain rule)
- $T$ is a monad with unit $\nu_0$ and sum $\theta$ (Leibniz)
- $c$ is natural (Schwarz)
- $l$ is natural (The differential is linear)
- $\sigma$ is natural (Morphisms are analytic)

$(T, \nu_0, \theta, \sigma, l, c)$ is a $c$-bimonad
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Elementary summability structure

Very similar theory.

\[ \mathbb{D} = 1 \& 1 \& \cdots \quad SX = \mathbb{D} \rightarrow X \]

In $\mathbb{P}^{coh}$.

\[ |\mathbb{D}| = \mathbb{N} \quad PD = [0, 1]^\mathbb{N} \]

and

\[ |SX| = \mathbb{N} \times |X| \quad PSX = \{ \sum_{i \in \mathbb{N}} x_i \in \mathbb{R}^{\mathbb{N} \times |X|} \mid \sum_i x_i \in PX \} \]

Bimonoid structure

Co-unit $p_0 \in \mathbb{P}^{coh}(\mathbb{D}, 1)$

Co-multitiplication $\tilde{\theta} \in \mathbb{P}^{coh}(\mathbb{D}, \mathbb{D} \otimes \mathbb{D})$ $\tilde{\theta}_{n,(i,j)} = \delta_{n,i+j}$

Unit $\Delta \in \mathbb{P}^{coh}(1, \mathbb{D})$ $\Delta_{*,n} = 1$

Multiplication $\tilde{1} \in \mathbb{P}^{coh}(\mathbb{D} \otimes \mathbb{D}, \mathbb{D})$ $\tilde{1}_{(n,m),k} = \delta_{n,m} \delta_{m,k}$
Taylor structure

Coalgebra $\tilde{\partial} \in \mathbf{Pcoh}(\mathbb{D}, !\mathbb{D})$

$$\tilde{\partial}_{n,[i_1,\ldots,i_k]} = \begin{cases} 1 & \text{if } n = i_1 + \cdots + i_k \\ 0 & \text{otherwise} \end{cases}$$

Induce $\partial \in \mathbf{Pcoh}(!SX, S!X)$. If $m = [a_1, \ldots, a_k]$

$$\partial_p,(n,m) = \begin{cases} \frac{m!}{p!} & \text{if } p = [(i_1, a_1), \ldots, (i_k, a_k)] \text{ and } \sum_{l=1}^k i_l = n \\ 0 & \text{otherwise} \end{cases}$$

If $s \in \mathbf{Pcoh}(!X, Y)$, $Ts \in \mathbf{Pcoh}(!SX, SY)$. If $p = [(i_1, a_1), \ldots, (i_k, a_k)]$

$$T(s)_{p,(n,b)} = \begin{cases} \frac{m!}{p!} s_{m,b} & \text{if } \sum_{l=1}^k i_l = n \\ 0 & \text{otherwise} \end{cases}$$
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Takeaway on coherent differentiation

- Axiomatization of differentiation with partial sums
- Axioms of differentiation: functoriality and naturality
- Nice theory of partial derivatives
- Coherent differential PCF
Takeaway on Taylor expansion

- Same theory as differentiation
- Same theory of partial derivative
- The coherent differential PCF should be adapted to this setting

Deterministic (or probabilistic) calculus with a Krivine machine that counts the number of time an input is used during run time.