Cartesian Coherent Differential Categories

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Differential λ-calculus

Models of LL suggest the existence of a derivation operation on terms

Derivation

If $\Gamma, x : A \vdash P : B$ and $\Gamma \vdash Q : A$

$$\Gamma, x : A \vdash \frac{\partial P}{\partial x} \cdot Q : B$$

substitutes in P **one** call of x by a call of Q.

Taylor expansion

$$\mathcal{T}(P[Q/x]) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n P}{\partial x^n} \cdot (\underbrace{Q, \dots, Q}_{\text{n times}}) \right) [0/x]$$

Term of rank n: part of computation that uses Q exactly n times.

Non-deterministic

$$\frac{\Gamma \vdash P : A \quad \Gamma \vdash Q : A}{\Gamma \vdash P + Q : A}$$

This sum arises in the definition of $\frac{\partial P}{\partial x} \cdot Q$ (Leibniz rule)

Differentiation in categorical semantics

- In Linear Logic: differential categories
- In cartesian (closed) categories: cartesian differential categories

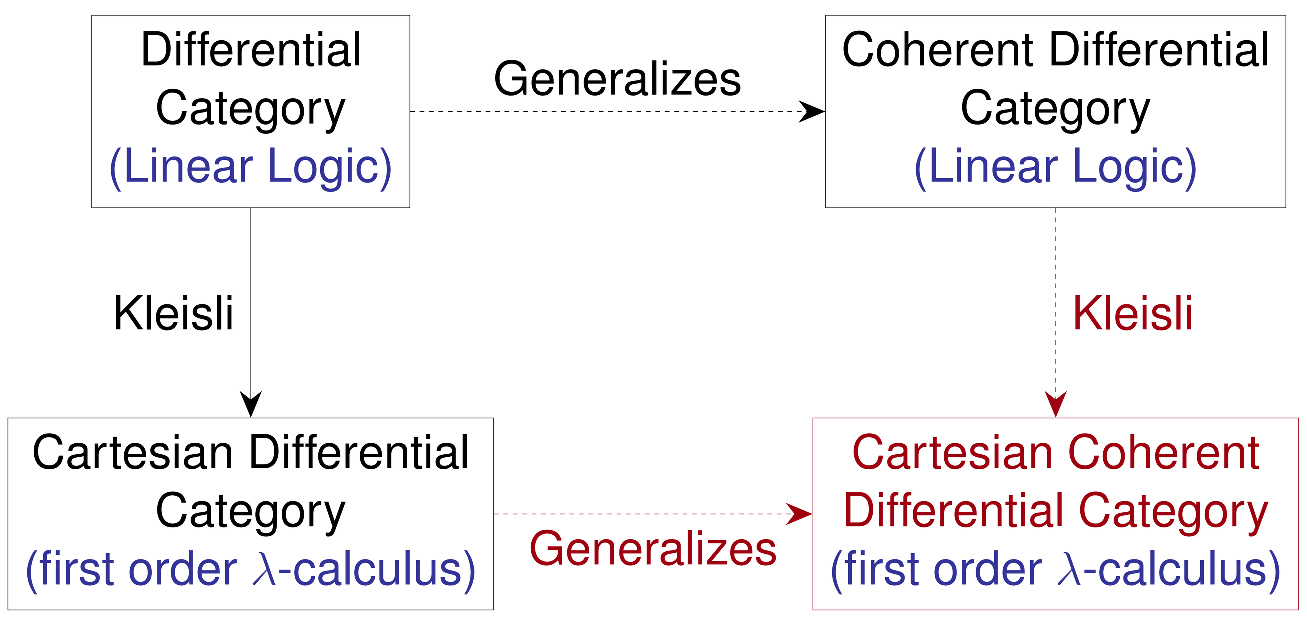
All models are (left) additive: hom-sets are commutative monoids and

- Left additivity: $(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$
- Additivity (only in LL): $g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$

We need an unrestricted sum. Operationally, this sum is non-determinism.

Coherent differentiation

Coherent differentiation extends differentiation to deterministic models of LL: coherence spaces, probabilistic coherence spaces, etc.



A partial notion of summation

Let $\mathcal{C}$ be a category with 0-morphisms.

A structure for partial sum

S : **Obj** $\to$ **Obj**. Intuitively, $\mathsf{S}X = \{\langle\!\langle x_0, x_1 \rangle\!\rangle \mid x_0 + x_1 \text{ is defined}\}$

- $\pi_0, \pi_1 \in \mathcal{C}(\mathsf{S}X, X)$ jointly monic $\qquad \pi_i : \langle\!\langle x_0, x_1 \rangle\!\rangle \mapsto x_i$
- Sum $\sigma \in \mathcal{C}(\mathsf{S}X, X)$ $\qquad \sigma : \langle\!\langle x_0, x_1 \rangle\!\rangle \mapsto x_0 + x_1$

$f_0, f_1 \in \mathcal{C}(X, Y)$ are **summable** if :

$$\exists \langle\!\langle f_0, f_1 \rangle\!\rangle \in \mathcal{C}(X, \mathsf{S}Y) \text{ s.t. } \pi_i \circ \langle\!\langle f_0, f_1 \rangle\!\rangle = f_i$$

$$\langle\!\langle f_0, f_1 \rangle\!\rangle : x \mapsto \langle\!\langle f_0(x), f_1(x) \rangle\!\rangle$$

The additivity of π_0, π_1, σ and some axioms on S give to hom-sets the structure of a finite **partially additive monoid** and morphisms are all left additive.

When the sum is total

$$\mathsf{S}X = X \,\&\, X \iff \text{Cartesian Left Additive Category}$$

Structure associated to S

Linear case

Additive morphisms are a subcategory $\mathcal{C}^{\mathrm{add}}$ of $\mathcal{C}$. The map S on objects extends to an endofunctor on $\mathcal{C}^{\mathrm{add}}$ making $\pi_0, \pi_1, \sigma : \mathsf{S} \Rightarrow \mathsf{Id}$ natural.

$$\mathsf{S}f : \langle\!\langle x_0, x_1 \rangle\!\rangle \mapsto \langle\!\langle f(x_0), f(x_1) \rangle\!\rangle$$

Define ι_0, θ, c and l natural transformations in $\mathcal{C}^{\mathrm{add}}$.

$$\begin{array}{ll} \iota_0 \in \mathcal{C}^{\mathrm{add}}(X, \mathsf{S}X) & \iota_0 \circ x = \langle\!\langle x, 0 \rangle\!\rangle \\ \theta \in \mathcal{C}^{\mathrm{add}}(\mathsf{S}^2X, \mathsf{S}X) & \boxed{\theta \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle = \langle\!\langle x, u + v \rangle\!\rangle} \\ \mathsf{c} \in \mathcal{C}^{\mathrm{add}}(\mathsf{S}^2X, \mathsf{S}^2X) & \mathsf{c} \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle = \langle\!\langle \langle\!\langle x, v \rangle\!\rangle, \langle\!\langle u, w \rangle\!\rangle \rangle\!\rangle \\ \mathsf{l} \in \mathcal{C}^{\mathrm{add}}(\mathsf{S}X, \mathsf{S}^2X) & \mathsf{l} \circ \langle\!\langle x, u \rangle\!\rangle = \langle\!\langle \langle\!\langle x, 0 \rangle\!\rangle, \langle\!\langle 0, u \rangle\!\rangle \rangle\!\rangle \end{array}$$

Then $(\mathsf{S}, \iota_0, \theta)$ is a monad, $(\mathsf{S}, \sigma, \mathsf{l})$ is a comonad, and $(\mathsf{S}, \iota_0, \theta, \sigma, \mathsf{l}, \mathsf{c})$ is a c-bimonad.
Notice: S is **not** a functor on $\mathcal{C}$

Differentiation as a functor T

Differentiation: functor T on $\mathcal{C}$ such that $\mathsf{T}X = \mathsf{S}X$.

$$\begin{array}{rcl} \mathsf{T}f : \mathsf{S}X & \to & \mathsf{S}Y \\ \langle\!\langle x, u \rangle\!\rangle & \mapsto & \langle\!\langle f(x), f'(x).u \rangle\!\rangle \end{array}$$

Axioms of differentiation: functoriality of T and naturality in $\mathcal{C}$.

- Chain rule: T is a functor
- Leibniz: $\iota_0 : \mathsf{Id} \Rightarrow \mathsf{T}$ and $\theta : \mathsf{T}^2 \Rightarrow \mathsf{T}$ are natural in $\mathcal{C}$
- Linearity of derivative: $\mathsf{l} : \mathsf{T} \Rightarrow \mathsf{T}^2$ is natural in $\mathcal{C}$
- Schwarz: $\mathsf{c} : \mathsf{T}^2 \Rightarrow \mathsf{T}^2$ is natural in $\mathcal{C}$.

$(\mathsf{T}, \iota_0, \mathsf{l})$ is a monad, but $(\mathsf{T}, \sigma, \mathsf{l})$ is not a comonad because σ is not natural.

Interaction with cartesian product

$$\mathsf{T}(X \,\&\, Y) \simeq \mathsf{T}X \,\&\, \mathsf{T}Y$$

Strength associated to this structure $\rightsquigarrow$ partial derivatives.

$$\text{if } f \in \mathcal{C}(X_1 \,\&\, X_2, Y), \text{ then } \mathsf{T}_1 f \in \mathcal{C}(\mathsf{T}X_1 \,\&\, X_2, \mathsf{T}Y)$$

The functor T performs a first order Taylor expansion. It should be possible to do something similar for all orders.

Coherent Taylor expansion

Infinitary Taylor functor

We introduce an infinitary counterpart of summability structures.

$$\mathsf{S}X = \{\langle\!\langle x_i \rangle\!\rangle_{i=0}^{\infty} \mid \sum_{i=0}^{\infty} x_i \text{ is defined}\}$$

As differentiation, Taylor expansion is a functor such that $\mathsf{T}X = \mathsf{S}X$. Same axioms except that σ is now natural too (account for analyticity of the morphisms).

Intuitively, if $f : \mathbb{R} \to \mathbb{R}$ is analytic

$$f(\sum_{n=0}^{\infty} x_n \epsilon^n) = \sum_{n=0}^{\infty} f_n(x_0, \dots, x_n) \epsilon^n$$

Where f_n can be computed by the Faà Di Bruno formula. Then $\mathsf{T}f$ can be seen as

$$\mathsf{T}f \ \langle\!\langle x_i \rangle\!\rangle_{i=0}^{\infty} = \langle\!\langle f_n(x_0, \dots, x_n) \rangle\!\rangle_{i=0}^{\infty}$$

A syntax for coherent Taylor expansion

- Using coherent differentiation, Ehrhard introduced a deterministic PCF with both fixpoints and differentiation, with a straightforward probabilistic extension.
- The recent discovery of a coherent Taylor expansion suggests that this calculus can feature the full Taylor expansion.

Algèbre et calcul