Cartesian Coherent Differential Categories

Thomas Ehrhard Aymeric Walch

IRIF

June 29, 2023

Plan

1. Differential λ-calculus and (Cartesian) Differential Categories
2. Coherent differentiation
3. Sum and differentiation in a partial setting
4. Compatibility with the Cartesian product
5. Conclusion and perspectives

Differential λ-calculus

A function $f : E \to F$ is differentiable in x if

$$\boxed{f(x+u) \simeq f(x) + f'(x) \cdot u}$$

With $f'(x) : E \to F$ a linear map.

Differential λ-calculus

A function $f : E \to F$ is differentiable in x if

$$\boxed{f(x+u) \simeq f(x) + f'(x) \cdot u}$$

With $f'(x) : E \to F$ a linear map.

Differential in terms

If $\Gamma, x : A \vdash P : B$ and $\Gamma \vdash Q : A$

$$\Gamma, x : A \vdash \frac{\partial P}{\partial x} \cdot Q : B$$

substitute <u>one</u> occurrence of x by Q in P.

Differential λ-calculus

A function $f : E \to F$ is differentiable in x if

$$\boxed{f(x+u) \simeq f(x) + f'(x) \cdot u}$$

With $f'(x) : E \to F$ a linear map.

Differential in terms

If $\Gamma, x : A \vdash P : B$ and $\Gamma \vdash Q : A$

$$\Gamma, x : A \vdash \frac{\partial P}{\partial x} \cdot Q : B$$

substitute one occurrence of x by Q in P.

Taylor Expansion

$$(\lambda x.P)Q \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n P}{\partial x^n} \cdot (\underbrace{Q, \ldots, Q}_{\text{n times}}) \right) [0/x]$$

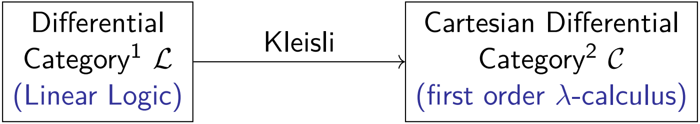
Differential Categories and Cartesian Differential Categories

Differential Category[1] $\mathcal{L}$
(Linear Logic)

[1] Blute, Cockett, and Seely 2006.
[2] Blute, Cockett, and Seely 2009.

Differential Categories and Cartesian Differential Categories

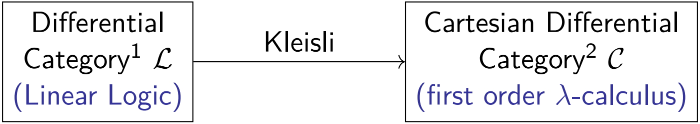
Recall: $\mathcal{L}_!(X, Y) := \mathcal{L}(!X, Y)$ is a CCC



[1] Blute, Cockett, and Seely 2006.
[2] Blute, Cockett, and Seely 2009.

Differential Categories and Cartesian Differential Categories

Recall: $\mathcal{L}_!(X, Y) := \mathcal{L}(!X, Y)$ is a CCC



[1] Blute, Cockett, and Seely 2006.
[2] Blute, Cockett, and Seely 2009.

Differential Categories and Cartesian Differential Categories

Recall: $\mathcal{L}_!(X, Y) := \mathcal{L}(!X, Y)$ is a CCC

Cartesian Differential Category[2] $\mathcal{C}$
(first order λ-calculus)

[1] Blute, Cockett, and Seely 2006.
[2] Blute, Cockett, and Seely 2009.

Differential Categories and Cartesian Differential Categories

Recall: $\mathcal{L}_!(X, Y) := \mathcal{L}(!X, Y)$ is a CCC

Cartesian Differential Category[2] $\mathcal{C}$
(first order λ-calculus)

- Compatibility with the CCC struture: models of differential λ-calculus Bucciarelli, Ehrhard, and Manzonetto 2010
- Models for Taylor expansion (qualitative setting) Manzonetto 2012

Example: relation model

[1] Blute, Cockett, and Seely 2006.
[2] Blute, Cockett, and Seely 2009.

(Left) additivity and non determinism

Leibniz: $f'(x, y) \cdot (u, v) = \partial_0 f(x, y) \cdot u + \partial_1 f(x, y) \cdot v$

(Left) additivity and non determinism

Leibniz: $f'(x, y) \cdot (u, v) = \partial_0 f(x, y) \cdot u + \partial_1 f(x, y) \cdot v$

A Differential Category $\mathcal{L}$ must be additive

- $\mathcal{L}(X, Y)$ is a commutative monoïd
- $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$ (left additive)
- $h \circ (f_1 + f_2) = h \circ f_1 + h \circ f_2$ (additive)

(Left) additivity and non determinism

Leibniz: $f'(x, y) \cdot (u, v) = \partial_0 f(x, y) \cdot u + \partial_1 f(x, y) \cdot v$

A Cartesian Differential Category $\mathcal{C}$ must be left additive

- $\mathcal{C}(X, Y)$ is a commutative monoïd
- $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$ (left additive)
- ~~$h \circ (f_1 + f_2) = h \circ f_1 + h \circ f_2$~~ (additive)

(Left) additivity and non determinism

$$\text{Leibniz: } f'(x,y)\cdot(u,v) = \partial_0 f(x,y)\cdot u + \partial_1 f(x,y)\cdot v$$

A Differential Category $\mathcal{L}$ must be additive

- $\mathcal{L}(X,Y)$ is a commutative monoïd
- $(f_1 + f_2)\circ g = f_1 \circ g + f_2 \circ g$ (left additive)
- $h \circ (f_1 + f_2) = h \circ f_1 + h \circ f_2$ (additive)

Non-deterministic: $\mathtt{true}, \mathtt{false} \in \mathcal{L}(1, 1 \oplus 1)$. What is $\mathtt{true} + \mathtt{false}$?

(Left) additivity and non determinism

$$\text{Leibniz: } f'(x,y)\cdot(u,v) = \partial_0 f(x,y)\cdot u + \partial_1 f(x,y)\cdot v$$

A Differential Category $\mathcal{L}$ must be additive

- $\mathcal{L}(X,Y)$ is a commutative monoïd
- $(f_1+f_2)\circ g = f_1\circ g + f_2\circ g$ (left additive)
- $h\circ(f_1+f_2) = h\circ f_1 + h\circ f_2$ (additive)

Non-deterministic: $\texttt{true}, \texttt{false} \in \mathcal{L}(1, 1\oplus 1)$. What is $\texttt{true} + \texttt{false}$?

(Left) additivity and non determinism

$$\text{Leibniz: } f'(x,y)\cdot(u,v) = \partial_0 f(x,y)\cdot u + \partial_1 f(x,y)\cdot v$$

A Differential Category $\mathcal{L}$ must be additive

- $\mathcal{L}(X,Y)$ is a commutative monoïd
- $(f_1+f_2)\circ g = f_1\circ g + f_2\circ g$ (left additive)
- $h\circ(f_1+f_2) = h\circ f_1 + h\circ f_2$ (additive)

Non-deterministic: $\mathtt{true},\mathtt{false}\in\mathcal{L}(1,1\oplus 1)$. What is $\mathtt{true}+\mathtt{false}$?

- If $(\lambda x.P)Q$ is well typed and reduces to a variable: only one member of $\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\partial^n P}{\partial x^n}\cdot(Q,\ldots,Q)\right)[0/x]$ is non zero.

(Left) additivity and non determinism

$$\text{Leibniz: } f'(x,y)\cdot(u,v) = \partial_0 f(x,y)\cdot u + \partial_1 f(x,y)\cdot v$$

A Differential Category $\mathcal{L}$ must be additive

- $\mathcal{L}(X, Y)$ is a commutative monoïd
- $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$ (left additive)
- $h \circ (f_1 + f_2) = h \circ f_1 + h \circ f_2$ (additive)

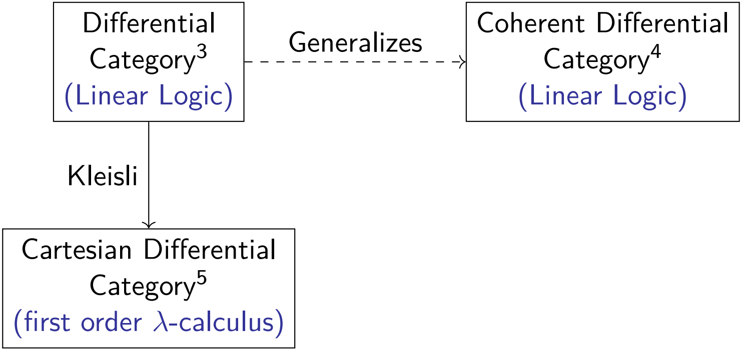
Non-deterministic: $\mathtt{true}, \mathtt{false} \in \mathcal{L}(1, 1 \oplus 1)$. What is $\mathtt{true} + \mathtt{false}$?

- If $(\lambda x.P)Q$ is well typed and reduces to a variable: only one member of $\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n P}{\partial x^n} \cdot (Q, \ldots, Q) \right) [0/x]$ is non zero.
- Interesting models $\mathcal{L}$ of LL in which $\mathcal{L}_!$ is a category with differentiable morphisms, with a partial addition

Plan

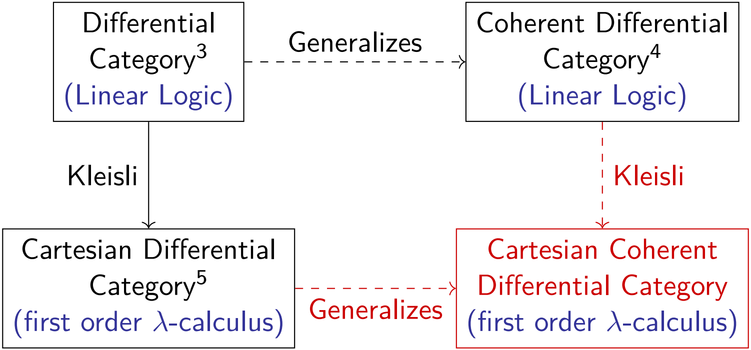
1. Differential λ-calculus and (Cartesian) Differential Categories
2. Coherent differentiation
3. Sum and differentiation in a partial setting
4. Compatibility with the Cartesian product
5. Conclusion and perspectives

Our work in this paper



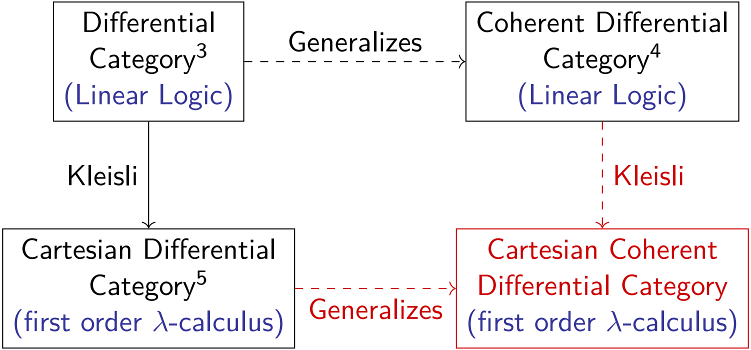
[3]Blute, Cockett, and Seely 2006.
[4]Ehrhard 2023.
[5]Blute, Cockett, and Seely 2009.

Our work in this paper



[3] Blute, Cockett, and Seely 2006.
[4] Ehrhard 2023.
[5] Blute, Cockett, and Seely 2009.

Our work in this paper



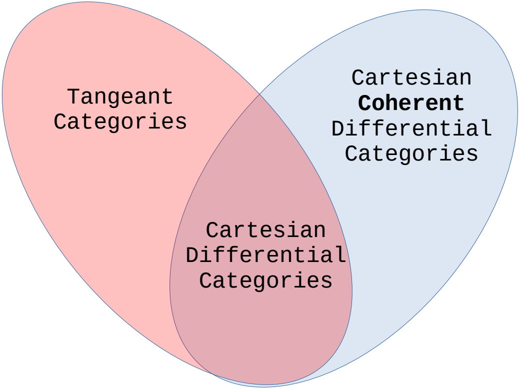
Models of a first order calculus with differentiation (subject reduction)

[3] Blute, Cockett, and Seely 2006.

[4] Ehrhard 2023.

[5] Blute, Cockett, and Seely 2009.

Comparison with tangeant category



- Tangeant Category: distinguish point/vector, total sum on vectors
- Coherent Differential Category: no distinction point/vector, but restricted sum

Plan

1 Differential λ-calculus and (Cartesian) Differential Categories

2 Coherent differentiation

3 Sum and differentiation in a partial setting

4 Compatibility with the Cartesian product

5 Conclusion and perspectives

A categorical axiomatization of partial sum

A structure for partial sum

$\widetilde{\mathrm{D}} : \mathbf{Obj}(\mathcal{C}) \to \mathbf{Obj}(\mathcal{C})$: $\widetilde{\mathrm{D}}X = \{\langle\langle x_0, x_1 \rangle\rangle | x_0 + x_1 \text{ is defined}\}$

- $\pi_0, \pi_1 \in \mathcal{C}(\widetilde{\mathrm{D}}X, X)$ jointly monic

A categorical axiomatization of partial sum

A structure for partial sum

$\widetilde{\mathrm{D}} : \mathbf{Obj}(\mathcal{C}) \to \mathbf{Obj}(\mathcal{C})$: $\widetilde{\mathrm{D}}X = \{\langle\!\langle x_0, x_1 \rangle\!\rangle | x_0 + x_1 \text{ is defined}\}$

- $\pi_0, \pi_1 \in \mathcal{C}(\widetilde{\mathrm{D}}X, X)$ jointly monic
- Sum $\sigma \in \mathcal{C}(\widetilde{\mathrm{D}}X, X)$ $\sigma : \langle\!\langle x_0, x_1 \rangle\!\rangle \mapsto x_0 + x_1$

A categorical axiomatization of partial sum

A structure for partial sum

$\widetilde{\mathsf{D}} : \mathbf{Obj}(\mathcal{C}) \to \mathbf{Obj}(\mathcal{C})$: $\widetilde{\mathsf{D}} X = \{\langle\!\langle x_0, x_1 \rangle\!\rangle | x_0 + x_1 \text{ is defined}\}$

- $\pi_0, \pi_1 \in \mathcal{C}(\widetilde{\mathsf{D}} X, X)$ jointly monic
- Sum $\sigma \in \mathcal{C}(\widetilde{\mathsf{D}} X, X)$ $\sigma : \langle\!\langle x_0, x_1 \rangle\!\rangle \mapsto x_0 + x_1$

$f_0, f_1 \in \mathcal{C}(X, Y)$ summable: $\exists \; \langle\!\langle f_0, f_1 \rangle\!\rangle \in \mathcal{C}(X, \widetilde{\mathsf{D}} Y)$ s.t. $\pi_i \circ \langle\!\langle f_0, f_1 \rangle\!\rangle = f_i$.

$$x \mapsto \langle\!\langle f_0(x), f_1(x) \rangle\!\rangle$$

A categorical axiomatization of partial sum

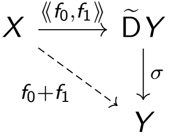
A structure for partial sum

$\widetilde{\mathrm{D}} : \mathbf{Obj}(\mathcal{C}) \to \mathbf{Obj}(\mathcal{C})$: $\widetilde{\mathrm{D}}X = \{\langle\langle x_0, x_1\rangle\rangle | x_0 + x_1 \text{ is defined}\}$

- $\pi_0, \pi_1 \in \mathcal{C}(\widetilde{\mathrm{D}}X, X)$ jointly monic
- Sum $\sigma \in \mathcal{C}(\widetilde{\mathrm{D}}X, X)$ $\sigma : \langle\langle x_0, x_1\rangle\rangle \mapsto x_0 + x_1$

$f_0, f_1 \in \mathcal{C}(X, Y)$ summable: $\exists \, \langle\langle f_0, f_1\rangle\rangle \in \mathcal{C}(X, \widetilde{\mathrm{D}}Y)$ s.t. $\pi_i \circ \langle\langle f_0, f_1\rangle\rangle = f_i$.

$$x \mapsto \langle\langle f_0(x), f_1(x)\rangle\rangle$$



A categorical axiomatization of partial sum

A structure for partial sum

$\widetilde{\mathrm{D}} : \mathbf{Obj}(\mathcal{C}) \to \mathbf{Obj}(\mathcal{C})$: $\widetilde{\mathrm{D}}X = \{\langle\!\langle x_0, x_1 \rangle\!\rangle | x_0 + x_1 \text{ is defined}\}$

- $\pi_0, \pi_1 \in \mathcal{C}(\widetilde{\mathrm{D}}X, X)$ jointly monic
- Sum $\sigma \in \mathcal{C}(\widetilde{\mathrm{D}}X, X)$ $\sigma : \langle\!\langle x_0, x_1 \rangle\!\rangle \mapsto x_0 + x_1$

$f_0, f_1 \in \mathcal{C}(X, Y)$ summable: $\exists \ \langle\!\langle f_0, f_1 \rangle\!\rangle \in \mathcal{C}(X, \widetilde{\mathrm{D}}Y)$ s.t. $\pi_i \circ \langle\!\langle f_0, f_1 \rangle\!\rangle = f_i$.

$$x \mapsto \langle\!\langle f_0(x), f_1(x) \rangle\!\rangle$$

$$\begin{array}{ccc} X & \xrightarrow{\langle\!\langle f_0, f_1 \rangle\!\rangle} & \widetilde{\mathrm{D}}Y \\ & \underset{f_0+f_1}{\searrow} & \downarrow \sigma \\ & & Y \end{array}$$

$$\boxed{\widetilde{\mathrm{D}}X = X \,\&\, X (= X \times X) \iff \text{Cartesian Left Additive Category}}$$

Summability structure

Compatibility with composition

If g_0 and g_1 are summable, then $g_0 \circ f$ and $g_1 \circ f$ are summable.

- $0 \circ f = 0$ and $(g_0 + g_1) \circ f = g_0 \circ f + g_1 \circ f$ (left additive)
- ~~$h \circ 0 = 0$ and $h \circ (f_0 + f_1) = h \circ f_0 + h \circ f_1$~~

Summability structure

Compatibility with composition

If g_0 and g_1 are summable, then $g_0 \circ f$ and $g_1 \circ f$ are summable.

- $0 \circ f = 0$ and $(g_0 + g_1) \circ f = g_0 \circ f + g_1 \circ f$ (left additive)
- ~~$h \circ 0 = 0$ and $h \circ (f_0 + f_1) = h \circ f_0 + h \circ f_1$~~

Left summability structure

Axioms that endows $\mathcal{C}(X, Y)$ with the structure of a partially additive monoid, see Arbib and Manes 1980

Summability structure

Compatibility with composition

If g_0 and g_1 are summable, then $g_0 \circ f$ and $g_1 \circ f$ are summable.

- $0 \circ f = 0$ and $(g_0 + g_1) \circ f = g_0 \circ f + g_1 \circ f$ (left additive)
- ~~$h \circ 0 = 0$ and $h \circ (f_0 + f_1) = h \circ f_0 + h \circ f_1$~~

Left summability structure

Axioms that endows $\mathcal{C}(X, Y)$ with the structure of a partially additive monoid, see Arbib and Manes 1980

$\widetilde{\mathrm{D}}$ is not a functor (yet)!

Differentiation

An operator for differentiation

Given $f \in \mathcal{C}(X, Y)$, there is $\widetilde{\mathrm{D}}f \in \mathcal{C}(\widetilde{\mathrm{D}}X, \widetilde{\mathrm{D}}Y)$ such that $\pi_0 \circ \widetilde{\mathrm{D}}f = f \circ \pi_0$

$$\begin{array}{rccc} \widetilde{\mathrm{D}}f : & \widetilde{\mathrm{D}}X & \to & \widetilde{\mathrm{D}}Y \\ & \langle\!\langle x, u \rangle\!\rangle & \mapsto & \langle\!\langle f(x), f'(x).u \rangle\!\rangle \end{array}$$

Differentiation

An operator for differentiation

Given $f \in \mathcal{C}(X, Y)$, there is $\widetilde{\mathrm{D}} f \in \mathcal{C}(\widetilde{\mathrm{D}} X, \widetilde{\mathrm{D}} Y)$ such that $\pi_0 \circ \widetilde{\mathrm{D}} f = f \circ \pi_0$

$$\begin{array}{rccc} \widetilde{\mathrm{D}} f : & \widetilde{\mathrm{D}} X & \to & \widetilde{\mathrm{D}} Y \\ & \langle\!\langle x, u \rangle\!\rangle & \mapsto & \langle\!\langle f(x), f'(x).u \rangle\!\rangle \end{array}$$

Define $f' = \pi_1 \circ \widetilde{\mathrm{D}} f \in \mathcal{C}(\widetilde{\mathrm{D}} X, Y)$

Differentiation

An operator for differentiation

Given $f \in \mathcal{C}(X, Y)$, there is $\widetilde{\mathrm{D}} f \in \mathcal{C}(\widetilde{\mathrm{D}} X, \widetilde{\mathrm{D}} Y)$ such that $\pi_0 \circ \widetilde{\mathrm{D}} f = f \circ \pi_0$

$$\begin{array}{rccc} \widetilde{\mathrm{D}} f : & \widetilde{\mathrm{D}} X & \to & \widetilde{\mathrm{D}} Y \\ & \langle\!\langle x, u \rangle\!\rangle & \mapsto & \langle\!\langle f(x), f'(x).u \rangle\!\rangle \end{array}$$

Define $f' = \pi_1 \circ \widetilde{\mathrm{D}} f \in \mathcal{C}(\widetilde{\mathrm{D}} X, Y)$

Axioms of differentiation: very structural properties

- $\widetilde{\mathrm{D}}$ is a functor (Chain rule)

Differentiation

An operator for differentiation

Given $f \in \mathcal{C}(X, Y)$, there is $\widetilde{\mathrm{D}}f \in \mathcal{C}(\widetilde{\mathrm{D}}X, \widetilde{\mathrm{D}}Y)$ such that $\pi_0 \circ \widetilde{\mathrm{D}}f = f \circ \pi_0$

$$\begin{array}{rccc} \widetilde{\mathrm{D}}f : & \widetilde{\mathrm{D}}X & \to & \widetilde{\mathrm{D}}Y \\ & \langle\!\langle x, u \rangle\!\rangle & \mapsto & \langle\!\langle f(x), f'(x).u \rangle\!\rangle \end{array}$$

Define $f' = \pi_1 \circ \widetilde{\mathrm{D}}f \in \mathcal{C}(\widetilde{\mathrm{D}}X, Y)$

Axioms of differentiation: very structural properties

- $\widetilde{\mathrm{D}}$ is a functor (Chain rule)
- π_0, π_1 are linear (h linear if $h'(x) \cdot u = h(u)$)
- σ is linear ($(f+g)' = f' + g'$)

Differentiation

An operator for differentiation

Given $f \in \mathcal{C}(X, Y)$, there is $\widetilde{\mathrm{D}} f \in \mathcal{C}(\widetilde{\mathrm{D}} X, \widetilde{\mathrm{D}} Y)$ such that $\pi_0 \circ \widetilde{\mathrm{D}} f = f \circ \pi_0$

$$\begin{array}{rccc} \widetilde{\mathrm{D}} f : & \widetilde{\mathrm{D}} X & \to & \widetilde{\mathrm{D}} Y \\ & \langle\!\langle x, u \rangle\!\rangle & \mapsto & \langle\!\langle f(x), f'(x).u \rangle\!\rangle \end{array}$$

Define $f' = \pi_1 \circ \widetilde{\mathrm{D}} f \in \mathcal{C}(\widetilde{\mathrm{D}} X, Y)$

Axioms of differentiation: very structural properties

- $\widetilde{\mathrm{D}}$ is a functor (Chain rule)
- π_0, π_1 are linear (h linear if $h'(x) \cdot u = h(u)$)
- σ is linear ($(f + g)' = f' + g'$)
- Leibniz + Schwarz + the differential is linear = naturality !

Define ι_0, θ, c and l

$$\iota_0 \circ x = \langle\!\langle x, 0 \rangle\!\rangle$$

$$\boxed{\theta \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle = \langle\!\langle x, u + v \rangle\!\rangle}$$

$$\mathsf{c} \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle = \langle\!\langle \langle\!\langle x, v \rangle\!\rangle, \langle\!\langle u, w \rangle\!\rangle \rangle\!\rangle$$

$$\mathsf{l} \circ \langle\!\langle x, u \rangle\!\rangle = \langle\!\langle \langle\!\langle x, 0 \rangle\!\rangle, \langle\!\langle 0, u \rangle\!\rangle \rangle\!\rangle$$

- $\widetilde{\mathsf{D}}$ is a monad with unit ι_0 and sum θ (The differential is additive = Leibniz)
- c is natural (Schwarz)
- l is natural (The differential is linear)

Define ι_0, θ, c and l

$$\iota_0 \circ x = \langle\!\langle x, 0 \rangle\!\rangle$$

$$\boxed{\theta \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle = \langle\!\langle x, u + v \rangle\!\rangle}$$

$$\mathsf{c} \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle = \langle\!\langle \langle\!\langle x, v \rangle\!\rangle, \langle\!\langle u, w \rangle\!\rangle \rangle\!\rangle$$

$$\mathsf{l} \circ \langle\!\langle x, u \rangle\!\rangle = \langle\!\langle \langle\!\langle x, 0 \rangle\!\rangle, \langle\!\langle 0, u \rangle\!\rangle \rangle\!\rangle$$

- $\widetilde{\mathrm{D}}$ is a monad with unit ι_0 and sum θ (The differential is additive = Leibniz)
- c is natural (Schwarz)
- l is natural (The differential is linear)

Cartesian Differential Categories

Naturality equations $\iff$ equations on the differential f'.
They are exactly the equations of Cartesian Differential Categories.

$$\text{Cartesian Differential Category} \iff \widetilde{\mathrm{D}}X = X \mathbin{\&} X$$

Plan

1 Differential λ-calculus and (Cartesian) Differential Categories

2 Coherent differentiation

3 Sum and differentiation in a partial setting

4 Compatibility with the Cartesian product

5 Conclusion and perspectives

Compatibility with Cartesian Product

Compatibility with the Cartesian product

- Product and sum : $\langle x, y\rangle + \langle u, v\rangle = \langle x + y, u + v\rangle$
- Product and differential: the projections of the cartesian product are linear, $\mathrm{D}\langle f, g\rangle = \langle \mathrm{D}f, \mathrm{D}g\rangle$

Compatibility with Cartesian Product

Compatibility with the Cartesian product

- Product and sum : $\langle x, y\rangle + \langle u, v\rangle = \langle x + y, u + v\rangle$
- Product and differential: the projections of the cartesian product are linear, $\mathrm{D}\langle f, g\rangle = \langle \mathrm{D}f, \mathrm{D}g\rangle$

In analysis (and Cartesian Differential Categories)

$$\boxed{\partial_0 f(x, y) \cdot u = f'(x, y) \cdot (u, 0)}$$

Compatibility with Cartesian Product

Compatibility with the Cartesian product

- Product and sum : $\langle x, y\rangle + \langle u, v\rangle = \langle x + y, u + v\rangle$
- Product and differential: the projections of the cartesian product are linear, $\mathrm{D}\langle f, g\rangle = \langle \mathrm{D}f, \mathrm{D}g\rangle$

In analysis (and Cartesian Differential Categories)

$$\boxed{\partial_0 f(x, y) \cdot u = f'(x, y) \cdot (u, 0)}$$

In our setting: strength $\Phi^0 \in \mathcal{C}(\widetilde{\mathrm{D}} X_0 \,\&\, X_1, \widetilde{\mathrm{D}}(X_0 \,\&\, X_1))$

$$\begin{array}{cccccc} \Phi^0 : & \widetilde{\mathrm{D}} X_0 \,\&\, X_1 & \to & \widetilde{\mathrm{D}} X_0 \,\&\, \widetilde{\mathrm{D}} X_1 & \simeq & \widetilde{\mathrm{D}}(X_0 \,\&\, X_1) \\ & \langle \langle\!\langle x, u\rangle\!\rangle, y\rangle & \mapsto & \langle \langle\!\langle x, u\rangle\!\rangle, \langle\!\langle y, 0\rangle\!\rangle\rangle & \mapsto & \langle\!\langle \langle x, y\rangle, \langle u, 0\rangle \rangle\!\rangle \end{array}$$

Compatibility with Cartesian Product

Compatibility with the Cartesian product

- Product and sum : $\langle x, y\rangle + \langle u, v\rangle = \langle x + y, u + v\rangle$
- Product and differential: the projections of the cartesian product are linear, $\mathrm{D}\langle f, g\rangle = \langle \mathrm{D}f, \mathrm{D}g\rangle$

In analysis (and Cartesian Differential Categories)

$$\boxed{\partial_0 f(x, y) \cdot u = f'(x, y) \cdot (u, 0)}$$

In our setting: strength $\Phi^0 \in \mathcal{C}(\widetilde{\mathrm{D}} X_0 \,\&\, X_1, \widetilde{\mathrm{D}}(X_0 \,\&\, X_1))$

$$\begin{array}{lccccc} \Phi^0 : & \widetilde{\mathrm{D}} X_0 \,\&\, X_1 & \to & \widetilde{\mathrm{D}} X_0 \,\&\, \widetilde{\mathrm{D}} X_1 & \simeq & \widetilde{\mathrm{D}}(X_0 \,\&\, X_1) \\ & \langle \langle\!\langle x, u\rangle\!\rangle, y\rangle & \mapsto & \langle \langle\!\langle x, u\rangle\!\rangle, \langle\!\langle y, 0\rangle\!\rangle\rangle & \mapsto & \langle\!\langle \langle x, y\rangle, \langle u, 0\rangle \rangle\!\rangle \end{array}$$

Partial derivative of $f \in \mathcal{C}(X_0 \,\&\, X_1, Y)$: $\widetilde{\mathrm{D}}_0 f \in \mathcal{C}(\widetilde{\mathrm{D}} X_0 \,\&\, X_1, \widetilde{\mathrm{D}} Y)$

$$\widetilde{\mathrm{D}} X_0 \,\&\, X_1 \xrightarrow{\Phi^0} \widetilde{\mathrm{D}}(X_0 \,\&\, X_1) \xrightarrow{\widetilde{\mathrm{D}} f} \widetilde{\mathrm{D}} Y$$

Leibniz and Schwarz

Leibniz

In analysis :

$$f'(x, y) \cdot (u, v) = \partial_0 f(x, y) \cdot u + \partial_1 f(x, y) \cdot v$$

In Cartesian Coherent Differential Categories

$$\widetilde{\mathrm{D}} f \circ \mathrm{c}_{\&}^{-1} = \theta \circ \widetilde{\mathrm{D}}_0 \widetilde{\mathrm{D}}_1 f = \theta \circ \widetilde{\mathrm{D}}_1 \widetilde{\mathrm{D}}_0 f$$

Plan

1. Differential λ-calculus and (Cartesian) Differential Categories
2. Coherent differentiation
3. Sum and differentiation in a partial setting
4. Compatibility with the Cartesian product
5. Conclusion and perspectives

Takeaway

- Axiomatization of differentiation with partial sums
- Axioms of differentiation: functoriality and naturality
- Nice theory of partial derivatives

TODO list

- Introduce closure to interpret a deterministic differential λ-calculus
- Deal with fixpoints to interpret the Coherent Differential PCF of Ehrhard
- Revisit syntactical Taylor expansion in a coherent setting
- It should provide generic denotational proofs of important results on syntactical Taylor expansion
- Is this construction insigthful for traditional analysis ?