A new categorical axiomatization of Taylor expansion

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IRIF

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Plan

1. Taylor expansion in the lambda calculus
2. The puzzling interpretation of sums
3. Coherent Differentiation
4. The whole Taylor expansion
5. Going back to syntax
6. Taylor expansion in the elementary case
7. Conclusion
Differentiable map and analytic maps

Differential: A function $f : E \to F$ is differentiable in $x$ if

$$f(x + u) \simeq f(x) + f'(x) \cdot u$$

With $f'(x) : E \to F$ a linear map, that is, $f' : E \to (E \to F)$
Differentiable map and analytic maps

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**Iterated differential:** We can differentiate $f'$

$$f^{(2)}(x) : (E \to E \to F)$$
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We iterate the process to get

$$f^{(n)}(x) : E \otimes \ldots \otimes E \to F$$
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Taylor expansion: A function $f : E \to F$ is analytic around $x$ if

$$f(x + u) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) \cdot (u, \ldots, u)^n$$
Reminder: if $L$ is a model of linear logic, then $L!$ where

$$L!(X, Y) = L(!X, Y)$$

is a model of $\lambda$-calculus (CCC)

In finiteness spaces and Koethe spaces
- morphisms in $L(X, Y)$ are linear maps
- morphisms in $L!(X, Y)$ are analytic maps

It means that programs are interpreted as analytic maps that can be computed by the sum of their derivatives!
Differential and Taylor expansion in calculus

Differential in terms

If $\Gamma, x : A \vdash P : B$ and $\Gamma \vdash Q : A$

$$\Gamma, x : A \vdash \frac{\partial P}{\partial x} \cdot Q : B$$

substitute one occurrence of $x$ by $Q$ in $P$. 
Differential and Taylor expansion in calculus

Differential in terms
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Taylor Expansion

$$(\lambda x. P) Q \leadsto \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\partial^n P}{\partial x^n} \cdot (Q, \ldots, Q) \right) [0/x]$$

We can do Taylor expansion in all applications at once
Taylor expansion

Lambda Term $M$ \quad \text{regular application} \quad \text{Taylor expansion} \quad \mapsto \quad \text{Infinite sum} \sum_{i \in I} r_i \cdot t_i

$t_i$ Ressource Terms, $r_i \in \mathbb{Q}$

\text{multilinear application}

Resource terms: $t, u = \lambda x.t \ | \ x \ | \langle t \rangle \cdot \bar{u}$

Bags/Multisets: $\bar{u} = [u_1, \ldots, u_n]$
Taylor expansion

Lambda Term $M$  \[\begin{array}{c}
\text{Taylor expansion} \\
\mathcal{T}
\end{array}\] \[\begin{array}{c}
\text{Infinite sum} \\
\sum_{i \in I} r_i \cdot t_i \\
\text{Ressource Terms}, r_i \in \mathbb{Q}
\end{array}\]

Multilinear application

Resource terms: $t, u = \lambda x.t \mid x \mid \langle t \rangle \cdot \overline{u}$

Bags/Multisets: $\overline{u} = [u_1, \ldots, u_n]$

Redex: if $x$ occurs $n$ times in $t$

\[
\langle \lambda x.t \rangle \cdot [u_1, \ldots, u_n] \rightarrow \sum_{\sigma} t[u_{\sigma(1)}/x_1, \ldots, u_{\sigma(n)}/x_n]
\]

Resource term are strongly normalizing: any resource term $t$ reduces to a finite sum $\text{nf}(t)$ of normal resource terms.
The sum in the Taylor expansion is not arbitrary

Taylor expansion has a normal form: if $T(M) = \sum_{i \in I} r_i \cdot t_i$

$$\text{nf}(T(M)) = \sum_{i \in I} r_i \cdot \text{nf}(t_i)$$
The sum in the Taylor expansion is not arbitrary

Taylor expansion has a normal form: if $\mathcal{T}(M) = \sum_{i \in I} r_i \cdot t_i$

$$\text{nf}(\mathcal{T}(M)) = \sum_{i \in I} r_i \cdot \text{nf}(t_i)$$

Crucial observation

If $\mathcal{T}M = \sum_{i \in I} r_i \cdot t_i$, the finite sums $\text{nf}(t_i)$ are of disjoint support.

Proof based on Uniformity properties.
Another proof of normalization by Lionel Vaux is based on finiteness spaces.
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Differential Categories and Cartesian Differential Categories

Recall:
\[ L ! (X, Y) := L (\! X, Y) \]
is a CCC

Differential Category\(^1\) \( \mathcal{L} \)
(Linear Logic)

Cartesian Differential Category\(^2\)
(first order \( \lambda \)-calculus)

Kleisli

Compatibility with the CCC structure: models of differential \( \lambda \)-calculus

Bucciarelli, Ehrhard, and Manzonetto 2010

Models for Taylor expansion Manzonetto 2012

Example: relation model, weighted relational model.

\(^1\)Blute, Cockett, and Seely 2006.

\(^2\)Blute, Cockett, and Seely 2009.
Recall: $\mathcal{L}(X, Y) := \mathcal{L}(!X, Y)$ is a CCC

\[\text{Differential Category}^1 \, \mathcal{L} \quad \text{(Linear Logic)} \xrightarrow{\text{Kleisli}} \quad \text{Cartesian Differential Category}^2 \, \mathcal{C} \quad \text{(first order \(\lambda\)-calculus)}\]

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$^1$Blute, Cockett, and Seely 2006.

Differential Categories and Cartesian Differential Categories

Recall: $\mathcal{L}(X, Y) := \mathcal{L}(!X, Y)$ is a CCC

- Differential Category\(^1\) $\mathcal{L}$
  - (Linear Logic)
- Kleisli
- Cartesian Differential Category\(^2\) $\mathcal{C}$
  - (first order $\lambda$-calculus)

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(Left) additivity and non determinism

Leibniz: \( f'(x, y) \cdot (u, v) = \partial_0 f(x, y) \cdot u + \partial_1 f(x, y) \cdot v \)
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A Differential Category \( \mathcal{L} \) must be left additive

- \( \mathcal{L}(X, Y) \) is a commutative monoid
- \( (f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g \) (left additive)
- \( h \circ (f_1 + f_2) = h \circ f_1 + h \circ f_2 \) (additive)

We exclude coherence spaces and probabilistic coherence spaces.

If we want Taylor expansion, we do even worse: we allow for arbitrary countable sums. We exclude finiteness spaces.
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A categorical axiomatization of partial sum

Let $\mathcal{L}$ be a model of linear logic.

A structure for partial sum

Functor $S$. Intuitively, $SX = \{ \langle x_0, x_1 \rangle | x_0 + x_1 \text{ is defined} \}$

$\pi_0, \pi_1 \in \mathcal{L}(SX, X)$ jointly monic $\pi_i : \langle x_0, x_1 \rangle \mapsto x_i$
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- Sum $\sigma \in \mathcal{L}(SX, X)$ $\sigma : \langle x_0, x_1 \rangle \mapsto x_0 + x_1$
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- Sum $\sigma \in \mathcal{L}(SX, X)$ $\sigma : \langle x_0, x_1 \rangle \mapsto x_0 + x_1$

- $f_0, f_1 \in \mathcal{L}(X, Y)$ summable: $\exists \langle f_0, f_1 \rangle \in \mathcal{L}(X, SY)$ s.t. $\pi_i \circ \langle f_0, f_1 \rangle = f_i$

  $\langle f_0, f_1 \rangle : x \mapsto \langle f_0(x), f_1(x) \rangle$
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- $f_0, f_1 \in \mathcal{L}(X, Y)$ **summable:** $\exists \langle f_0, f_1 \rangle \in \mathcal{L}(X, SY)$ s.t. $\pi_i \circ \langle f_0, f_1 \rangle = f_i$
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- $f_0 + f_1 := \sigma \circ \langle f_0, f_1 \rangle : x \mapsto f_0(x) + f_1(x)$
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$SX = X \& X \iff \text{Cartesian Additive Category}$
Partial commutative monoid

summability structure \((S, \pi_0, \pi_1, \sigma)\) and some axioms:

- A zero morphism 0 neutral with regard to the sum: \(f + 0 = 0 + f = f\)
- Commutativity of sums \(f + g \cong g + f\)
- Associativity \(f + (g + h) \cong (f + g) + h\)

\(\mathcal{L}(X, Y)\) is a partial commutative monoid. This is the finite counterpart of the notion of \(\Sigma\)-monoid found in literature.
Partial commutative monoid

summability structure: \((S, \pi_0, \pi_1, \sigma)\) and some axioms:

▶ A zero morphism 0 neutral with regard to the sum: \(f + 0 = 0 + f = f\)
▶ Commutativity of sums \(f + g \simeq g + f\)
▶ Associativity \(f + (g + h) \simeq (f + g) + h\)

\(\mathcal{L}(X, Y)\) is a partial commutative monoid. This is the finite counterpart of the notion of \(\Sigma\)-monoid found in literature.

▶ Naturality of \(\pi_i\): \(Sf : \langle x_0, x_1 \rangle \mapsto \langle f \cdot x_0, f \cdot x_1 \rangle\)
▶ Naturality of \(\sigma\): \(f \circ (h_0 + h_1) \sqsubseteq f \circ h_0 + f \circ h_1\)
How to formalize differentiation

We want $f \in \mathcal{L}(!X, Y)$ to be differentiable.

$$f(x + u) \simeq f(x) + f'(x) \cdot u$$

So differentiation should be interpreted as an operator $T$ that maps $f \in \mathcal{L}(!X, Y)$ to $Tf \in \mathcal{L}(!SX, SY)$

$$Tf : SX \rightarrow SY$$

$$\langle x, u \rangle \mapsto \langle f(x), f'(x) \cdot u \rangle$$

It performs a first order Taylor expansion.
How to formalize differentiation

We want \( f \in \mathcal{L}(!X, Y) \) to be differentiable.

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Tf : \quad \langle x, u \rangle \mapsto \langle f(x), f'(x) \cdot u \rangle
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It performs a first order Taylor expansion.

- **Chain rule:** \( T \) is a functor in \( \mathcal{L}_! \);
- **Leibniz + Schwarz + Linearity of the derivatives:** naturality assumptions.
\begin{align*}
\nu_0 &\in \mathcal{L}(X, SX) \\
\theta &\in \mathcal{L}(S^2X, SX) \\
c &\in \mathcal{L}(S^2X, S^2X) \\
l &\in \mathcal{L}(SX, S^2X)
\end{align*}
\begin{align*}
\nu_0 \circ x &= \langle x, 0 \rangle \\
\theta \circ \langle \langle x, u \rangle, \langle v, w \rangle \rangle &= \langle x, u + v \rangle \\
c \circ \langle \langle x, u \rangle, \langle v, w \rangle \rangle &= \langle \langle x, v \rangle, \langle u, w \rangle \rangle \\
l \circ \langle x, u \rangle &= \langle \langle x, 0 \rangle, \langle 0, u \rangle \rangle
\end{align*}

All natural in \( \mathcal{L} \), \((S, \nu_0, \theta, \sigma, l, c)\) is a c-bimonad.
\( \nu_0 \in \mathcal{L}(X, SX) \) \hspace{1cm} \( \nu_0 \circ x = \langle x, 0 \rangle \)

\[ \theta \in \mathcal{L}(S^2X, SX) \] \hspace{1cm} \( \theta \circ \langle \langle x, u \rangle, \langle v, w \rangle \rangle = \langle x, u + v \rangle \)

\[ c \in \mathcal{L}(S^2X, S^2X) \] \hspace{1cm} \( c \circ \langle \langle x, u \rangle, \langle v, w \rangle \rangle = \langle \langle x, v \rangle, \langle u, w \rangle \rangle \)

\[ l \in \mathcal{L}(SX, S^2X) \] \hspace{1cm} \( l \circ \langle x, u \rangle = \langle \langle x, 0 \rangle, \langle 0, u \rangle \rangle \)

All natural in \( \mathcal{L} \), \( (S, \nu_0, \theta, \sigma, l, c) \) is a c-bimonad.

Any \( f \in \mathcal{L}(X, Y) \) can be sent to a morphism in \( \mathcal{L}_!(X, Y) \) as follows:

\[
\text{Der } f = \quad !X \xrightarrow{\text{der}} X \xrightarrow{f} Y
\]

But if \( \alpha : S^k \Rightarrow \mathcal{L} S^l \) is a natural transformation, \( \text{Der } \alpha \) has no reason to be a natural transformation \( T^k \Rightarrow \mathcal{L}_! T^l \).
Axioms of differentiation as naturality

Axioms of differentiation

- Der $\iota_0$ and Der $\theta$ are natural in $\mathcal{L}_!$ (Leibniz)
- Der $c$ is natural in $\mathcal{L}_!$ (Schwarz)
- Der $l$ is natural in $\mathcal{L}_!$ (The differential is linear)

$$SX = X \& X \iff \mathcal{L}_! \text{ is a Cartesian Differential Category}$$

Der $\sigma$ is not natural in $\mathcal{L}_!$: $f(x + u) \neq f(x) + f'(x) \cdot u$
Axioms of differentiation as naturality

Axioms of differentiation

- \( \text{Der}_{\nu_0} \) and \( \text{Der}_\theta \) are natural in \( L_! \) (Leibniz)
- \( \text{Der}_c \) is natural in \( L_! \) (Schwarz)
- \( \text{Der}_I \) is natural in \( L_! \) (The differential is linear)

\[ SX = X \& X \iff L_! \text{ is a Cartesian Differential Category} \]

\( \text{Der}_\sigma \) is not natural in \( L_! \): \( f(x + u) \neq f(x) + f'(x) \cdot u \)

Observe that the axioms above are only about \( L_! \): it is possible to directly axiomatize coherent differentiation in any CCC.

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Coherent Differential Category
(Linear Logic)  Kleisli  Cartesian Coherent Differential Category
(first order \( \lambda \)-calculus)
Distributive law

If $f \in \mathcal{L}(!X, Y)$, $Tf \in \mathcal{L}(!SX, SY)$.

Chain Rule

$T$ is a functor in $\mathcal{L}$.
Distributive law

If $f \in \mathcal{L}(!X, Y)$, $Tf \in \mathcal{L}(!SX, SY)$.

**Chain Rule**

$T$ is a functor in $\mathcal{L}_!$ that extend $S$ to $\mathcal{L}_!$:

$$T(\text{Der } f) = \text{Der } (Sf)$$

The differential of a linear map is the map itself

The existence of such functor is equivalent to the existence of a distributive law $\partial \in \mathcal{L}(!SX, S!X)$.

$$Tf = !SX \xrightarrow{\partial} S!X \xrightarrow{Sf} SY$$

Naturality of $\text{Der } (\alpha)$ in $\mathcal{L}_!$ is equivalent to a coherence diagrams between $\alpha$ and $\partial$. 
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An order 2 coherent Taylor expansion

Coherent differentiation is a first order Taylor expansion.

\[ \mbox{T}_f : \mathcal{S}X \rightarrow \mathcal{S}Y \]
\[ \langle x, u \rangle \mapsto \langle f(x), f'(x) \cdot u \rangle \]
An order 2 coherent Taylor expansion

Coherent differentiation is a first order Taylor expansion.

\[ T_f : \quad SX \rightarrow SY \]
\[ \langle x, u \rangle \mapsto \langle f(x), f'(x) \cdot u \rangle \]

But we can do the same for second order

\[ T_f : \quad SX \rightarrow SY \]
\[ \langle x, u, v \rangle \mapsto \langle f(x), f'(x) \cdot u, \frac{1}{2}f''(x) \cdot (u, u) + f'(x) \cdot v \rangle \]
An order 2 coherent Taylor expansion

Coherent differentiation is a first order Taylor expansion.

\[ Tf : \quad SX \rightarrow SY \quad \langle x, u \rangle \mapsto \langle f(x), f'(x) \cdot u \rangle \]

But we can do the same for second order

\[ Tf : \quad SX \rightarrow SY \quad \langle x, u, v \rangle \mapsto \langle f(x), f'(x) \cdot u, \frac{1}{2} f''(x) \cdot (u, u) + f'(x) \cdot v \rangle \]

We can express order 2 Taylor expansion

\[ Tf \, \langle x, u, 0 \rangle = \langle f(x), f'(x) \cdot u, \frac{1}{2} f''(x) \cdot (u, u) \rangle \]

but the red term is necessary for compositionality.
The whole Taylor expansion

Going to infinity

Introduce **infinitary summability** structures:

\[
SX = \{ \langle x_i \rangle_{i=0}^{\infty} \mid \sum_{i=0}^{\infty} x_i \text{ is defined} \}
\]

Taylor expansion is still a functor

\[
Tf \langle x_i \rangle_{i=0}^{\infty} = \langle \sum_{m \in \mathcal{M}(n)} \frac{1}{m!} \frac{d^{|m|} f}{d|m|x}(x_0) \cdot \vec{x}_m \rangle_{n=0}^{\infty}
\]

- \( \mathcal{M}(n) \) is the set of multiset\( s m \in \mathcal{M}_{\text{fin}}(\mathbb{N}^*) \) s.t. \( \sum_{i \in \mathbb{N}^*} i \cdot m(i) = n \)
- \( m! = \prod_{i \in \mathbb{N}^*} m(i)! \)
- \( \vec{x}_m = (x_1, \ldots, x_1, \ldots, x_i, \ldots, x_i, \ldots, x_n, \ldots, x_n) \)
  - \( m(1) \) times
  - \( m(i) \) times
  - \( m(n) \) times
Going to infinity

We recover the usual Taylor expansion.

\[ T_f \left\langle x, u, 0, \ldots \right\rangle = \left\langle \frac{1}{n!} \frac{d^n f}{d^n x}(x) \cdot (u, \ldots, u) \right\rangle_{i=0}^{\infty} \]

Morphisms are analytic: naturality of \( \text{Der} \ \sigma \)

\[ f \circ (x + u) = f \circ \text{Der} \ \sigma \circ \left\langle x, u, 0, \ldots \right\rangle = \text{Der} \ \sigma \circ T_f \circ \left\langle x, u, 0, \ldots \right\rangle = \sum_{n \in \mathbb{N}} \frac{1}{n!} \frac{d^n f}{d^n x}(x) \cdot (u, \ldots, u) \]
The whole Taylor expansion

Same axioms as Coherent Differentiation

- $T$ is a functor that extends $S$ (Chain rule)
- $\text{Der} \nu_0$ and $\text{Der} \theta$ are natural in $\mathcal{L}_!$
- $\text{Der} c$ is natural in $\mathcal{L}_!$
- $\text{Der} l$ is natural in $\mathcal{L}_!$
- $\text{Der} \sigma$ is natural in $\mathcal{L}_!$ (Morphisms are analytic)

Taylor expansion $= (T, \nu_0, \theta, \sigma, l, c)$ is a $c$-bimonad in $\mathcal{L}_!$
Same axioms as Coherent Differentiation

- $T$ is a functor that extends $S$ (Chain rule)
- $\text{Der} \, \iota_0$ and $\text{Der} \, \theta$ are natural in $\mathcal{L}_!$
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- $\text{Der} \, l$ is natural in $\mathcal{L}_!$
- $\text{Der} \, \sigma$ is natural in $\mathcal{L}_!$ (Morphisms are analytic)

Taylor expansion $= (T, \iota_0, \theta, \sigma, l, c)$ is a c-bimonad in $\mathcal{L}_!$

Again, we can give the axioms in any CCC.
The axioms of infinitary summability structures are essentially the same as
the axioms of summability structure.

\[ \mathcal{L}(X, Y) \text{ is a } \Sigma\text{-monoid.} \]

In particular, coefficients are positive: if \( x + y = 0 \) then \( x = y = 0 \).
The axioms of infinitary summability structures are essentially the same as the axioms of summability structure.

\[ L(X, Y) \text{ is a } \Sigma\text{-monoid.} \]

In particular, coefficients are positive: if \( x + y = 0 \) then \( x = y = 0 \).

It excludes finiteness spaces and Köethe spaces.

- I am working on a new axiomatization of summability to include those
- In this axiomatization, \( L(X, Y) \) is a Partial Commutative Monoid (PCM), see Hines 2013
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Ehrhard introduced a coherent differential PCF whose semantic is based on coherent differentiation. We can adapt this calculus to get a PCF with Taylor expansion.

We can define $\mathcal{T}$ that performs Taylor expansion in 0.

- If $\Gamma \vdash M : A$ then $\Gamma \vdash \mathcal{T}(M) : TA$
- $[\mathcal{T}(M)] = T[M] \circ 0, \text{id}, 0 \ldots$.

So we can compile a term into an infinite sum of terms

$$M \mapsto \sum_{i \in \mathbb{N}} \pi_i(\mathcal{T}(M))$$

In any analytic category

$$[M] = \sum_{i \in \mathbb{N}} [\pi_i(\mathcal{T}(M))]$$
Link between summability and normalization

The $\pi_i(T(M))$ are not resource term. Fortunately, they should be **normalizing** and reduce to finite sums of normal terms (Work In Progress).

**Crucial observation**

The $\text{nf}(\pi_i(T(M)))$ have disjoint support

We can show it in 3 lines using normalization and the fact that the $\llbracket \pi_i(T(M)) \rrbracket$ are summable in **coherent spaces**.
Plan

1. Taylor expansion in the lambda calculus
2. The puzzling interpretation of sums
3. Coherent Differentiation
4. The whole Taylor expansion
5. Going back to syntax
6. Taylor expansion in the elementary case
7. Conclusion
Elementary summability structure

It turns out that often:

- **Binary summability structure:** $SX = 1 \& 1 \rightarrow X$
- **Infinitary summability structure:** $SX = \&_{i \in \mathbb{N}}1 \rightarrow X$

We call such summability structures **elementary.**
Elementary summability structure

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**Example**

The set \([0, 1] \) is a partial commutative monoid: \( x_0, x_1 \in [0, 1] \) are summable if \( x_0 + x_1 \leq 1 \).

This is equivalent to the existence of \( \varphi : [0, 1]^2 \to X \) such that

\[
    x_0 = \varphi(1, 0) \quad x_1 = \varphi(0, 1) \quad x_0 + x_1 = \varphi(1, 1)
\]

Take \( \varphi : (\lambda, \mu) \mapsto \lambda x_0 + \mu x_1 \).
Elementary summability structure

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The bimonoid structure of $\mathbb{D}$

Let $\mathbb{D} = \&_\mathbb{N} 1$. The same results work for $1 \& 1$ for the exact same reasons.

In $\text{Rel}$

$$\mathbb{D} = \mathbb{N}$$

and

$$\text{SX} = \mathbb{N} \times X$$

Bimonoid structure

Co-unit $p_0 \in \mathcal{L}(\mathbb{D}, 1)$ $p_0 = \{(0, *)\}$

Co-multitiplication $\tilde{\theta} \in \mathcal{L}(\mathbb{D}, \mathbb{D} \otimes \mathbb{D})$ $\tilde{\theta} = \{(n, (i, j)) \mid n = i + j\}$

Unit $\Delta \in \mathcal{L}(1, \mathbb{D})$ $\Delta = \{(*, n) \mid n \in \mathbb{N}\}$

Multiplication $\tilde{l} \in \mathcal{L}(\mathbb{D} \otimes \mathbb{D}, \mathbb{D})$ $\tilde{l} = \{((n, n), n) \mid n \in \mathbb{N}\}$
Relating the bimonad $S$ to a bimonoid structure on $D$

$SX = D \rightarrow X$ is the right adjoint of the functor $\_ \otimes D$.

<table>
<thead>
<tr>
<th>Bimonoid $D$</th>
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<th>Bimonad $S$</th>
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<tbody>
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Relating the bimonad $S$ to a bimonoid structure on $\mathcal{D}$

$SX = \mathcal{D} \to X$ is the right adjoint of the functor $\_ \otimes \mathcal{D}$.

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Coalgebra

$\tilde{\partial} \in \mathcal{L}(\mathcal{D}, !\mathcal{D})$

Distributive law

$\mathcal{L}(!X \otimes \mathcal{D}, !(X \otimes \mathcal{D}))$

Distributive law $\partial$

$\mathcal{L}(!SX, S!X)$

Compatibility between bimonoid and coalgebra

$\Leftrightarrow$ Compatibility between bimonad and $\partial$. 
The coalgebra structure of $\mathbb{D}$

In \textbf{Rel}: $\tilde{\partial} \in \text{Rel}(\mathbb{D}, \!\mathbb{D})$

$$\tilde{\partial} = \{ (n, [i_1, \ldots, i_k]) \mid n = i_1 + \cdots + i_k \}$$

Induce $\partial \in \mathcal{L}(\!S\!X, S\!\!X)$. Explicitely, it is the curryfication of

$$!(\mathbb{D} \to X) \otimes \mathbb{D} \xrightarrow{!(\mathbb{D} \to X) \otimes \tilde{\partial}} !(\mathbb{D} \to X) \otimes \!\mathbb{D} \xrightarrow{\mu^2} !((\mathbb{D} \to X) \otimes \mathbb{D}) \xrightarrow{!\text{ev}} !X$$
The coalgebra structure of $D$

In $\text{Rel}$: $\tilde{\partial} \in \text{Rel}(D, !D)$

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Induce $\partial \in \mathcal{L}(!SX, S!X)$. Explicitely, it is the curryfication of

$$!(D \to X) \otimes D \xrightarrow{!(D \to X) \otimes \tilde{\partial}} !(D \to X) \otimes !D \xrightarrow{\mu^2} !((D \to X) \otimes D) \xrightarrow{!\text{ev}} !X$$

To sum up

If $S$ is elementary then the category is analytic if and only if there is a coalgebra $\tilde{\partial}$ compatible with the bimonoid structure of $\tilde{\partial}$.

In particular: this is always the case if $!\_!$ is Lafont.
A curious isomorphism

Comonoid $\mathbb{D}$ in $\text{Rel}$

Co-unit $\rho_0 \in \text{Rel}(\mathbb{D}, 1)$ $\rho_0 = \{(0, \ast)\}$

Co-multitipation $\tilde{\theta} \in \text{Rel}(\mathbb{D}, \mathbb{D} \otimes \mathbb{D})$ $\tilde{\theta} = \{(n, (i, j)) \mid n = i + j\}$

Comonoid $!1$ in $\text{Rel}$: $!1 = \{[*], \ldots, [*] \mid n \in \mathbb{N}\} = \{n \cdot [*] \mid n \in \mathbb{N}\}$

$wk \in \text{Rel}(!1, 1)$ $\{([], \ast)\}$

$ctr \in \text{Rel}(!1, !1 \otimes !1)$ $\{(n \cdot [*], (i \cdot [*], j \cdot [*])) \mid n = i + j\}$

$\mathbb{D} \equiv !1$
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Takeaway

- Axiomatization of Taylor expansion with partial countable sums (Σ-monoids or PCM)
- Axioms of differentiation: functoriality and naturality
- It covers a wide range of models, especially those with Lafont exponential
- In syntax, Taylor expansion produce an infinite sums of normalizing terms whose semantics is summable in any model
- Different models can provide different insights on how the syntactical sum behaves, especially coherence spaces