A new categorical axiomatization of Taylor expansion

Thomas Ehrhard   Aymeric Walch

IRIF

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Plan

Differentiable map and analytic maps

Differential: A function $f : E \to F$ is differentiable in x if

$$\boxed{f(x+u) \simeq f(x) + f'(x) \cdot u}$$

With $f'(x) : E \multimap F$ a linear map, that is, $f' : E \to (E \multimap F)$

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Taylor expansion: A function $f : E \to F$ is analytic around x if

$$\boxed{f(x+u) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) \cdot (\underbrace{u, \ldots, u}_{n \text{ times}})}$$

Differentiation in Linear Logic

Reminder: if $\mathcal{L}$ is a model of linear logic, then $\mathcal{L}_!$ where

$$\mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$$

is a model of λ-calculus (CCC)

In finiteness spaces and Koethe spaces

- morphisms in $\mathcal{L}(X, Y)$ are linear maps
- morphisms in $\mathcal{L}_!(X, Y)$ are analytic maps

It means that programs are interpreted as analytic maps that can be computed by the sum of their derivatives!

Differential and Taylor expansion in calculus

Differential in terms

If $\Gamma, x : A \vdash P : B$ and $\Gamma \vdash Q : A$

$$\Gamma, x : A \vdash \frac{\partial P}{\partial x} \cdot Q : B$$

substitute <u>one</u> occurrence of x by Q in P.

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Taylor Expansion

$$(\lambda x.P)Q \rightsquigarrow \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n P}{\partial x^n} \cdot (\underbrace{Q, \ldots, Q}_{\text{n times}}) \right) [0/x]$$

We can do Taylor expansion in all applications at once

Taylor expansion

$$\begin{array}{l} \text{Lambda Term } M \\ \text{regular application} \end{array} \xrightarrow[\mathcal{T}]{\text{Taylor expansion}} \begin{array}{l} \text{Infinite sum } \sum_{i \in I} r_i \cdot t_i \\ t_i \text{ Ressource Terms, } r_i \in \mathbb{Q} \\ \text{multilinear application} \end{array}$$

Resource terms: $t, u = \lambda x.t \mid x \mid \langle t \rangle \cdot \overline{u}$

Bags/Multisets: $\overline{u} = [u_1, \ldots, u_n]$

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Redex: if x occurs n times in t

$$\langle \lambda x.t \rangle \cdot [u_1, \ldots, u_n] \to \sum_{\sigma} t[u_{\sigma(1)}/x_1, \ldots, u_{\sigma(n)}/x_n]$$

Resource term are strongly normalizing: any resource term t reduces to a finite sum $\mathrm{nf}(t)$ of normal resource terms.

The sum in the Taylor expansion is not arbitrary

Taylor expansion has a normal form: if $\mathcal{T}(M) = \sum_{i \in I} r_i \cdot t_i$

$$\mathsf{nf}(\mathcal{T}(M)) = \sum_{i \in I} r_i \cdot \mathsf{nf}(t_i)$$

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Crucial observation

If $\mathcal{T} M = \sum_{i \in I} r_i \cdot t_i$, the finite sums $\mathrm{nf}(t_i)$ are of disjoint support.

Proof based on Uniformity properties.
Another proof of normalization by Lionel Vaux is based on finiteness spaces.

Plan

Differential Categories and Cartesian Differential Categories

Differential Category[1] $\mathcal{L}$
(Linear Logic)

[1] Blute, Cockett, and Seely 2006.
[2] Blute, Cockett, and Seely 2009.

Differential Categories and Cartesian Differential Categories

Recall: $\mathcal{L}_!(X, Y) := \mathcal{L}(!X, Y)$ is a CCC

$$\begin{tikzcd} \boxed{\begin{array}{c}\text{Differential} \\ \text{Category}^1\ \mathcal{L} \\ \text{(Linear Logic)}\end{array}} \arrow[r, "\text{Kleisli}"] & \boxed{\begin{array}{c}\text{Cartesian Differential} \\ \text{Category}^2\ \mathcal{C} \\ \text{(first order } \lambda\text{-calculus)}\end{array}} \end{tikzcd}$$

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Differential Categories and Cartesian Differential Categories

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$\boxed{\text{Cartesian Differential Category}^2\ \mathcal{C}\ \text{(first order } \lambda\text{-calculus)}}$

- Compatibility with the CCC struture: models of differential λ-calculus Bucciarelli, Ehrhard, and Manzonetto 2010
- Models for Taylor expansion Manzonetto 2012

Example: relation model, weighted relational model.

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(Left) additivity and non determinism

Leibniz: $f'(x,y)\cdot(u,v) = \partial_0 f(x,y)\cdot u + \partial_1 f(x,y)\cdot v$

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A Differential Category $\mathcal{L}$ must be left additive

- $\mathcal{L}(X, Y)$ is a commutative monoïd
- $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$ (left additive)
- $h \circ (f_1 + f_2) = h \circ f_1 + h \circ f_2$ (additive)

We exclude coherence spaces and probabilistic coherence spaces

If we want Taylor expansion, we do even worse: we allow for arbitrary countable sums. We exclude finiteness spaces.

Plan

A categorical axiomatization of partial sum

Let $\mathcal{L}$ be a model of linear logic.

A structure for partial sum

Functor S. Intuitively, $\mathsf{S}X = \{\langle\!\langle x_0, x_1 \rangle\!\rangle \mid x_0 + x_1 \text{ is defined}\}$

- $\pi_0, \pi_1 \in \mathcal{L}(\mathsf{S}X, X)$ jointly monic $\pi_i : \langle\!\langle x_0, x_1 \rangle\!\rangle \mapsto x_i$

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- $f_0, f_1 \in \mathcal{L}(X, Y)$ summable: $\exists\ \langle\!\langle f_0, f_1 \rangle\!\rangle \in \mathcal{L}(X, \mathrm{S}Y)$ s.t. $\pi_i \circ \langle\!\langle f_0, f_1 \rangle\!\rangle = f_i$
  $$\langle\!\langle f_0, f_1 \rangle\!\rangle : x \mapsto \langle\!\langle f_0(x), f_1(x) \rangle\!\rangle$$

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- $f_0, f_1 \in \mathcal{L}(X, Y)$ <u>summable</u>: $\exists\ \langle\!\langle f_0, f_1 \rangle\!\rangle \in \mathcal{L}(X, \mathsf{S}Y)$ s.t. $\pi_i \circ \langle\!\langle f_0, f_1 \rangle\!\rangle = f_i$
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$$\boxed{\mathsf{S}X = X \,\&\, X \iff \text{Cartesian Additive Category}}$$

Partial commutative monoid

summability structure : $(\mathsf{S}, \pi_0, \pi_1, \sigma)$ and some axioms:

- A zero morphism 0 neutral with regard to the sum: $f + 0 = 0 + f = f$
- Commutativity of sums $f + g \cong g + f$
- Associativity $f + (g + h) \cong (f + g) + h$

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- Naturality of π_i: $\mathsf{S}f : \langle\langle x_0, x_1 \rangle\rangle \mapsto \langle\langle f \cdot x_0, f \cdot x_1 \rangle\rangle$
- Naturality of σ: $f \circ (h_0 + h_1) \sqsubseteq f \circ h_0 + f \circ h_1$

How to formalize differentiation

We want $f \in \mathcal{L}(!X, Y)$ to be differentiable.

$$f(x+u) \simeq f(x) + f'(x) \cdot u$$

So differentiation should be interpreted as an operator T that maps $f \in \mathcal{L}(!X, Y)$ to $\mathsf{T}f \in \mathcal{L}(!\mathsf{S}X, \mathsf{S}Y)$

$$\begin{array}{cccc} \mathsf{T}f : & \mathsf{S}X & \to & \mathsf{S}Y \\ & \langle\!\langle x, u \rangle\!\rangle & \mapsto & \langle\!\langle f(x), f'(x).u \rangle\!\rangle \end{array}$$

It performs a first order Taylor expansion.

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- Chain rule: T is a functor in $\mathcal{L}_!$;
- Leibniz + Schwarz + Linearity of the derivatives: naturality assumptions.

$$\iota_0 \in \mathcal{L}(X, \mathsf{S}X) \qquad \iota_0 \circ x = \langle\!\langle x, 0 \rangle\!\rangle$$

$$\boxed{\theta \in \mathcal{L}(\mathsf{S}^2 X, \mathsf{S}X)} \qquad \theta \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle , \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle = \langle\!\langle x, u + v \rangle\!\rangle$$

$$\mathsf{c} \in \mathcal{L}(\mathsf{S}^2 X, \mathsf{S}^2 X) \qquad \mathsf{c} \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle , \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle = \langle\!\langle \langle\!\langle x, v \rangle\!\rangle , \langle\!\langle u, w \rangle\!\rangle \rangle\!\rangle$$

$$\mathsf{l} \in \mathcal{L}(\mathsf{S}X, \mathsf{S}^2 X) \qquad \mathsf{l} \circ \langle\!\langle x, u \rangle\!\rangle = \langle\!\langle \langle\!\langle x, 0 \rangle\!\rangle , \langle\!\langle 0, u \rangle\!\rangle \rangle\!\rangle$$

All natural in $\mathcal{L}$, $(\mathsf{S}, \iota_0, \theta, \sigma, \mathsf{l}, \mathsf{c})$ is a c-bimonad.

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All natural in $\mathcal{L}$, $(\mathsf{S}, \iota_0, \theta, \sigma, \mathsf{l}, \mathsf{c})$ is a c-bimonad.

Any $f \in \mathcal{L}(X, Y)$ can be sent to a morphism in $\mathcal{L}_!(X, Y)$ as follows:

$$\mathsf{Der}\, f = \; !X \xrightarrow{\mathsf{der}} X \xrightarrow{f} Y$$

But if $\alpha : \mathsf{S}^k \Rightarrow_{\mathcal{L}} \mathsf{S}^l$ is a natural transformation, $\mathsf{Der}\,\alpha$ has no reason to be a natural transformation $\mathsf{T}^k \Rightarrow_{\mathcal{L}_!} \mathsf{T}^l$.

Axioms of differentiation as naturality

Axioms of differentiation

- $\mathrm{Der}\,\iota_0$ and $\mathrm{Der}\,\theta$ are natural in $\mathcal{L}_!$ (Leibniz)
- $\mathrm{Der}\,\mathsf{c}$ is natural in $\mathcal{L}_!$ (Schwarz)
- $\mathrm{Der}\,\mathsf{l}$ is natural in $\mathcal{L}_!$ (The differential is linear)

$$SX = X \,\&\, X \iff \mathcal{L}_! \text{ is a Cartesian Differential Category}$$

$\mathrm{Der}\,\sigma$ is not natural in $\mathcal{L}_!$: $f(x+u) \neq f(x) + f'(x) \cdot u$

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Der σ is not natural in $\mathcal{L}_!$: $f(x+u) \neq f(x) + f'(x) \cdot u$

Observe that the axioms above are only about $\mathcal{L}_!$: it is possible to directly axiomatize coherent differentiation in any CCC.

$$\boxed{\text{Coherent Differential Category (Linear Logic)}} \xrightarrow{\text{Kleisli}} \boxed{\text{Cartesian Coherent Differential Category (first order } \lambda\text{-calculus)}}$$

Distributive law

If $f \in \mathcal{L}(!X, Y)$, $\mathsf{T}f \in \mathcal{L}(!\mathsf{S}X, \mathsf{S}Y)$.

Chain Rule

T is a functor in $\mathcal{L}_!$

Distributive law

If $f \in \mathcal{L}(!X, Y)$, $\mathsf{T}f \in \mathcal{L}(!\mathsf{S}X, \mathsf{S}Y)$.

Chain Rule

T is a functor in $\mathcal{L}_!$ that extend S to $\mathcal{L}_!$:

$$\mathsf{T}(\mathrm{Der}\, f) = \mathrm{Der}\,(\mathsf{S}f)$$

The differential of a linear map is the map itself

The existence of such functor is equivalent to the existence of a distributive law $\partial \in \mathcal{L}(!\mathsf{S}X, \mathsf{S}!X)$.

$$\mathsf{T}f = \; !\mathsf{S}X \xrightarrow{\partial} \mathsf{S}!X \xrightarrow{\mathsf{S}f} \mathsf{S}Y$$

Naturality of $\mathrm{Der}\,(\alpha)$ in $\mathcal{L}_!$ is equivalent to a coherence diagrams between α and ∂.

Plan

An order 2 coherent Taylor expansion

Coherent differentiation is a first order Taylor expansion.

$$\begin{array}{rccc} \mathsf{T}f : & \mathsf{S}X & \to & \mathsf{S}Y \\ & \langle\!\langle x, u \rangle\!\rangle & \mapsto & \langle\!\langle f(x), f'(x) \cdot u \rangle\!\rangle \end{array}$$

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But we can do the same for second order

$$\begin{array}{rccc} \mathrm{T}f: & \mathrm{S}X & \to & \mathrm{S}Y \\ & \langle\!\langle x, u, v \rangle\!\rangle & \mapsto & \left\langle\!\!\left\langle f(x),\ f'(x)\cdot u,\ \frac{1}{2}f''(x)\cdot(u,u) + f'(x)\cdot v \right\rangle\!\!\right\rangle \end{array}$$

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$$\begin{array}{rccc} \mathsf{T}f: & SX & \to & SY \\ & \langle\!\langle x, u \rangle\!\rangle & \mapsto & \langle\!\langle f(x), f'(x)\cdot u \rangle\!\rangle \end{array}$$

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We can express order 2 Taylor expansion

$$\mathsf{T}f\ \langle\!\langle x, u, 0 \rangle\!\rangle = \left\langle\!\!\left\langle f(x),\ f'(x)\cdot u,\ \frac{1}{2}f''(x)\cdot(u,u) \right\rangle\!\!\right\rangle$$

but the red term is necessary for compositionality.

Going to infinity

Introduce infinitary summability structures:

$$\mathsf{S}X = \{\langle\!\langle x_i \rangle\!\rangle_{i=0}^{\infty} \mid \sum_{i=0}^{\infty} x_i \text{ is defined}\}$$

Taylor expansion is still a functor

$$\mathsf{T}f \ \langle\!\langle x_i \rangle\!\rangle_{i=0}^{\infty} = \langle\!\langle \sum_{m \in \mathcal{M}(n)} \frac{1}{m!} \frac{\mathrm{d}^{|m|} f}{\mathrm{d}^{|m|} x}(x_0) \cdot \vec{x}_m \rangle\!\rangle_{n=0}^{\infty}$$

- $\mathcal{M}(n)$ is the set of multisets $m \in \mathcal{M}_{fin}(\mathbb{N}^*)$ s.t. $\sum_{i \in \mathbb{N}^*} i \ m(i) = n$
- $m! = \prod_{i \in \mathbb{N}^*} m(i)!$
- $\vec{x}_m = (\underbrace{x_1, \ldots, x_1}_{m(1) \text{ times}}, \ldots, \underbrace{x_i, \ldots, x_i}_{m(i) \text{ times}}, \ldots, \underbrace{x_n, \ldots, x_n}_{m(n) \text{ times}})$

Going to infinity

We recover the usual Taylor expansion.

$$\mathrm{T}f \ \langle\!\langle x, u, 0, \ldots \rangle\!\rangle = \langle\!\langle \frac{1}{n!}\frac{\mathrm{d}^n f}{\mathrm{d}^n x}(x) \cdot (u, \ldots, u) \rangle\!\rangle_{i=0}^{\infty}$$

Morphisms are analytic: naturality of $\mathrm{Der}\,\sigma$

$$\begin{aligned} f \circ (x+u) &= f \circ \mathrm{Der}\,\sigma \circ \langle\!\langle x, u, 0, \ldots \rangle\!\rangle \\ &= \mathrm{Der}\,\sigma \circ \mathrm{T}f \circ \langle\!\langle x, u, 0, \ldots \rangle\!\rangle \\ &= \sum_{n \in \mathbb{N}} \frac{1}{n!}\frac{\mathrm{d}^n f}{\mathrm{d}^n x}(x) \cdot (u, \ldots, u) \end{aligned}$$

Same axioms as Coherent Differentiation

- T is a functor that extends S (Chain rule)
- $\mathrm{Der}\,\iota_0$ and $\mathrm{Der}\,\theta$ are natural in $\mathcal{L}_!$
- $\mathrm{Der}\,\mathsf{c}$ is natural in $\mathcal{L}_!$
- $\mathrm{Der}\,\mathsf{l}$ is natural in $\mathcal{L}_!$
- $\mathrm{Der}\,\sigma$ is natural in $\mathcal{L}_!$ (Morphisms are analytic)

Taylor expansion $= (\mathsf{T}, \iota_0, \theta, \sigma, \mathsf{l}, \mathsf{c})$ is a c-bimonad in $\mathcal{L}_!$

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$$\text{Taylor expansion} = (\mathsf{T}, \iota_0, \theta, \sigma, \mathsf{l}, \mathsf{c}) \text{ is a c-bimonad in } \mathcal{L}_!$$

Again, we can give the axioms in <u>any</u> CCC.

$$\boxed{\begin{array}{c}\text{Analytic Category}\\ \text{(Linear Logic)}\end{array}} \xrightarrow{\text{Kleisli}} \boxed{\begin{array}{c}\text{Cartesian Closed}\\ \text{Analytic Category}\\ \text{(first order } \lambda\text{-calculus)}\end{array}}$$

Notes on summability

The axioms of infinitary summability structures are essentially the same as the axioms of summability structure.

$$\mathcal{L}(X, Y) \text{ is a } \Sigma\text{-monoid.}$$

In particular, coefficients are positive: if $x + y = 0$ then $x = y = 0$.

Notes on summability

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It excludes finiteness spaces and Köethe spaces.

- I am working on a new axiomatization of summability to include those
- In this axiomatization, $\mathcal{L}(X, Y)$ is a Partial Commutative Monoid (PCM), see Hines 2013

Plan

PCF with probability and Taylor expansion

Ehrhard introduced a coherent differential PCF whose semantic is based on coherent differentiation. We can adapt this calculus to get a PCF with Taylor expansion.

We can define $\mathcal{T}$ that performs Taylor expansion in 0.

- If $\Gamma \vdash M : A$ then $\Gamma \vdash \mathcal{T}(M) : \mathrm{T}A$
- $[\![\mathcal{T}(M)]\!] = \mathrm{T}[\![M]\!] \circ \langle\!\langle 0, \mathrm{id}, 0 \ldots \rangle\!\rangle$.

So we can compile a term into an infinite sum of terms

$$M \mapsto \sum_{i \in \mathbb{N}} \pi_i(\mathcal{T}(M))$$

In any analytic category

$$[\![M]\!] = \sum_{i \in \mathbb{N}} [\![\pi_i(\mathcal{T}(M))]\!]$$

Link between summability and normalization

The $\pi_i(\mathcal{T}(M))$ are not resource term. Fortunately, they should be normalizing and reduce to finite sums of normal terms (Work In Progress).

Crucial observation

The $\mathrm{nf}(\pi_i(\mathcal{T}(M)))$ have disjoint support

We can show it in 3 lines using normalization and the fact that the $[\![\pi_i(\mathcal{T}(M))]\!]$ are summable in coherent spaces.

Plan

Elementary summability structure

It turns out that often:

$$\text{Binary summability structure: } SX = 1 \,\&\, 1 \multimap X$$

$$\text{Infinitary summability structure: } SX = \&_{i\in\mathbb{N}} 1 \multimap X$$

We call such summability structures <u>elementary</u>.

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Example

The set $[0,1]$ is a partial commutative monoid: $x_0, x_1 \in [0,1]$ are summable if $x_0 + x_1 \leq 1$.

This is equivalent to the existence of $\varphi : [0,1]^2 \multimap X$ such that

$$x_0 = \varphi(1,0) \qquad x_1 = \varphi(0,1)$$

$$x_0 + x_1 = \varphi(1,1)$$

Take $\varphi : (\lambda, \mu) \mapsto \lambda x_0 + \mu x_1$.

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The bimonoid structure of $\mathbb{D}$

Let $\mathbb{D} = \&_{\mathbb{N}} 1$. The same results work for $1 \& 1$ for the exact same reasons.
In **Rel**

$$\mathbb{D} = \mathbb{N}$$

and

$$\mathsf{S}X = \mathbb{N} \times X$$

Bimonoid structure

Co-unit	$\mathsf{p}_0 \in \mathcal{L}(\mathbb{D}, 1)$	$p_0 = \{(0, *)\}$
Co-multitiplication	$\widetilde{\theta} \in \mathcal{L}(\mathbb{D}, \mathbb{D} \otimes \mathbb{D})$	$\widetilde{\theta} = \{(n, (i, j)) \mid n = i + j\}$
Unit	$\Delta \in \mathcal{L}(1, \mathbb{D})$	$\Delta = \{(*, n) \mid n \in \mathbb{N}\}$
Multiplication	$\widetilde{\mathsf{I}} \in \mathcal{L}(\mathbb{D} \otimes \mathbb{D}, \mathbb{D})$	$\widetilde{\mathsf{I}} = \{((n, n), n) \mid n \in \mathbb{N}\}$

Relating the bimonad S to a bimonoid structure on $\mathbb{D}$

$\mathsf{S}X = \mathbb{D} \multimap X$ is the right adjoint of the functor $\_ \otimes \mathbb{D}$.

Bimonoid $\mathbb{D}$		Bimonad $\_ \otimes \mathbb{D}$		Bimonad S
Unit Δ		Unit		co-Unit σ
Multiplication $\widetilde{\mathsf{I}}$		Multiplication		co-Multiplication I
co-Unit p_0	$\Leftrightarrow$	co-Unit	$\overset{\text{mates}}{\Leftrightarrow}$	Unit ι_0
co-Multiplication $\widetilde{\theta}$		co-Multiplication		Multiplication θ
Commutativity γ		Commutativity		Commutativity c

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Bimonoid $\mathbb{D}$		Bimonad $\_ \otimes \mathbb{D}$		Bimonad S
Unit Δ Multiplication $\widetilde{\mathsf{I}}$ co-Unit p_0 co-Multiplication $\widetilde{\theta}$ Commutativity γ	$\Leftrightarrow$	Unit Multiplication co-Unit co-Multiplication Commutativity	$\overset{\text{mates}}{\Leftrightarrow}$	co-Unit σ co-Multiplication I Unit ι_0 Multiplication θ Commutativity c
Coalgebra $\widetilde{\partial} \in \mathcal{L}(\mathbb{D}, !\mathbb{D})$	$\Leftrightarrow$	Distributive law $\mathcal{L}(!X \otimes \mathbb{D}, !(X \otimes \mathbb{D}))$	$\overset{\text{mates}}{\Leftrightarrow}$	Distributive law ∂ $\mathcal{L}(!\mathsf{S}X, \mathsf{S}!X)$

Compatibility between bimonoid and coalgebra
$\iff$ Compatibility between bimonad and ∂.

The coalgebra structure of $\mathbb{D}$

In **Rel**: $\widetilde{\partial} \in \mathbf{Rel}(\mathbb{D}, !\mathbb{D})$

$$\widetilde{\partial} = \{(n, [i_1, \ldots, i_k]) \mid n = i_1 + \cdots + i_k\}$$

Induce $\partial \in \mathcal{L}(!\mathsf{S}X, \mathsf{S}!X)$. Explicitely, it is the curryfication of

$$!(\mathbb{D} \multimap X) \otimes \mathbb{D} \xrightarrow{!(\mathbb{D} \multimap X) \otimes \widetilde{\partial}} !(\mathbb{D} \multimap X) \otimes !\mathbb{D} \xrightarrow{\mu^2} !((\mathbb{D} \multimap X) \otimes \mathbb{D}) \xrightarrow{!\mathbf{ev}} !X$$

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To sum up

If S is elementary then the category is analytic if and only if there is a coalgebra $\widetilde{\partial}$ compatible with the bimonoid structure of $\widetilde{\partial}$.

In particular: this is always the case if $!\_$ is Lafont.

A curious isomorphism

Comonoid $\mathbb{D}$ in **Rel**

Co-unit $\quad \mathrm{p}_0 \in \mathbf{Rel}(\mathbb{D}, 1) \quad p_0 = \{(0, *)\}$

Co-multititiplication $\quad \widetilde{\theta} \in \mathbf{Rel}(\mathbb{D}, \mathbb{D} \otimes \mathbb{D}) \quad \widetilde{\theta} = \{(n, (i, j)) \mid n = i + j\}$

Comonoid !1 in **Rel**: $!1 = \{\underbrace{[*, \ldots, *]}_{n} \mid n \in \mathbb{N}\} = \{n \cdot [*] \mid n \in \mathbb{N}\}$

$$\mathrm{wk} \in \mathbf{Rel}(!1, 1) \quad \{([], *)\}$$

$$\mathrm{ctr} \in \mathbf{Rel}(!1, !1 \otimes !1) \quad \{(n \cdot [*], (i \cdot [*], j \cdot [*])) \mid n = i + j\}$$

$$\mathbb{D} \equiv !1$$

Plan

Takeaway

- Axiomatization of Taylor expansion with partial countable sums (Σ-monoids or PCM)
- Axioms of differentiation: functoriality and naturality
- It covers a wide range of models, especially those with Lafont exponential
- In syntax, Taylor expansion produce an infinite sums of normalizing terms whose semantics is summable in any model
- Different models can provide different insights on how the syntactical sum behaves, especially coherence spaces