Low symbolic discrepancy in the Tribonacci shift

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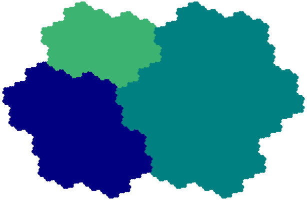
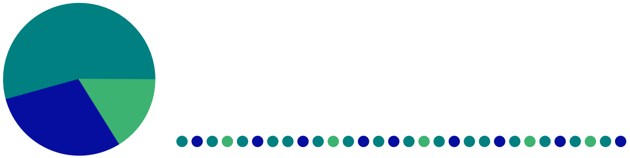


Outline

1. Chairperson Problem
2. Questions
3. Tribonacci Shift (and Fibonacci)

The chairperson assignement problem



- d states form a congress
- Each year a president for the congress is selected

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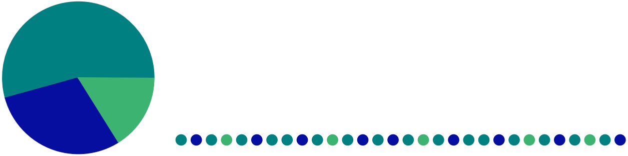

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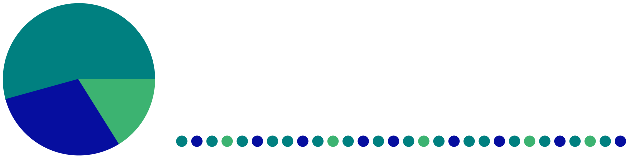
How to get a fair assignment?
What is fair?

The chairperson assignement problem



- d states form a congress
- Each year a president for the congress is selected

How to get a fair assignment?
What is fair?

- At any time, for each state, the accumulated number of presidents from that state is proportional to its weight

Modelling with symbolic dynamics

Consider $\mathcal{A} = \{\texttt{0,1, ..., d-1}\}$ an alphabet.

Take a sequence/word $(u_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$.

A frequency vector $\vec{\alpha} \in \mathbb{R}^d$ is such that

$$\sum_{a\in\mathcal{A}} \alpha_a = 1 \text{ and, for every } a \in \mathcal{A},\ \alpha_a \geq 0.$$

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The discrepancy of a word u given a frequency vector $\vec{\alpha}$ is:

$$\Delta_{\vec{\alpha}}(u) := \max_{a\in\mathcal{A}} \sup_{n\in\mathbb{N}} \left|\Delta^a(u_0 u_1 \cdots u_{n-1})\right|,$$

where, for every letter a,

$$\Delta^a(u_0 u_1 \cdots u_{n-1}) := \#\{0 \leq i < n : u_i = a\} - n\alpha_a.$$

Symbolic dynamics notation

The abelianization or Parikh vector of a finite word w is

$$\vec{l}(w) = (|w|_1, |w|_2, \ldots, |w|_d),$$

where $|w|_i$ denote the number of times the letter i occurs in w.

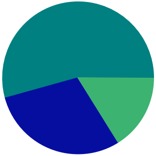
A factor of an infinite word u is a finite word denoted by $u_{[i,j)}$, $i < j$, with

$$u_{[i,j)} = u_i u_{i+1} \cdots u_{j-1}.$$

The frequency of a letter a in an infinite word u (if the limit exists) is:

$$\alpha_a = \lim_{n\to\infty} \frac{\#\{0 \leq i < n : u_i = a\}}{n}.$$

Modelling with symbolic dynamics



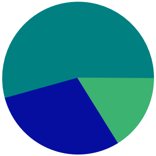
0102010010201010201001020102010201

0

$$\begin{cases} \Delta^0(0) = 1 - 0.54 = 0.46 \\ \Delta^1(0) = 0 - 0.30 = -0.30 \\ \Delta^2(0) = 0 - 0.16 = -0.16 \end{cases}$$

$$\max_{a \in \mathcal{A}} |\Delta^a(0)| = 0.46$$

Modelling with symbolic dynamics



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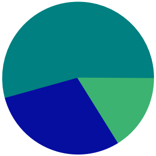
01

$$\begin{cases} \Delta^0(01) = 1 - 2 \cdot 0.54 = -0.08 \\ \Delta^1(01) = 1 - 2 \cdot 0.30 = 0.40 \\ \Delta^2(01) = 0 - 2 \cdot 0.16 = -0.32 \end{cases}$$

$$\max_{n \in \{1,2\}} \max_{a \in \mathcal{A}} |\Delta^a(u_{[0,n)})| = 0.46$$

Modelling with symbolic dynamics



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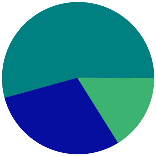
010

$$\begin{cases} \Delta^0(010) = 2 - 3 \cdot 0.54 = 0.38 \\ \Delta^1(010) = 1 - 3 \cdot 0.30 = 0.10 \\ \Delta^2(010) = 0 - 3 \cdot 0.16 = 0.48 \end{cases}$$

$$\max_{n \in \{1,2,3\}} \max_{a \in \mathcal{A}} |\Delta^a(u_{[0,n)})| = 0.48$$

Modelling with symbolic dynamics



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0102

$$\begin{cases} \Delta^0(0102) = 2 - 4 \cdot 0.54 = -0.16 \\ \Delta^1(0102) = 1 - 4 \cdot 0.30 = -0.20 \\ \Delta^2(0102) = 1 - 4 \cdot 0.16 = 0.36 \end{cases}$$

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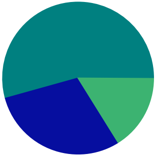
Modelling with symbolic dynamics



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$$\max_{n \in \{1,2,3\}} \max_{a \in \mathcal{A}} |\Delta^a(u_{[0,n)})| = 0.48$$

$$\Delta_\alpha(u) := \max_{a \in \mathcal{A}} \sup_{n \in \mathbb{N}} |\Delta_a(n)| \geq 0.48$$

Questions

Question. Given a word u, what is the asymptotic behaviour of its discrepancy? Is it bounded?

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Theorem. (B. Adamczewski'03,'04) The discrepancy is bounded in a substitution word of Pisot type.

A substitution is Pisot if its Perron-Frobenius eingenvalue has module bigger than one and all its algebraic conjugates have module less that 1 and its characterisct polynomial is irreductible.

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Question. Can we bound D_d?

Historical Development

Question. Can we bound $D_d = \sup_\alpha \inf_{u \in \mathcal{A}^{\mathbb{N}}} \Delta_\alpha(u)$?

- Niederreiter'72: $D_d \leq d - 1$
- Tijdeman'73: $D_d \leq 1 - \dfrac{1}{2d-2}$
- Meijer'73: $D_d = 1 - \dfrac{1}{2d-2}$
- Schneider'96: [$\vec{\alpha}$ coordinates are $\mathbb{Q}-$linearly independent]

$$\Delta_\alpha(u) \geq 1 - \frac{1}{d}$$

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Question. What are the shift dynamical systems (X_u, S) generated by a word u with d letters that contain an infinite word $v \in X_u$ that attains low discrepancy?

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Question. (refined) Is it true for the Tribonacci shift, i.e, is there $v \in X_u$ where $\Delta_{\vec{\alpha}}(v) < \frac{3}{4}$?

Tribonacci word

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$$u = 0102010010201\cdots$$

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Fact. $\sigma^{n+3}(0) = \sigma^{n+2}(0) \cdot \sigma^{n+1}(0) \cdot \sigma^n(0)$

Tribonacci Shift

Theorem (in preparation)

There exists a word w that belongs to the Tribonacci shift that achieves low discrepancy, that is, $\Delta_{\vec{\alpha}}(w) < \frac{3}{4}$.
The word w can be explicitly described as:

$$w = \lim_{k \to +\infty} 01\sigma(1)\sigma^2(2) \cdots \sigma^{4k}(1)\sigma^{4k+1}(1)\sigma^{4k+2}(2).$$

Discrepancy and balancedness: a friendship

The balancedness constant B_u of a word u is defined as:

$$\inf\{B > 0 \mid \forall v, w \text{ factors of } u, |v| = |w|, \forall a \in \mathcal{A},\ ||v|_a - |w|_a| \leq B\}$$

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Fact (modified B. Adamczweski'03)

For a minimal shift (X_u, S),

$$\sup_{v \in X_u} \Delta_{\vec{\alpha}}(v) \leq B_u \leq 4 \inf_{v \in X_u} \Delta_{\vec{\alpha}}(v)$$

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Corollary (G. Richomme, K. Saari, L. Zamboni'10)

The Tribonacci word is 2-balanced.

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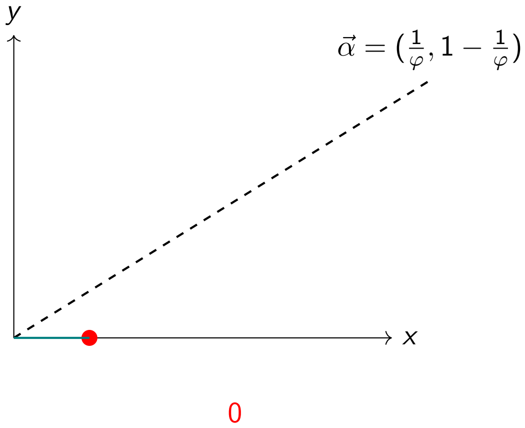
Proof.

$B_u < 4 \inf_{v \in X_u} \Delta_{\vec{\alpha}}(v) < 4 \cdot \frac{3}{4}$ and $|101|_1 - |020|_1 = 2$. □

Quick detour: Viewing discrepancy geometrically

Fibonacci word $u = 01001010010010100\cdots$, golden number φ



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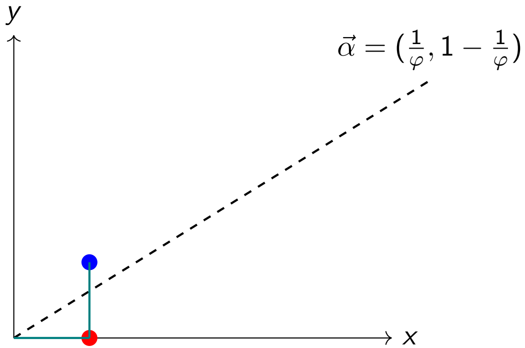

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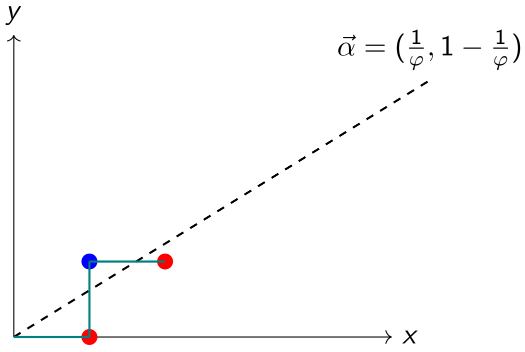

010

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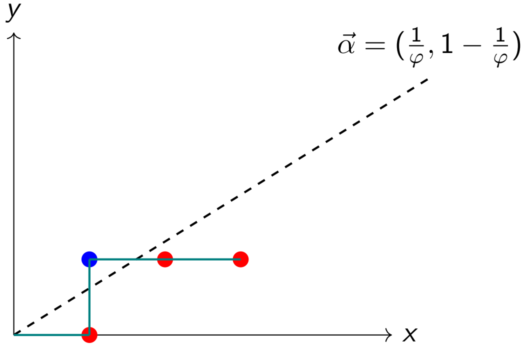

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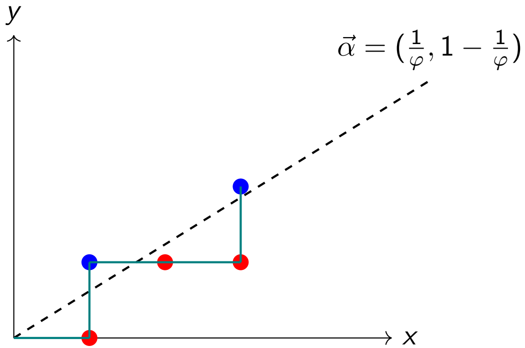

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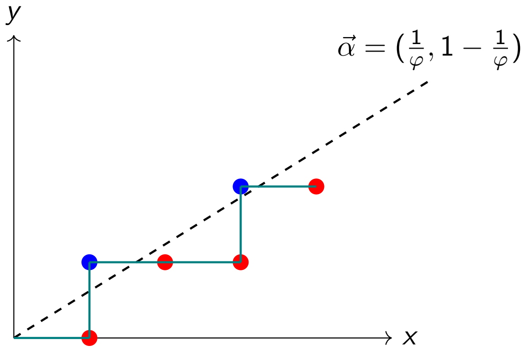

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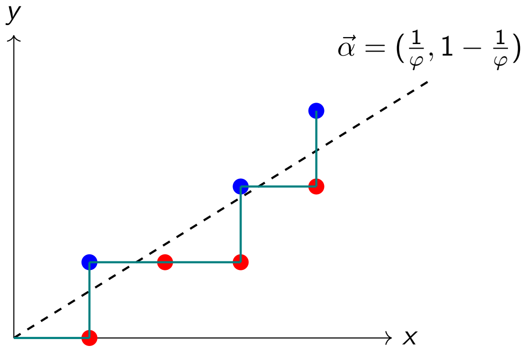

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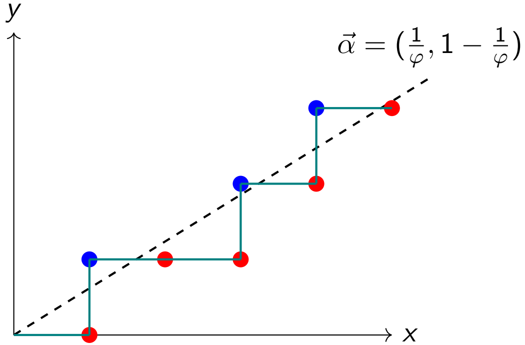

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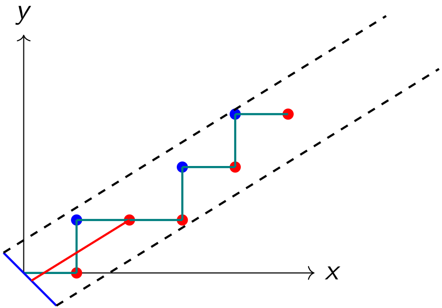
Viewing discrepancy geometrically

Each point is represented by a Parikh vector $\vec{l}(u_{[0,n)})$

Project along $\vec{\alpha}$ in $\mathbf{1}^{\perp}$: $\pi_{\alpha}(\vec{l}(u_{[0,n)})) = \vec{l}(u_{[0,n)}) - n\vec{\alpha}$



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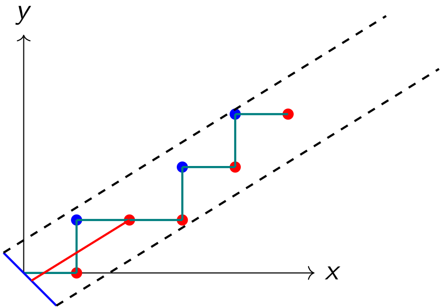
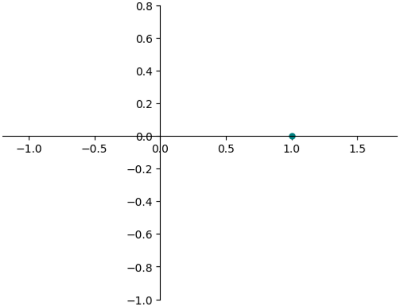


(Discrepancy) $\Delta_\alpha(u) = \sup_n \|\pi_{\vec{\alpha}}(\vec{l}(u_{[0,n)}))\|_\infty$

Tribonacci case: Rauzy Fractal

Geometry of the discrepancy of the Tribonacci word u:



$$u = 0 \cdots$$

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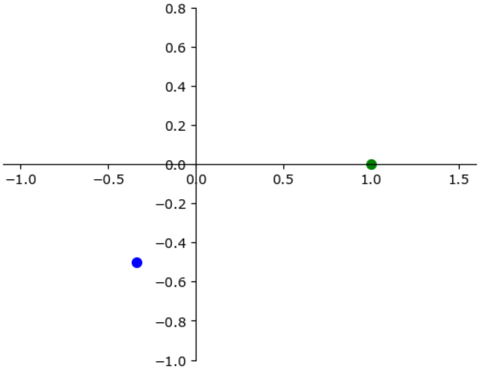

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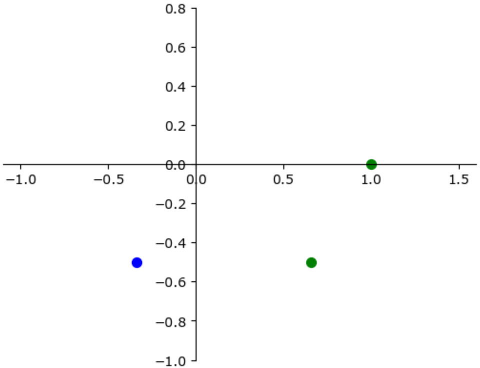

$$u = 010\cdots$$

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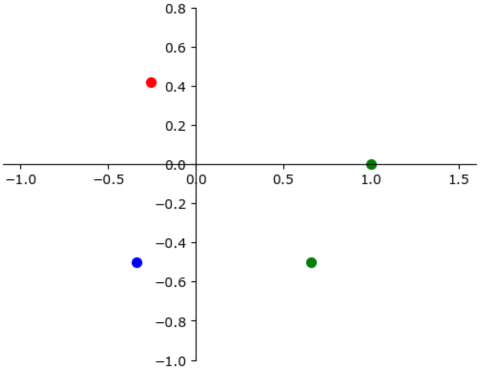

$$u = 0102\cdots$$

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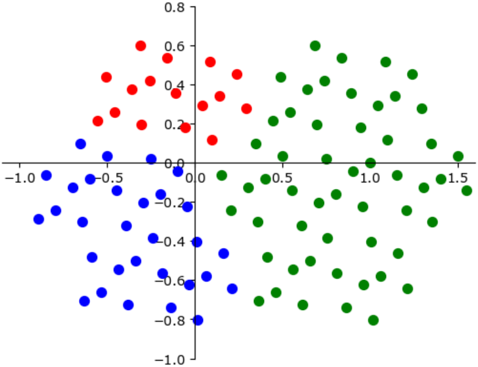

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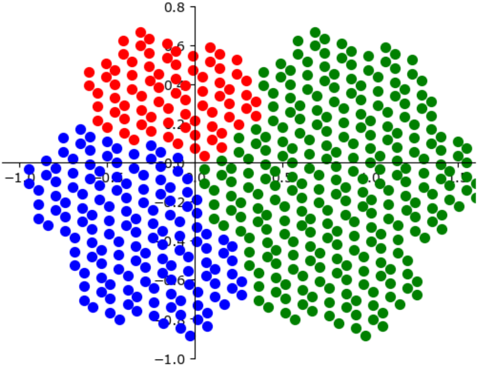

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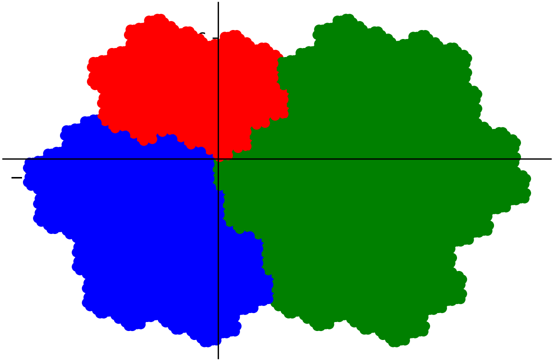
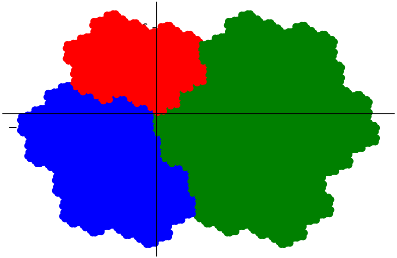


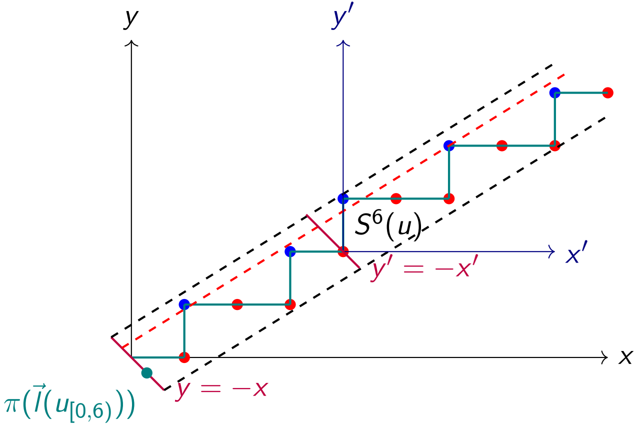
Main idea

- Encode the symmetrical point of the Rauzy Tribonacci fractal
- Prefix-suffix automaton
- (1989) Dumont-Thomas decomposition
- (2001) Canterini-Siegel decomposition
- Understanding the discrepancy with the Rauzy fractal



Tools

Question. What happens with the discrepancy when we shift u?



Proposition

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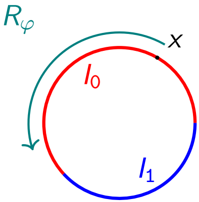
Let u be an infinite word. Consider the dynamical system (X_u, S) generated by u. Let $\vec{\alpha}$ be a frequency vector. Then, for every $k \in \mathbb{N}$,

$$\Delta_{\vec{\alpha}}(S^k(u)) = \sup_{n \in \mathbb{N}} |\|\pi_{\vec{\alpha}}(\vec{l}(u_{[0,n+k)}))\|_\infty - \|\pi_{\vec{\alpha}}(\vec{l}(u_{[0,k)}))\|_\infty|.$$

Fibonacci case: pause for rotation



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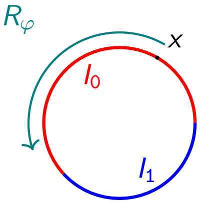

$$R_\varphi : [\varphi - 1, 2 - \varphi) \to [\varphi - 1, 2 - \varphi)$$
$$x \to x + \varphi \mod 1$$

Recall: $\frac{1}{\varphi} = \varphi - 1$

Orbit of x under R_φ coded by the word $w \in \{0,1\}^{\mathbb{Z}}$

$$w_i = a \iff R_\varphi^i(x) \in I_a$$

Fibonacci case

Complete understanding of discrepancy of every word in the shift:

Theorem (in preparation)

Let w be an infinite word in the Fibonacci shift encoded by the of orbit of x under the rotation R_φ. Its discrepancy satisfies:

$$\Delta_{\vec{\alpha}}(w) = \max(|\varphi - 1 - x|, |\varphi - x|).$$

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Corollary

There exists an infinite word w in the Fibonacci shift that atains minimum discrepancy, that is $\Delta_{\vec{\alpha}}(w) = \frac{1}{2}$. The word w can be described as:

$$w = \lim_{k \to +\infty} 01\sigma^2(0)\sigma^5(0)\ldots\sigma^{3k+2}(0).$$

Research direction

- Can it be generalized for *m*-bonacci shifts?
- What about other classes of substitutions? Can we find words in the shift which achieve low discrepancy, that is:

$$\exists\, v \in X_u, \ \ \Delta_{\vec{\alpha}}(v) \leq 1 - \frac{1}{2d-2},$$

where d is the number of letters?

- Do subshifts that achieve low discrepancy have other common (and nice) spectral properties, as pure discrete spectrum?

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Thank you!