Connectivity Guarantees for Wireless Networks with Directional Antennas

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Abstract

We study a combinatorial geometric problem related to the design of wireless networks with directional antennas. Specifically, we are interested in necessary and sufficient conditions on such antennas that enable one to build a connected communication network, and in efficient algorithms for building such networks when possible.

We formulate the problem by a set $\mathcal{P}$ of n points in the plane, indicating the positions of n transceivers. Each point is equipped with an α-degree directional antenna, and one needs to adjust the antennas (represented as wedges), by specifying their directions, so that the resulting (undirected) communication graph G is connected. (Two points $p, q \in \mathcal{P}$ are connected by an edge in G, if and only if q lies in p's wedge and p lies in q's wedge.) We prove that if $\alpha = 60^\circ$, then it is always possible to adjust the wedges so that G is connected, and that $\alpha \geq 60^\circ$ is sometimes necessary to achieve this. Our proof is constructive and yields an $O(n \log k)$ time algorithm for adjusting the wedges, where k is the size of the convex hull of $\mathcal{P}$.

Sometimes it is desirable that the communication graph G contain a Hamiltonian path. By a result of Fekete and Woeginger [8], if $\alpha = 90^\circ$, then it is always possible to adjust the wedges so that G contains a Hamiltonian path. We give an alternative proof to this, which is interesting, since it produces paths of a different nature than those produced by the construction of Fekete and Woeginger. We also show that for any n and $\varepsilon > 0$, there exist sets of points such that G cannot contain a Hamiltonian path if $\alpha = 90^\circ - \varepsilon$.

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1 Introduction

Most wireless networks use low-gain *omni-directional* antennas. Such antennas radiate power in all directions. In contrast, high-gain *directional antennas* radiate more power in a particular direction. The coverage area of a directional antenna is often modeled by a wedge, whose direction (and sometimes also angle) can be adjusted. Directional antennas enable energy conservation and interference reduction, which increases the network's efficiency, e.g., by increased throughput and reduced congestion. As in any network, the question of connectivity is fundamental. In wireless networks with directional antennas, we say that two stations u and v can (directly) communicate with each other if and only if u lies in the wedge defined by the antenna at v, and v lies in the wedge defined by the antenna at u (see Section 1.1 for a formal definition).

In this work we are interested in building a wireless network, using directional antennas, such that the resulting network is connected, i.e., any two stations can communicate with each other (possibly via other stations). Clearly this can be done using omni-directional antennas (which can be viewed as the extreme case of directional antennas). Therefore, the problem that arises is what is the smallest value α, for which a connected network with directional antennas of α degrees can be built. We prove (in Section 2) that for *any* set of n points in the plane, a connected network can be built with antennas of 60 degrees. Furthermore, we give an efficient $O(n \log k)$-time algorithm for directing these antennas, so that the resulting network is connected, where k is the size of the convex hull of the underlying set of points. This result is complemented by the (simple) observation that there exist sets of n points, for which one cannot build a connected network with antennas of less than 60 degrees, regardless of how one directs them. To the best of our knowledge this is the first work that deals with questions of connectivity in the present model, which we believe is the natural one (see Section 1.2). The results of this work are significant both from the combinatorial and computational geometric perspective, and from the point of view of designing wireless networks.

We also study a related (secondary) problem, in which one has to direct the antennas, so that the resulting communication graph contains a Hamiltonian path. By a result of Fekete and Woeginger [8], this is always possible with antennas of 90 degrees, since, as they show, for any set of n points in the plane, one can draw a (not necessarily simple) polygonal line, whose set of vertices is the given set of points, such that the (smaller) angle at each internal vertex is less than 90°. We present (in Section 3) an alternative proof of this statement, which is interesting, since it produces paths of a different nature than those produced by the construction of Fekete and Woeginger. The paths produced by our construction tend to have shorter edges and fewer self crossings. We also show that for any $\varepsilon > 0$, there exists a set of n points, such that any polygonal line through the points must make a turn of angle greater than $90^\circ - \varepsilon$.

1.1 Model

We consider a set $\mathcal{P}$ of points in the Euclidean plane, each point represents a communication station. Given an angle α, one can place at each node $v \in \mathcal{P}$ a *wedge* of angle α, centered at v. We say that u *sees* v if u lies in the the wedge centered at v. The *communication graph* is an undirected graph that consists of the node set $\mathcal{P}$ and the set of edges $E = \{(u,v) \,|\, u \text{ sees } v \text{ and } v \text{ sees } u\}$. One seeks to build *connected* communication graphs with, sometimes, additional properties.

1.2 Related work

Several previous papers deal with routing and topology control issues in networks of transceivers equipped with directional antennas, e.g., [15, 16]. These papers consider predominantly ad-hoc networks, where energy efficiency is an important issue. Several papers address connectivity and energy efficiency in ad-hoc networks when the network is built using omni-directional antennas, e.g., [4, 6, 11, 13]. To the best of our knowledge, only Caragiannis et al. [5] and van Nijnatten [14] address the problem of connectivity (and energy efficiency) in the case of directional antennas. However, the model used in [5] (as well as in [12, 14]) differs from ours, since it allows station v to transmit to station u if and only if u lies in the wedge of v. Thus Caragiannis et al. consider a *directed* communication graph and the *strong* connectivity property. We believe that our model is more natural in that it models the fact that two stations need to hear *each other* in order to communicate directly (e.g., to send a message and receive an acknowledgement).

The Hamiltonian path problem that we (and previously Fekete and Woeginger [8]) study, is the "complement" of a problem posed by Fekete and Woeginger [8] and recently studied by Bárány et al. [1]. In this latter problem, one has to draw a polygonal line through a given set of points in the plane, such that the (smaller) angle at each internal vertex is at least some constant θ (independent of the given set). Bárány et al. proved that this can always be done for $\theta = 20^\circ$.

2 60-degree directional antennas

Let $\mathcal{P}$ be a set of points in the plane. In this section, we prove that one can place, at each point $p \in \mathcal{P}$, a single 60-degree directional antenna, such that the resulting communication graph is connected. Our proof consists of two stages. In the first stage we place the antennas, such that the resulting communication graph might still consist of several connected components, and in the second stage, we adjust the initial placement, if necessary, to achieve a connected graph.

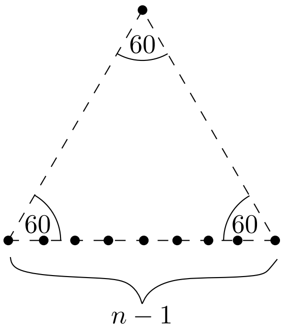

Figure 1: For any $\alpha < 60^\circ$, it is impossible to obtain a connected communication graph for the n points above with α-degree directional antennas.

Before proving this theorem, observe that 60 degrees is the best one can hope for, since there exist sets of points for which it is impossible to obtain a connected communication graph using α-degree directional antennas, for $\alpha < 60^\circ$; see Figure 1 for an example.

2.1 Stage I

For each $p \in \mathcal{P}$, let $f(p)$ denote the farthest point from p among the points in $\mathcal{P}$. For each $p \in \mathcal{P}$, let $\mathcal{F}_p = \{q \in \mathcal{P} \mid f(q) = p\}$. In this stage, we construct an initial communication graph, by placing, at each $p \in \mathcal{P}$, a 60-degree antenna, such that all points in $\{f(p)\} \cup \mathcal{F}_p$ are covered by it. We next prove that this is possible.

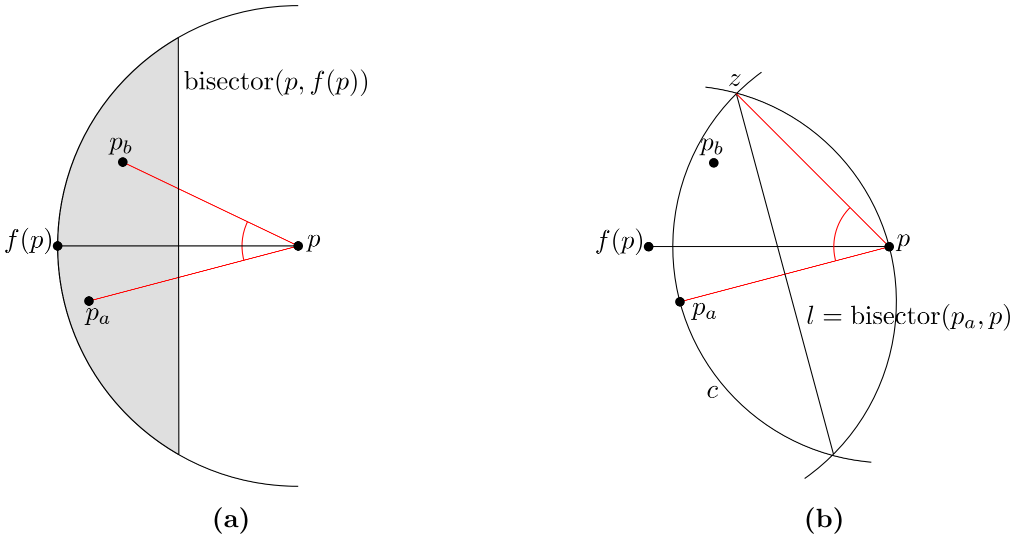

Figure 2: (a) $\mathcal{F}_p$ is contained in the grey region. (b) $\angle p_a p z = 60°$.

Let $p \in \mathcal{P}$. W.l.o.g. we draw $f(p)$ on the horizontal line through p and to the left of p; see Figure 2(a). Since for each $q \in \mathcal{F}_p$ (i) $d(q,p) \geq d(q, f(p))$, where $d(a,b)$ is the Euclidean distance between points a and b, and (ii) $d(p,q) \leq d(p, f(p))$, we conclude that $\mathcal{F}_p$ is contained in the region indicated in Figure 2(a). Let p_a be the point in $\mathcal{F}_p$ below $pf(p)$, for which the angle $\angle p_a p f(p)$ is maximal. Similarly, let p_b be the point in $\mathcal{F}_p$ above $pf(p)$, for which the angle $\angle f(p) p p_b$ is maximal. If there is no point in $\mathcal{F}_p$ below (resp. above) $pf(p)$, then set $p_a = f(p)$ (resp. $p_b = f(p)$).

Lemma 2.1. *The angle $\angle p_a p p_b$ is not greater than 60 degrees.*

Proof. Assume, w.l.o.g., that $d(p_a, p) \geq d(p_b, p)$. We show that p_b lies in the 60-degree wedge at p whose left side contains pp_a; see Figure 2(b). Let l be the bisector of p_a and p. Then p_b lies to the left of l. Now, draw the circle c of radius $d(p_a, p)$ centered at p. By our assumption p_b is in the disk bounded by c. Let z be the intersection point between c and l (above pp_a). The angle $\angle p_a p z$ is of 60 degrees and p_b lies in it; hence, $\angle p_a p p_b \leq 60°$. ■

2.2 Stage II

Consider the communication graph obtained in the first stage, and let $C_1, \ldots, C_k$ be its connected components. If $k = 1$, then we are done. Assume therefore that $k \geq 2$. For $i = 1, \ldots, k$, let $e_i = (u_i, v_i)$ be the longest edge in C_i, and set $r_i = d(u_i, v_i)$. Notice that $f(u_i) = v_i$ and $f(v_i) = u_i$ (since, if, e.g., $f(u_i) \neq v_i$, then we would have the edge $(u_i, f(u_i))$ which is longer than

e_i). We say that e_i is the *representative* of component C_i. Let $lune(u_i, v_i)$ denote the region $D_{u_i}(r_i) \cap D_{v_i}(r_i)$, where $D_p(r)$ is the disk centered at p of radius r. Then $\mathcal{P} \subseteq lune(u_i, v_i)$, for $i = 1, \ldots, k$.

Claim 2.2. *Let C, C' be two connected components (of the communication graph obtained in the first stage), and let $e = (x, y)$ and $e' = (u, v)$ be their representative edges. Then e and e' cross each other.*

Proof. Assume, e.g., that u lies in the upper half of $lune(x, y)$. If v also lies in the upper half of $lune(x, y)$, then at least one of the four distances $d(u, x), d(u, y), d(v, x), d(v, y)$ is greater than $d(u, v)$. But, this is impossible, since $f(u) = v$ and $f(v) = u$. ■

An angle α is considered *good* if $\alpha \leq 60°$. Let C, C' be two connected components and consider their representative edges $e = (x, y)$ and $e' = (u, v)$. Assume, e.g., that u lies above e and that v lies below e. Consider the quadrilateral x, v, y, u; see Figure 3. The edges e and e' divide the quadrilateral into four triangles. The next lemma states that at least one of these triangles is *good*, in the sense that both its angles opposite e and e' are good.

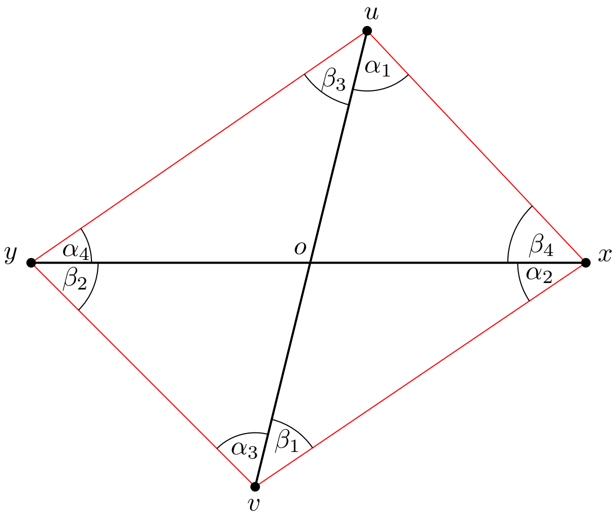

Figure 3: At least one of the triangles is good.

Lemma 2.3. *Consider Figure 3, and let o denote the intersection point between e and e'. Then, at least one of the four triangles sharing o as a corner is good.*

Proof. Assume that all four triangles are bad. Consider the triangle Δuxo and assume, e.g., that $\angle oux$ (marked as α_1 in Figure 3) is bad. Then $\angle ovx$ (marked as β_1) is necessarily good. To see this consider the triangle Δuxv. Since $f(v) = u$, the angle opposite the edge uv is greater than α_1 (which is assumed to be bad), therefore β_1 must be good. Now, since β_1 is good, α_2 must be bad (otherwise, we are done). This implies that β_2 is good, etc. Eventually, we get that $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are bad and $\beta_1, \beta_2, \beta_3, \beta_4$ are good.

Next, we show that this is impossible. Since $\alpha_1 > \beta_4$, we have $|ox| > |ou|$. Similarly, we have $|ou| > |oy|$, $|oy| > |ov|$, and $|ov| > |ox|$. Writing the four inequalities as a chain of inequalities, we get that $|ox| > |ox|$, which is of course impossible. We conclude that there exists a good triangle. ■

We are ready to show how to readjust the wedges at the points in $\mathcal{P}$ in order to obtain a single connected component. Notice that $\mathcal{P} \subseteq lune(x,y) \cap lune(u,v)$ (since, as mentioned above, $\mathcal{P} \subseteq lune(u_i, v_i)$, for each connected component C_i). Set $D = lune(x,y) \cap lune(u,v)$.

By Lemma 2.3, at least one of the four triangles formed by x, v, y, u is good. Assume, w.l.o.g. that the upper right triangle is good, i.e., $\angle uxy \leq 60°$ and $\angle xuv \leq 60°$. Then, at least one of the two angles $\angle uvy$ and $\angle xyv$ is not greater than $60°$ (since $\angle voy = \angle uox \geq 60°$). Assume, w.l.o.g. that $\angle uvy \leq 60°$.

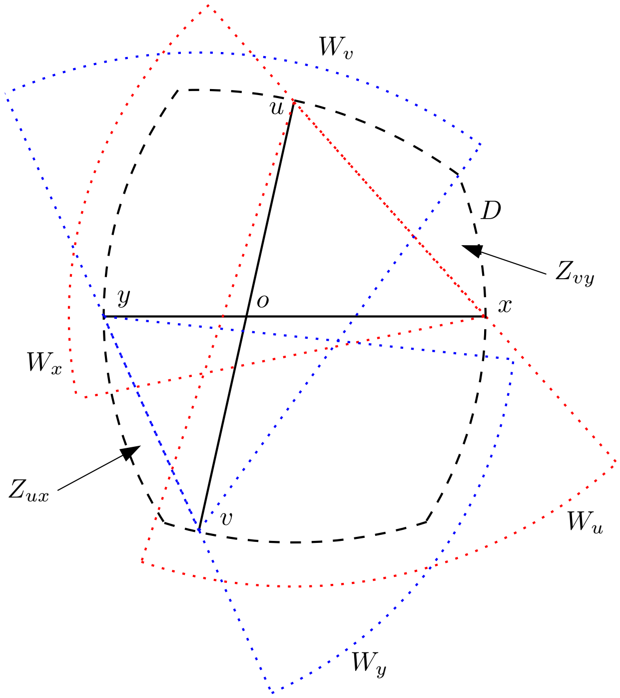

Figure 4: Initial placement of wedges W_u, W_x, W_v, W_y.

Consider the following four 60-degree wedges; see Figure 4. Wedge W_u with apex u and x on its left border (when looking from u into the wedge). By assumption W_u covers v. Wedge W_x with apex x and u on its right border. By assumption W_x covers y. Wedge W_v with apex v and y on its left border. By assumption W_v covers u. Wedge W_y with apex y and v on its right border. (W_y covers x if and only if $\angle xyv \leq 60°$.) Notice that the communication graph induced by x, y, u, v with wedges W_x, W_y, W_u, W_v is connected, since it includes the edges $(u,v), (u,x), (v,y)$. Thus, if $\mathcal{P}$ is contained in the union of these four wedges, then we are done, since any point in $\mathcal{P} \setminus \{x, y, u, v\}$ can transmit to an apex of a wedge in which it lies.

Assume therefore that there exists at least one point that is not covered by the union of the wedges W_x, W_y, W_u, W_v, and let p be such a point. Let $R_{ux}, R_{uy}, R_{vx}, R_{vy}$ denote the wedges corresponding to the angles $\angle uox, \angle uoy, \angle vox, \angle voy$, respectively. Then, p cannot lie in wedge R_{uy} or in wedge R_{vx}. (Notice that $D \cap R_{uy} \subseteq W_x$ and $D \cap R_{vx} \subseteq W_u$.) Thus, p is either between the wedges W_x, W_u (and in the wedge R_{vy}), or p is between the wedges W_y, W_v (and in the wedge R_{ux}). (Notice that although the intersection point between wedges W_y, W_v might lie in wedge R_{vx}, any point between these wedges and in wedge R_{vx} is already covered by W_u.) Let $Z_{ux} \subset \mathcal{P}$ be the subset of points of $\mathcal{P}$ that are not covered and lie between W_x, W_u (and in R_{vy}), and let $Z_{vy} \subset \mathcal{P}$ be the subset of points of $\mathcal{P}$ that are not covered and lie between W_y, W_v (and in R_{ux}); see Figure 4. We distinguish between two cases.

Case 1: $\angle xyv \leq 60^\circ$.

In this case, the lower left triangle is also good, and the wedge W_y covers x. Let z be a point that is not covered by the union of the four wedges W_x, W_y, W_u, W_v. W.l.o.g., assume z belongs to Z_{vy}. Consider triangle Δyzv. In Δyzv the angle $\angle yzv < 60^\circ$ (since, the other two angles in this triangle are greater than 60°, otherwise z would be covered by $W_y \cup W_v$). Thus, a 60-degree wedge with apex z can cover all the points in Z_{ux}, including y and v (since $D \cap R_{vy}$ is contained in the wedge corresponding to $\angle yzv$). We now claim that at least one of the two angles $\angle xyz$ and $\angle uvz$ is not greater than 60°. Assume this is true (see Lemma 2.5 below), then the following setting of 60-degree wedges consists of a solution to our problem. Wedge W_u with apex u and x on its left border. Wedge W_x with apex x and u on its right border. Wedge W_v^* with apex v and u on its left border. Wedge W_y^* with apex y and x on its right border. Wedge W_z with apex z and v on its left border.

Claim 2.4. *$\mathcal{P}$ is contained in the union of the five wedges $W_x, W_y^*, W_u, W_v^*, W_z$.*

Proof. First notice that the communication graph induced by x, y, u, v, z with wedges W_x, W_y^*, W_u, W_v^*, W_z is connected. (By Lemma 2.5, at least one of the edges (y, z) and (v, z) is present.) Now let $p \in \mathcal{P} \setminus \{x, y, u, v, z\}$. Then, if p lies in the wedge R_{vy}, then it is covered by W_z. If p lies in the wedge R_{vx}, then it is covered by W_u. If p lies in the wedge R_{uy}, then it is covered by W_x. Finally, if p lies in the wedge R_{ux}, then we distinguish between two subcases. If $p \notin Z_{vy}$, then it is clearly covered by both W_x and W_u, and if $p \in Z_{vy}$, then we may apply Lemma 2.5 (replacing z by p) and conclude that p is covered either by W_y^* or by W_v^*. ■

Lemma 2.5. *At least one of the angles $\angle xyz$ and $\angle uvz$ is not greater than 60 degrees.*

Proof. First notice that x, y, u, v are vertices of the convex hull of $\mathcal{P}$. (This is true, since for any $p \in \mathcal{P}$, the point $f(p)$ is a vertex of $CH(\mathcal{P})$.) Assume, w.l.o.g., that $|uv| \geq |xy|$. If $\angle xyz \leq 60^\circ$, then we are done. Otherwise, consider the upper half of $lune(x, y)$ and draw the equilateral triangle xyo_{xy}, where o_{xy} is the top point of the upper lune; see Figure 5. z necessarily lies in the left shoulder of the upper lune (i.e., in the left part of the region obtained by subtracting the equilateral triangle from the upper lune). Since z lies $D \cap R_{ux}$, we know that u must also lie in the left shoulder of the upper lune, to the left of yz.

Now consider the right half of $lune(u, v)$ and its corresponding equilateral triangle uvo_{uv}. We have $|uo_{uv}| = |uv| \geq |xy| = |yo_{xy}|$. Therefore, o_{uv} lies outside the left shoulder of $lune(x, y)$ to the right of yo_{xy} (since v is below xy). This implies that the left shoulder of $lune(x, y)$ and the right shoulder of $lune(u, v)$ are disjoint. Thus, z cannot lie in the right shoulder of $lune(u, v)$ and therefore $\angle uvz \leq 60^\circ$ (since $\angle uvz > 60^\circ$ implies that z must lie in the right shoulder of $lune(u, v)$). ■

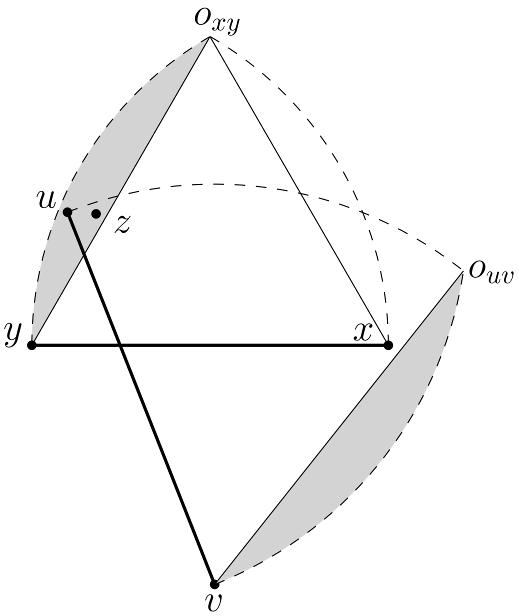

Figure 5: Proof of Lemma 2.5. The left shoulder of (the upper half of) $lune(x, y)$ and the right shoulder of (the right half of) $lune(u, v)$ are shown in grey.

Case 2: $\angle xyv > 60°$.
We distinguish between two subcases. If $Z_{vy} \neq \emptyset$, then consider Case 2.1; otherwise consider Case 2.2.

Case 2.1: $Z_{vy} \neq \emptyset$.
Let $z \in Z_{vy}$, and consider triangle Δyzv. As above, in Δyzv the angle $\angle yzv \leq 60°$. Thus, a 60-degree wedge with apex z can cover all points in Z_{ux}, including y and v. Moreover, observe that $\angle xyz \leq 60°$. Indeed, consider the triangle Δyzv. In this triangle, the angle at v is at least $60°$ (since z is not in W_v), and so is $\angle xyv$ (by assumption). Therefore, the sum of the angles $\angle yzv$ and $\angle xyz$ is at most $60°$, and, in particular, $\angle xyz \leq 60°$. The following placement of 60-degree wedges covers P. Wedge W_u with apex u and x on its left border. Wedge W_x with apex x and u on its right border. Wedge W_v with apex v and y on its left border. Wedge W_y^* with apex y and x on its right border. Wedge W_z with apex z and v on its left border.

Claim 2.6. *$\mathcal{P}$ is contained in the union of the five wedges $W_x, W_y^*, W_u, W_v, W_z$.*

Proof. First notice that the communication graph induced by x, y, u, v, z with wedges W_x, W_y^*, W_u, W_v, W_z is connected. (In particular, it includes the edge (y, z).) Now let $p \in \mathcal{P} \setminus \{x, y, u, v, z\}$. If p lies in the wedge R_{ux} and $p \notin Z_{vy}$, then it is covered by W_x and W_u, if p lies in Δuox, and by W_v, otherwise. If p lies in the wedge R_{ux} and $p \in Z_{vy}$, then it is covered by W_y^* since $\angle xyp \leq 60°$ (replacing z by p in the argument above). If p lies in the wedge R_{uy}, then it is covered by W_v. If p lies in the wedge R_{vx}, then it is covered by W_u. If p lies in the wedge R_{vy}, then it is covered by W_z. ■

Case 2.2: $Z_{vy} = \emptyset$.
The following placement of 60-degree wedges covers P. Wedge W_u with apex u and x on its left

border. Wedge W_x with apex x and u on its right border. Wedge W_v with apex v and y on its left border. Wedge W_y^* with apex y that has v on its *left* border.

Claim 2.7. *$\mathcal{P}$ is contained in the union of the four wedges W_x, W_y^*, W_u, W_v.*

Proof. First notice that the communication graph induced by x, y, u, v with wedges W_x, W_y^*, W_u, W_v is connected. Also notice that Z_{vy} remains empty, since we readjusted only the wedge at y. Now let $p \in \mathcal{P} \setminus \{x, y, u, v\}$. If p lies in the wedge R_{vy}, then it is covered by either W_v or by W_y^*, since the $\angle xyp$ cannot exceed $90°$. (Recall that we are assuming $\angle xyv > 60°$.) ■

The following theorem summarizes the main result of this section.

Theorem 2.8. *Given a set $\mathcal{P}$ of n points in the plane, one can position a 60-degree wedge at each of the points in $\mathcal{P}$, so that the resulting communication graph is connected. This can be done in $O(n \log k)$ time, where k is the number of vertices of $CH(\mathcal{P})$.*

Proof. It remains to establish the upper bound on the running time. It is easy to see that for each $p \in \mathcal{P}$, the point $f(p)$ is a vertex of $CH(\mathcal{P})$. Thus, we first compute $CH(\mathcal{P})$ in $O(n \log k)$ time, where k is the number of vertices of $CH(\mathcal{P})$ (see [7,10]). Next, we compute the farthest-neighbor Voronoi diagram of the set of vertices of $CH(\mathcal{P})$ and preprocess it for efficient ($O(\log k)$-time) point location queries in $O(k \log k)$ time (see [2]).

Consider the graph G in which there is an edge between $p \in \mathcal{P}$ and $q \in \mathcal{P}$ if and only if $q = f(p)$ or $p = f(q)$. Notice that the communication graph obtained in Stage I contains G. (This follows from Lemma 2.1.) One can construct G, in $O(n \log k)$ time, by performing n point location queries in the farthest-neighbor Voronoi diagram. We now determine whether G is connected or not. Observe that G is connected if and only if the subgraph of G induced by the k vertices of the convex hull is connected. This is true, since each point that is not a vertex of $CH(\mathcal{P})$ is connected by a single edge to a point that is a vertex of $CH(\mathcal{P})$, and since the edge (p, q) is not in G, if both p and q are not vertices of $CH(\mathcal{P})$. Thus, one can determine whether G is connected in $O(k \log k)$ time.

If G is connected, compute the placement of antennas as described in Stage I in $O(n \log k)$ time. If G consists of two or more connected components, compute the placement of antennas as described in Stage II in $O(n)$ time. Notice that in the latter case, it is possible that the graph computed in Stage I is connected, but we still may apply Stage II; we obtain edges xy and uv as required, by simply picking the longest edge of any two components of G, respectively. ■

3 Paths with no obtuse angles

Let $\mathcal{P}$ be a set of n points in the plane. In this section we prove that one can draw a (not necessarily simple) polygonal path π whose set of vertices is $\mathcal{P}$, such that the (smaller) angle at each internal vertex of π is at most $90°$. Thus, if we place at each vertex v of π a transceiver equipped with a 90-degree directional antenna, and adjust the antenna so that v's two neighbors (or only one if v is an extreme vertex) lie in the corresponding wedge, then π is a Hamiltonian path in the resulting communication graph. As mentioned above, this statement has been proven by Fekete and Woeginger [8]. We present an alternative construction that produces paths that tend to have shorter edges and fewer self crossings; see Figure 6 for an extreme example, and remark at the end of this section for a possible explanation.

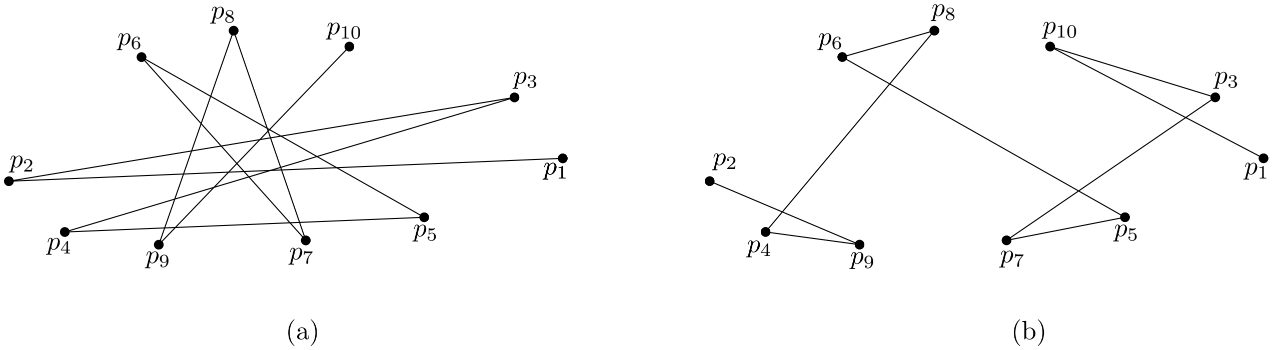

Figure 6: (a) The path obtained by the algorithm of Fekete and Woeginger, starting at p_1. (b) The path obtained by Algorithm *NoObtuse* below.

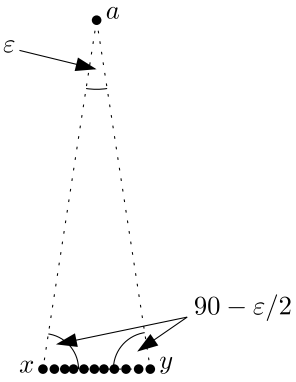

Figure 7: Any polygonal line has an internal vertex with angle at least $90^\circ - \varepsilon$.

We first show that for any $n \geq 4$ and $\varepsilon > 0$, there exists a set of n points, for which any polygonal path has an internal vertex with angle greater than $90° - \varepsilon$. Consider Figure 7. In this figure, points x, a, y define an isosceles triangle, such that the angle near a is of size ε. It is easy to see that any polygonal path through the points has an internal vertex with angle at least $90° - \varepsilon$.

Algorithm *NoObtuse* below, computes a polygonal path with no obtuse angles for a given set of points $\mathcal{P}$. A polygonal path obtained by this algorithm is depicted in Figure 8.

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NoObtuse($\mathcal{P}$)
  Let $p_1$ be the rightmost point in $\mathcal{P}$
  Let $a$ be the next vertex of CH($\mathcal{P}$) when moving counterclockwise from $p_1$
  flag $\leftarrow$ counterclockwise
  for $i \leftarrow 1$ to $n-2$
      $b \leftarrow NextPoint(p_i, a, flag)$
      if $\angle p_i a b \leq 90°$
          $p_{i+1} \leftarrow a$
          $a \leftarrow b$
      else
          $p_{i+1} \leftarrow b$
          $flag \leftarrow \overline{flag}$
      $\mathcal{P} \leftarrow \mathcal{P} \setminus \{p_i\}$
  $p_n \leftarrow a$
  output $\pi = p_1 p_2 \ldots p_n$
```

```
NextPoint($p_i, a, flag$)
  Let $\rho$ be a ray attached to $a$ and continuing the directed segment $p_i a$
  Let $b$ be the first point encountered by $\rho$ when rotating it around $a$
   in the direction indicated by flag
  return $b$
```

We now prove that the polygonal paths computed by the algorithm above have no obtuse angles. First observe that at the beginning of the i'th iteration, the segment $p_i a$ is an edge of the convex hull of the current set $\mathcal{P}$, and the point b returned by *NextPoint* is the next vertex of the convex hull, when moving in the direction from p_i to a.

Lemma 3.1. *Let $\pi = p_1 p_2 \ldots p_n$ be a polygonal path computed by the algorithm above. Then $\angle p_{i-1} p_i p_{i+1} \leq 90°$, for $i = 2, \ldots, n-1$.*

Proof. We prove that at the end of the i'th iteration $\angle p_i p_{i+1} a \leq 90°$. This is clearly true for the first iteration. Assume it is true for iteration $i-1$, that is, at the end of iteration $i-1$, $\angle p_{i-1} p_i a \leq 90°$. Consider the i'th iteration. At the beginning of this iteration b is found. Now, if $\angle p_i a b \leq 90°$, then we may add the edge $p_i a = p_i p_{i+1}$ (since by our assumption $90° \geq \angle p_{i-1} p_i a = \angle p_{i-1} p_i p_{i+1}$), and at the end of the iteration, $\angle p_i p_{i+1} a = \angle p_i p_{i+1} b \leq 90°$. If, on the other hand, $\angle p_i a b > 90°$, then we may add the edge $p_i b = p_i p_{i+1}$ (since, by the observation just above the lemma, $\angle p_{i-1} p_i p_{i+1} = \angle p_{i-1} p_i b \leq \angle p_{i-1} p_i a \leq 90°$). Moreover, since the angle $\angle p_i a b > 90°$, the angle $\angle p_i b a = \angle p_i p_{i+1} a \leq 90°$. ■

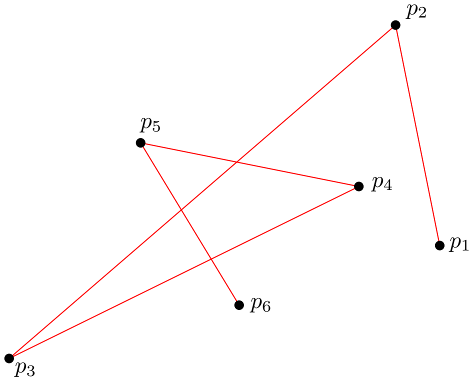

Figure 8: The polygonal line obtained by applying the algorithm to the 6 points above. In the first call to *NextPoint* p_5 is returned, in the second call p_3, then p_4, and finally p_6.

Theorem 3.2. *Given a set* $\mathcal{P}$ *of* n *points in the plane, one can position a 90-degree wedge at each of the points in* $\mathcal{P}$*, so that the resulting communication graph contains a Hamiltonian path. This can be done in* $O(n \log n)$ *time.*

Proof. The time bound follows from the observation just above Lemma 3.1. At the very beginning, we compute the convex hull of $\mathcal{P}$ in $O(n \log n)$ time (see [2]). Afterwards, whenever a point is removed from $\mathcal{P}$ (i.e., immediately after line 12), we update the convex hull in $O(\log n)$ time, using the dynamic convex hull algorithm of Brodal and Jacob [3] (or the semi-dynamic algorithm of Hershberger and Suri [9]). ■

Remark. As mentioned above, the paths produced by Algorithm *NoObtuse* tend to have shorter edges and fewer self crossings than the corresponding paths produced by the algorithm of Fekete and Woeginger [8]. A possible explanation of this tendency is that while the latter algorithm is based on repeatedly picking the farthest point among the remaining points, Algorithm *NoObtuse* picks the next vertex of the convex hull of the remaining points, if possible.

4 Conclusions and open problems

In this work we adopt the non-directed graph model to address problems in the design of wireless networks with directional antennas. This model is based on the approach that two stations need to hear *each other* in order to (directly) communicate. We then address the problem of designing a connected communication network for an arbitrary set of points in the plane, and the problem of designing such a network which also contains a Hamiltonian path. For both problems we give tight bounds on the angle required (for the directional antennas) to build the corresponding network for any set of points. We further give efficient algorithms to build such networks. (As mentioned, a completely different upper bound construction for the latter problem has been given by Fekete and Woeginger [8].)

The problem of computing the minimum required angle to achieve connectivity (or connectivity with a Hamiltonian path) for a given set of points is interesting and remains open. Also, for a given set of points, the problem of computing the minimum required range r to achieve connectivity using 60-degree antennas of range at most r is interesting. Finally, the problem of assigning both an angle and a range to each of the points to achieve connectivity, while minimizing the total power consumption (measured, e.g., as the sum of the areas of the n sectors) is intriguing. We leave these optimization problems for future research.

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