Diagrammatic sets and rewriting in weak higher categories

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There is a draft, but I am rewriting it from scratch. Some definitions have changed. Some results I will mention do not hold with the old definitions. The new version should be out before the end of the month.
In homotopy theory, algebraic geometry, ...:
Higher categories for all

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- If we do, it should feel familiar.

\(\implies\) Segal spaces, complicial sets... pick your favourite.
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Bialgebra equation
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*How do we interpret this?*
Pasting theorems

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There is a lack of pasting theorems
for models of weak higher categories.
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The golden age of strict $\omega$-categories

- **1987**: Ross Street’s *The algebra of oriented simplexes* is out, sparking an interest in the combinatorics of higher-dimensional categorical diagrams.
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Then several works on the combinatorics of *pasting diagrams* and their *pasting theorems* in strict $n$-categories:

- **1988**: John Power
- **1989**: Michael Johnson
- **1991**: Ross Street, John Power
- **1993**: Richard Steiner
We can associate to a cell complex its face poset...
Directed complexes

We can associate to a cell complex its face poset...

and to a pasting diagram its oriented face poset.
An orientation on a finite poset $P$ is an edge-labelling $o : \mathcal{H}P_1 \to \{+, -\}$ of its Hasse diagram.
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An oriented graded poset is a finite graded poset with an orientation.
Technical interlude #1: Directed complexes

- An orientation on a finite poset $P$ is an edge-labelling $o : \mathcal{HP}_1 \to \{+, -\}$ of its Hasse diagram.

- An oriented graded poset is a finite graded poset with an orientation.

- If $U \subseteq P$ is (downward) closed, $\alpha \in \{+, -\}$, $n \in \mathbb{N}$,

$$\Delta^\alpha_n U := \{x \in U \mid \text{dim}(x) = n \text{ and if } y \in U \text{ covers } x, \text{ then } o(y \to x) = \alpha\},$$

$$\partial^\alpha_n U := \text{cl}(\Delta^\alpha_n U) \cup \{x \in U \mid \text{for all } y \in U, \text{ if } x \leq y, \text{ then } \text{dim}(y) \leq n\},$$

$$\Delta_n U := \Delta^+_n U \cup \Delta^-_n U, \quad \partial_n U := \partial^+_n U \cup \partial^-_n U.$$
If $U$ is a closed subset of $P$, then $U$ is a *molecule* if either

- $U$ has a greatest element, in which case we call it an *atom*, or
- there exist molecules $U_1$ and $U_2$, both properly contained in $U$, and $n \in \mathbb{N}$ such that $U_1 \cap U_2 = \partial^n U_1 = \partial^n U_2$ and $U = U_1 \cup U_2$. 
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An oriented graded poset \( P \) is a *directed complex* if, for all \( x \in P \) and \( \alpha, \beta \in \{+, -\} \), if \( n = \dim(x) \),

1. \( \partial^\alpha x \) is a molecule, and
2. \( \partial^\alpha (\partial^\beta x) = \partial^\alpha_{n-2} x \).
Directed complexes

Steiner 1993 (rephrased)

*Every molecule in a directed complex is the oriented face poset of a pasting diagram.*
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All directed complexes present $\omega$-categories — fewer present polygraphs, that is, $\omega$-categories that are freely generated by some of their cells.
Directed complexes

Let $P, Q$ be oriented graded posets. We can take their cartesian product as posets.
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We can give it an orientation as in the tensor product of chain complexes.

A variant of this was used to define the Gray product of $\omega$-categories (Steiner 2004, Ara-Maltsiniotis 2017).
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$P \otimes Q$, the (lax) Gray product of $P$ and $Q$. 
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If $P$ has dim $n$ and $Q$ has dim $k$, $P \otimes Q$ has dim $n + k$. 
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Gray products and diagrammatic algebra

Around this time, I start seeing Gray products everywhere in diagrammatic algebra

$$2d + 2d = 4d$$
Gray products and diagrammatic algebra

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Around this time, I start seeing Gray products everywhere in diagrammatic algebra

(Fortunately I was not the only one)
Example: Biunitary equations

Used by Jamie Vicary and Mike Stay to unify quantum and encrypted communication protocols. They are models of a Gray product of 2-categories.
Example: Distributive laws of monads

They are models in $\textbf{Cat}$ of a Gray product of 2-categories.
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**monoidal category** $\rightsquigarrow$ 2-category with one 0-cell
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\textbf{monoidal category} \rightsquigarrow 2-category with one 0-cell
\textbf{PRO} \rightsquigarrow 2-cat with one 0-cell, one 1-generator

These are naturally pointed objects in $\omega$-Cat. With pointed objects, it is natural to take smash products $\wedge$.

$\text{PRO} \wedge \text{PRO} \rightsquigarrow 4$-cat with one 0-cell

Morally this should be a braided monoidal category. But in strict $\omega$-categories, it is a commutative monoidal category. This breaks everything.
Gray products and diagrammatic algebra

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1998: Carlos Simpson proves that the result is false (without pointing to a specific mistake).
1991: Mikhail Kapranov and Vladimir Voevodsky publish \( \infty \)-groupoids and homotopy types, claiming a proof that strict higher categories model all homotopy types in the sense of the homotopy hypothesis.

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The core of the argument relies on the fact that “doubly monoidal” degenerates to “commutative” in strict 3-categories (strict Eckmann-Hilton).
...still contained some good ideas

Good takeaway #1 from Kapranov-Voevodsky:

*homotopy types may have semistrict algebraic models with weak units*

- **2006**: André Joyal and Joachim Kock in dim 3
...still contained some good ideas

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- **2017**: Simon Henry and I come up independently with the *regularity* constraint as a way of avoiding the pitfall of strict Eckmann-Hilton
- **2018**: Henry proves the homotopy hypothesis for “regular \(\omega\)-groupoids”.
Diagrams with spherical boundary

*Regularity:* only $n$-diagrams with spherical boundary have a composite
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*Regularity:* only $n$-diagrams with *spherical boundary* have a composite

These are the ones whose face poset is the face poset of a regular CW $n$-ball of the appropriate dimension
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\( \sim \) “are homeomorphic to \( n \)-balls”
Diagrams with spherical boundary

but not
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$$\partial_k^+ U \cap \partial_k^- U = \partial_{k-1} U.$$
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A directed complex is **regular** if all atoms have spherical boundary.
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- The geometric realisation* of a regular directed complex \( P \) is a regular CW complex with one cell for each atom of \( P \).
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* simplicial nerve of poset + realisation of simplicial sets
More in general, let $C$ be a class of molecules closed under isomorphism, boundaries, and inclusion of atoms, and included in the class $S$ of (regular) molecules with spherical boundary.
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- A $C$-directed complex is a directed complex whose atoms are all in $C$. 

...and more good ideas

Good takeaway #2 from Kapranov-Voevodsky:

Diagrammatic sets
Kapranov-Voevodsky pass from spaces to $\omega$-categories through an intermediate notion of "spaces locally modelled on combinatorial pasting diagrams", they call diagrammatic sets.
...and more good ideas

Good takeaway #2 from Kapranov-Voevodsky:

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...and more good ideas

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Regular molecules with spherical boundary works.
But we take a more axiomatic approach.
A map $f : P \to Q$ of $C$-directed complexes is a function that satisfies

$$\partial_n^\alpha f(x) = f(\partial_n^\alpha x)$$

for all $x \in P$, $n \in \mathbb{N}$, and $\alpha \in \{+, -\}$. 
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A map factors essentially uniquely as a surjection followed by an inclusion.
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A map factors essentially uniquely as a surjection followed by an inclusion.

Let $f : P \to Q$ be a map. Then $f$ is a closed, order-preserving, dimension-non-increasing function of the underlying posets.
A \textit{\(C\)-functor} \(f : P \leftrightarrow Q\) of \(C\)-directed complexes is a function \(f : \mathcal{C}\ell(P) \to \mathcal{C}\ell(Q)\) such that

1. \(f\) preserves all unions and binary intersections,
2. \(\partial_n^\alpha f(\text{cl}\{x\}) = f(\partial_n^\alpha x)\), and
3. \(f(\text{cl}\{x\})\) is a \(C\)-molecule

for all \(x \in P\), \(n \in \mathbb{N}\), and \(\alpha \in \{+, -\}\).
A $\mathcal{C}$-functor $f : P \leftrightarrow Q$ of $\mathcal{C}$-directed complexes is a function $f : \mathcal{C}\ell(P) \to \mathcal{C}\ell(Q)$ such that:

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A class $\mathcal{C}$ is algebraic if $\mathcal{C}$-functors compose. We assume that $\mathcal{C}$ is algebraic.
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A class $\mathcal{C}$ is algebraic if $\mathcal{C}$-functors compose. We assume that $\mathcal{C}$ is algebraic.

A $\mathcal{C}$-functor factors e.u. as a subdivision followed by an inclusion.
Technical interlude #3a: Morphisms of directed complexes

A span of inclusions of subcategories:

\[ \text{DCpx}^C_{\text{in}} \quad \exists \quad \text{DCpx}^C \quad \exists \quad \text{DCpx}^C_{\text{fun}} \]
Let $C \subseteq S$ be an algebraic class of molecules with spherical boundary.

We say that $C$ is a *convenient* if it satisfies the following axioms:
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1. $C$ contains $\bullet$;
2. if $U \in C$ and $J \subseteq \mathbb{N} \setminus \{0\}$, then $D_J U \in C$;
Technical interlude #3b: Convenient classes

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3. if $U, V \in C$ and $U \Rightarrow V$ is defined, then $U \Rightarrow V \in C$;
4. if $U_1, U_2 \in C$ and the pasting $U_1 \cup U_2$ along $V \subset \partial U_2$ is defined, then $U_1 \cup U_2 \in C$;
5. if $U \in C$ and $V \subseteq \partial U$ is a closed subset, then $O_1 \otimes U / \sim V \in C$;
6. if $U, V \in C$, then $U \otimes V \in C$ and $U \star V \in C$.

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4. if $U_1, U_2 \in C$ and the pasting $U_1 \cup U_2$ along $V \subseteq \partial^\alpha U_2$ is defined, then $U_1 \cup U_2 \in C$;

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We write $\mathcal{C}$ for a skeleton of the full subcategory of $\mathbf{DCpx}^\mathcal{C}$ on the atoms of every dimension.
Diagrammatic sets

We fix a convenient class of molecules $\mathcal{C}$.

We write $\mathcal{C}_{\bullet}$ for a skeleton of the full subcategory of $\text{DCpx}^\mathcal{C}$ on the atoms of every dimension.

- A *diagrammatic set* $X$ is a presheaf on $\mathcal{C}_{\bullet}$. 
Diagrammatic sets

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The Yoneda embedding $\mathcal{C} \hookrightarrow \mathcal{C}\text{Set}$ extends to an embedding $\mathbf{DCpx}^\mathcal{C} \hookrightarrow \mathcal{C}\text{Set}$. 
Diagrammatic sets

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The Yoneda embedding $\mathcal{O} \hookrightarrow \mathcal{O}\mathbf{Set}$ extends to an embedding $\mathbf{DCpx}^\mathcal{C} \hookrightarrow \mathcal{O}\mathbf{Set}$.

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We write $\otimes$ for a skeleton of the full subcategory of $\mathbf{DCpx}^C$ on the atoms of every dimension.

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The Yoneda embedding $\otimes \hookrightarrow \otimes \mathbf{Set}$ extends to an embedding $\mathbf{DCpx}^C \hookrightarrow \otimes \mathbf{Set}$.

- A *diagram* in $X$ is a morphism $x : U \to X$ where $U$ is a molecule.

- It is *composable* if $U \in \mathcal{C}$, and a *cell* if $U$ is an atom.
Fixing half of KV’s proof

- A Kan diagrammatic set has fillers of all “horns of atoms”.

[9x247]Fixing half of KV’s proof

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There is a realisation of Kan diagrammatic sets that is surjective on homotopy types, together with natural isomorphisms between the homotopy groups of a pointed Kan diagrammatic set and those of its realisation.
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The silver age of strict $\omega$-categories

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This started the French school of rewriting with polygraphs (Yves Lafont, Philippe Malbos, Yves Guiraud, Samuel Mimram...) and related work on $\omega$-categories (François Métayer, Georges Maltsiniotis, Dimitri Ara...)
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The silver age of strict $\omega$-categories

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The silver age of strict $\omega$-categories

Many of the core ideas in polygraphic rewriting rest on an analogy between

polygraphs and CW complexes,
“presented $\omega$-categories” and “presented spaces”.

This analogy is limited by the fact that strict $\omega$-categories do not model all spaces.
A suggestion: rewriting in diagrammatic sets

A similar feel to working with polygraphs, but:

1. Better combinatorial grip on rewriting operations like substitution, surgery of diagrams, etc.
2. "Essential" separation between diagrams and cells.
3. Analogy with CW complexes becomes a functor.
4. Diagrams can be interpreted in models of all homotopy types for rewriting homotopies.
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A suggestion: rewriting in diagrammatic sets

The smash product of pointed diagrammatic sets produces this equation, the way it should.
Equivalences and weak composites

Need a model of weak higher categories as “semantic universe”.
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If $C = S$, we can interpret every regular diagram and compose every diagram with spherical boundary.

“Stuff” a diagram with units and it becomes regular.
If \((x_1, x_2) \Rightarrow [x_1, x_2]\) exhibits \([x_1, x_2]\) as a weak composite:

\[
\forall \ y
\]

And this equivalence should be witnessed by 3-dimensional equivalence diagrams...
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\[
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\]

\[
\xymatrix{ & y \\
 x_1 \ar[ur] & x_2 \ar[ur] \\
 & x_1 \ar[ur] & x_2 \ar[ur] 
}
\]

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\[
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If \((x_1, x_2) \Rightarrow \lfloor x_1, x_2 \rfloor\) exhibits \([x_1, x_2]\) as a weak composite:

\[
\forall y \exists z \sim [x_1, x_2]
\]

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- All *degenerate* composable diagrams are equivalences.
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- Morphisms of diagrammatic sets preserve equivalences.
- In a Kan diagrammatic set, all composable diagrams are equivalences.
A semistrict algebraic model

In the span

\[
\begin{array}{ccc}
\text{DCpx}^C_{in} & \xleftarrow{\quad} & \text{DCpx}^C_{fun} \\
\text{DCpx}^C & \xleftarrow{\quad} &
\end{array}
\]

the two functors preserve the set $\Gamma$ of colimit diagrams containing the initial object and all pushouts of inclusions.
A semistrict algebraic model

In the span

\[
\begin{array}{ccc}
\text{Set} & \xrightarrow{\text{DCpx}_{\text{in}}^{C}} & \text{DCpx}_{\text{fun}}^{C} \\
\downarrow & & \downarrow \\
\text{DCpx}^{C} & \xleftarrow{} & \text{DCpx}^{C}
\end{array}
\]

the two functors preserve the set $\Gamma$ of colimit diagrams containing the initial object and all pushouts of inclusions.

\[\text{Set}\] is equivalent to the category $\text{PSh}_{\Gamma}(\text{DCpx}_{\text{fun}}^{C})$ of $\Gamma$-continuous presheaves on $\text{DCpx}^{C}$.
Applying $PSh_{Γ}(-)$, we obtain a cospan

Pol$C$ is a category of “combinatorial $C$-polygraphs” (only faces, no units or compositions)

ω$Cat$ is a category of “non-unital $C$-$ω$-categories” (only faces and compositions, no units)
Applying $\text{PSh}_\Gamma(-)$, we obtain a cospan

$$
\begin{array}{ccc}
\text{Pol}^C & \xleftarrow{\otimes \text{Set}} & \text{Set} \\
\downarrow & & \downarrow \\
\omega \text{Cat}_{nu} & \xrightarrow{\omega \text{Cat}_{nu}^C} & \omega \text{Cat}_{nu}^C
\end{array}
$$

of restriction functors, where $\text{Pol}^C := \text{PSh}_\Gamma(\text{DCpx}_{in}^C)$ and $\omega \text{Cat}_{nu}^C := \text{PSh}_\Gamma(\text{DCpx}_{fun}^C)$. 
A semistrict algebraic model

Applying $\text{PSh}_\Gamma(-)$, we obtain a cospan

$$\begin{array}{ccc}
\text{Set} & \overset{\text{Pol}^C}{\longrightarrow} & \text{ωCat}_{nu}^C \\
\downarrow & & \downarrow \\
\text{Pol}^C & \overset{\text{ωCat}_{nu}^C}{\longleftarrow} & \text{ωCat}_{nu}^C
\end{array}$$

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- $\text{Pol}^C$ is a category of “combinatorial $C$-polygraphs” (only faces, no units or compositions)
- $\omega\text{Cat}^C_{nu}$ is a category of “non-unital $C$-$\omega$-categories” (only faces and compositions, no units)
Units and compositions interact nicely *separately* with faces. If they are let to interact fully with each other, they produce strict Eckmann-Hilton.
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- Idea: put them together with only a modicum of interaction.
A semistrict algebraic model

A *diagrammatic* $\omega$-*category* has a separate “diagrammatic set” and “non-unital $\omega$-category” structure on the same underlying combinatorial polygraph, with a compatibility condition ensuring that certain composites of units are units on composites.
A semistrict algebraic model

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- $\otimes \text{Cat}$, $\otimes \text{Set}$, $\omega \text{Cat}_{nu}^C$ are all Eilenberg-Moore categories of finitary monads on $\text{Pol}^C$, and all the restriction functors have left adjoints.
A semistrict algebraic model

A *diagrammatic ω-category* has a separate “diagrammatic set” and “non-unital ω-category” structure on the same underlying combinatorial polygraph, with a compatibility condition ensuring that certain composites of units are units on composites.

- ⊗Cat, ⊗Set, ωCat_{nu} are all Eilenberg-Moore categories of finitary monads on Pol^{C}, and all the restriction functors have left adjoints.
- The underlying diagrammatic set of a diagrammatic ω-category has weak composites.
A semistrict algebraic model

A *diagrammatic* $\omega$-*category* has a separate “diagrammatic set” and “non-unital $\omega$-*category*” structure on the same underlying combinatorial polygraph, with a compatibility condition ensuring that certain composites of units are units on composites.

- $\mathcal{O} \mathbf{Cat}$, $\mathcal{O} \mathbf{Set}$, $\omega \mathbf{Cat}^C_{nu}$ are all Eilenberg-Moore categories of finitary monads on $\mathbf{Pol}^C$, and all the restriction functors have left adjoints.

- The underlying diagrammatic set of a diagrammatic $\omega$-*category* has weak composites.

Idea: take a unit on a composable diagram, and fully compose the boundary only on one side.
A semistrict algebraic model

Say that $C$ is *algebraically free* if all $C$-directed complexes present polygraphs.
A semistrict algebraic model

Say that $C$ is *algebraically free* if all $C$-directed complexes present polygraphs.

If $C$ is algebraically free, then $\omega\text{Cat}$ embeds as a full subcategory into $\otimes\text{Cat}$. 
Two conjectures

1. Conjecture: If $X$ is a diagrammatic set with weak composites, its inclusion in the free diagrammatic $\omega$-category on $X$ is a weak equivalence.
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1. Conjecture: If $X$ is a diagrammatic set with weak composites, its inclusion in the free diagrammatic $\omega$-category on $X$ is a weak equivalence.

2. Conjecture: Every convenient class $C$ is algebraically free.
Higher-dimensional rewriting is packed with notions suggestive of a directed homotopy theory.
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The appearance of smash products in diagrammatic algebra seems to me another piece of a puzzle.
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My hope is that diagrammatic sets can make the link between rewriting and homotopy theory tighter, on our way to figuring out what the right notions are.
Directed homotopy theory: a tinkerer’s approach

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Work in progress:
a model of computation in diagrammatic sets based on a “directed homotopy extension property”.
Thanks for listening!