A combinatorial-topological shape category for polygraphs

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CW complexes and polygraphs

Higher categories as **directed spaces**

Top	ω**Cat**

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classical model structure	LMW model structure

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n-polygraph $\otimes$ k-polygraph $=$ $(n+k)$-polygraph

Tensor products and universal algebra

Tensor products are omnipresent in (higher-dimensional) universal algebra

(QPL, HDRA 2016; Chapter 2 of my thesis)

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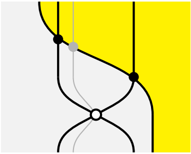
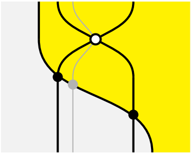


$\Rrightarrow$



$I \otimes X$

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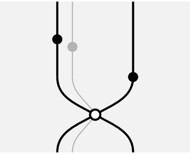

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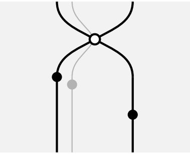

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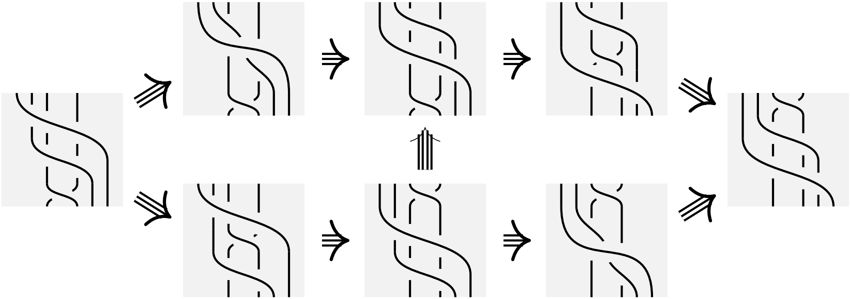
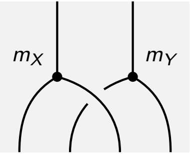
Tensor products and universal algebra



A quotient of $I \otimes I \otimes I \otimes I$

Tensor products and universal algebra



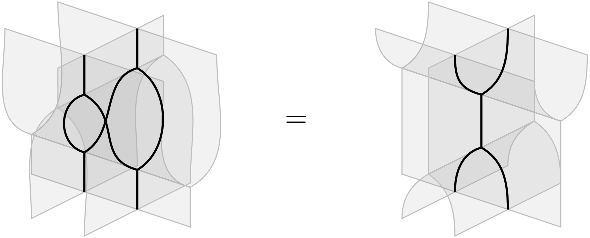
Distributive laws of monads

$=$

maps $Mon \otimes Mon \to \mathbf{Cat}$

A problem with strictness

The equations of **bialgebras** can be found in a quotient of *Mon* $\otimes$ *Mon* (a kind of smash product)...



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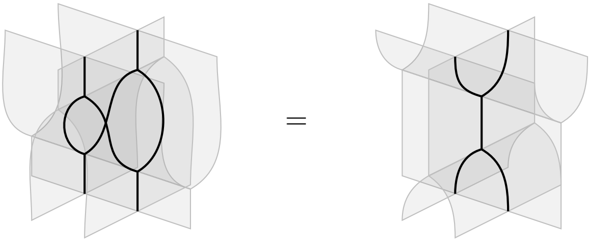

but the obvious "quotient by a subspace" leads to degeneracy!

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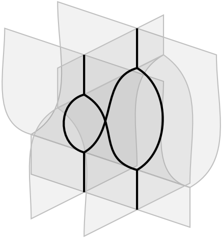

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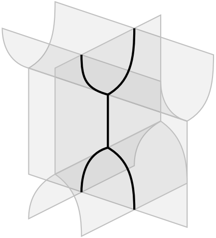

but the obvious "quotient by a subspace" leads to degeneracy!

Cells with degenerate boundaries are ill-behaved in strict ω-categories

A related problem

Tensor products are hard to calculate with the algebra of strict ω-categories

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A link between the two problems:

(Makkai-Zawadowski)

Polygraphs and cellular maps do not form a presheaf category

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Find a shape category for a class of polygraphs that

1. is "expressive enough"

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Tensor products are hard to calculate with the algebra of strict ω-categories

A link between the two problems:

(Makkai-Zawadowski)

Polygraphs and cellular maps do not form a presheaf category

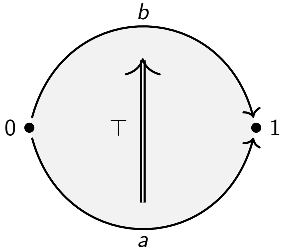
Find a shape category for a class of polygraphs that

1. is "expressive enough"
2. has easily computed tensor products

Incidence posets

Tensor products are easily computed for spaces characterised by their oriented incidence poset

$$
\begin{tikzcd}
 & \top \arrow[dl, "-"'] \arrow[dr, "+"] & \\
a \arrow[d, "-"'] \arrow[drr, "+" near start] & & b \arrow[d, "+"] \arrow[dll, "-" near end] \\
0 & & 1
\end{tikzcd}
$$



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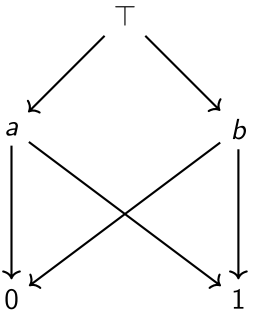

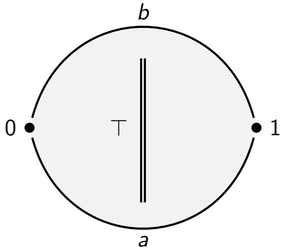

In the undirected case: **regular CW complexes**

CW posets

Two combinatorial criteria (Björner, Wachs):

1. **Thinness** $\rightsquigarrow$ the space is locally manifold-like

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We need stronger conditions in the directed case: both input and output k-boundaries must be homeomorphic to k-disks, for all k

Globular posets

1. **Oriented thinness**: all length-2 intervals $[x, y]$ are of the form

$$\begin{tikzcd} & y \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\ z_1 \arrow[dr, "\beta_1"'] & & z_2 \arrow[dl, "\beta_2"] \\ & x & \end{tikzcd}$$

with $\alpha_1\beta_1 = -\alpha_2\beta_2$

Globular posets

$U \subseteq X$ pure, closed, n-dimensional, $\alpha \in \{+, -\}$

$$\Delta^\alpha U := \{x \in U \mid \dim(x) = n - 1 \text{ and, for all } y \in U, \\ \text{if } y \longrightarrow x, \text{ then } y \xrightarrow{\alpha} x\}$$

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We define when U is a **globe**, inductively on dimension and number of maximal elements

Globular posets

$U = \{x\}$ is a 0-dimensional globe

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U is an n-dimensional globe when $\partial^+ U, \partial^- U$ are $(n-1)$-dimensional globes, and either

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U is an n-dimensional globe when $\partial^+ U, \partial^- U$ are $(n-1)$-dimensional globes, and either

- U has a single n-dimensional element (**atomic** globe), or
- $U = U_1 \cup U_2$ (non-trivially), U_1, U_2 are n-dimensional globes, and

  $$U_1 \cap U_2 = \partial^{\alpha} U_1 \cap \partial^{-\alpha} U_2$$

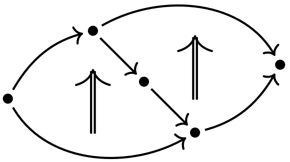
  is an $(n-1)$-dimensional globe

Globular posets

2. **Globularity**: for all $x \in X$, $\mathrm{cl}\{x\}$ is a globe

Globular poset

An oriented poset satisfying oriented thinness and globularity



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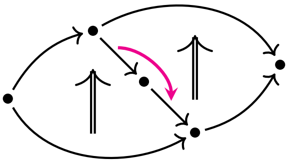

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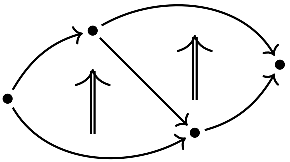

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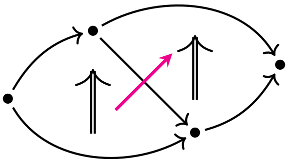

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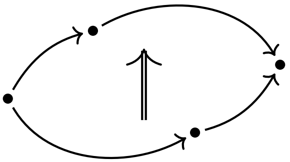

Results on globular posets

Theorem

X n-globe, $\alpha \in \{+, -\}$. Then

$$\partial^\alpha(\partial^+ X) = \partial^\alpha(\partial^- X).$$

Results on globular posets

Theorem

The underlying poset of a globular poset X is the incidence poset of a regular CW complex.

$$\simeq$$

for all n-globes U, $|U|$ is an n-disk, and $|\partial U|$ an $(n-1)$-sphere

Results on globular posets

Theorem

If X, Y are globular posets, $X \otimes Y$ is a globular poset.

(The underlying poset of $X \otimes Y$ is just $X \times Y$.)

A couple of open questions

A candidate for the shape category: globes (one per isomorphism class) and inclusions of sub-globes. Presheaves are called **regular polygraphs**

Recursive enumeration of isomorphism classes of globes?

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Recursive enumeration of isomorphism classes of globes?

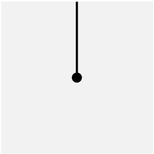
Conjecture

Globular posets are directed complexes (as in Steiner 1993)

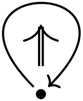
$\rightsquigarrow$ Translation into algebra of ω-categories

Bridging the gap with string diagrams

Two ways of looking at string diagrams:



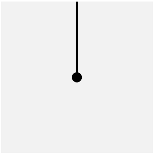
$\rightsquigarrow$



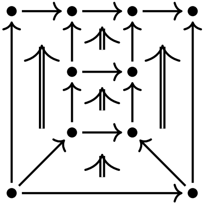
1. Duals of pasting diagrams

Bridging the gap with string diagrams

Two ways of looking at string diagrams:



$\rightsquigarrow$



1. Duals of pasting diagrams
2. Pasting diagrams filled up with (possibly weak) **unit cells**

Weakness via representability

Coherence via universality (Hermida): weak higher algebraic structure is subsumed by cells satisfying universal properties

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Coherence via universality (Hermida): weak higher algebraic structure is subsumed by cells satisfying universal properties

For **compositions**, overlap with opetopic higher categories (Baez-Dolan); but units need a different approach!

Worked out in low dimensions, with algebraic strict composition of 2-cells (*regular poly-bicategories*)

Saavedra units

(For simplicity: "multicategorical", one-sided version)

$1_x : x \to x$ is a **unit** if for all $a : x \to y$, $b : z \to x$, there exist

$$
\begin{tikzcd}
x \arrow[rr, "a"] \arrow[dr, "1_x"'] & & y \\
& x \arrow[ur, "a"'] \arrow[u, phantom, "\Uparrow l_a"] &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
z \arrow[rr, "b"] \arrow[dr, "b"'] & & x \\
& x \arrow[ur, "1_x"'] \arrow[u, phantom, "\Uparrow r_b"] &
\end{tikzcd}
$$

that exhibit a as both $1_x \otimes a$ and $1_x \multimap a$, and b as both $b \otimes 1_x$ and $b \mathbin{\circ\!\!-} 1_x$

Leads to the correct bicategorical units (triangle equations, etc.)

From equivalences to units

$e : x \to x'$ is **left divisible** if, for all $a : x \to y$, $a' : x' \to y$, there exist

$$
\begin{tikzcd}
x \arrow[rr, "a"] \arrow[dr, "e"'] & \arrow[d, phantom, "\Uparrow e^R_{e,a}"] & y \\
& x' \arrow[ur, "e \multimap a"'] &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
x \arrow[rr, "e \otimes a'"] \arrow[dr, "e"'] & \arrow[d, phantom, "\Uparrow \iota_{e,a'}"] & y \\
& x' \arrow[ur, "a'"'] &
\end{tikzcd}
$$

that exhibit a as $e \otimes (e \multimap a)$ and a' as $e \multimap (e \otimes a')$

From equivalences to units

$e : x \to x'$ is **right divisible** if, for all $b : z \to x$, $b' : z \to x'$, there exist

$$
\begin{tikzcd}
z \arrow[rr, "b'"] \arrow[dr, "b' \multimapinv e"'] & & x' \\
& x \arrow[ur, "e"'] \arrow[u, phantom, "\Uparrow e^L_{e,b'}"] &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
z \arrow[rr, "b \otimes e"] \arrow[dr, "b"'] & & x' \\
& x \arrow[ur, "e"'] \arrow[u, phantom, "\Uparrow \iota_{b,e}"] &
\end{tikzcd}
$$

that exhibit b' as $(b' \multimapinv e) \otimes e$ and b as $(b \otimes e) \multimapinv e$

From equivalences to units

$e : x \to x'$ is **divisible** if it is right and left divisible.

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$e : x \to x'$ is **divisible** if it is right and left divisible.

$\rightsquigarrow$ a notion of equivalence independent of the existence of units

(similar, but not the same as universal 1-cells in opetopic sets)

From equivalences to units

Theorem

The following are equivalent in a regular poly-bicategory:

- for all 0-cells x, there exists a unit $1_x : x \to x$;
- for all 0-cells x, there exist 0-cells $\overline{x}, \underline{x}$ and divisible 1-cells $e : x \to \overline{x}$, $e' : \underline{x} \to x$.

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If enough equivalences exist, units exist!

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(Relation with Univalence?)

What is to be done?

Work in progress: representability in arbitrary dimensions

Elucidate the combinatorics of shapes and of divisibility to obtain better semi-strictification theorems