Units without degeneracy, 
from polycategories to sequent calculi

Amar Hadzihasanovic
(ハジハサノヴィチ・アマル)

RIMS, Kyoto University

Kanazawa, 6 March 2018
1991. Kapranov, Voevodsky claim: all homotopy types are equivalent to **strict** homotopy types.


But Conjecture: All homotopy types are equivalent to ones that are strict, except for the units (2006. Joyal, Kock: $d = 3$)

1989. Danos, Regnier: proof equivalence for MLL without units decidable in P time, with proof nets

2014. Heijltjes, Houston: proof equivalence for MLL with units is PSPACE-complete

No proof nets for MLL with units
Trouble with units in topology and logic

- 1991. Kapranov, Voevodsky claim: all homotopy types are equivalent to **strict** homotopy types.

- 1989. Danos, Regnier: proof equivalence for MLL without units decidable in P time, with proof nets
- 2014. Heijltjes, Houston: proof equivalence for MLL with units is PSPACE-complete

No proof nets for MLL with units
Trouble with units in topology and logic

  claim: all homotopy types
  are equivalent to **strict**
  homotopy types.

  False for $d \geq 3$.

But Conjecture:

*All homotopy types are equivalent to ones that are strict, except for the units*
1991. Kapranov, Voevodsky claim: all homotopy types are equivalent to strict homotopy types.


But Conjecture:

All homotopy types are equivalent to ones that are strict, except for the units

Trouble with units in topology and logic

  claim: all homotopy types are equivalent to strict homotopy types.

  False for $d \geq 3$.

But Conjecture:
*All homotopy types are equivalent to ones that are strict, except for the units*

- 1989. Danos, Regnier:
  proof equivalence for MLL without units decidable in P time, with proof nets

- 2014. Heijltjes, Houston:
  proof equivalence for MLL with units is PSPACE-complete

  No proof nets for MLL with units
Trouble with units in topology and logic

- 1991. Kapranov, Voevodsky: claim all homotopy types are equivalent to strict homotopy types.


  But Conjecture: *All homotopy types are equivalent to ones that are strict, except for the units*  

- 1989. Danos, Regnier: proof equivalence for MLL without units decidable in P time, with proof nets

- 2014. Heijltjes, Houston: proof equivalence for MLL with units is PSPACE-complete
Trouble with units in topology and logic

- 1991. Kapranov, Voevodsky: claim: all homotopy types are equivalent to **strict** homotopy types.
  But Conjecture:
  *All homotopy types are equivalent to ones that are strict, except for the units*

- 1989. Danos, Regnier: proof equivalence for MLL **without units** decidable in P time, with proof nets
- 2014. Heijltjes, Houston: proof equivalence for MLL with units is \textit{PSPACE-complete}
  *No proof nets for MLL with units*
Poly-bicategories (Cockett-Koslowski-Seely)

- 0-cells $x, y, \ldots$

Topology: points; Logic: a unique 0-cell (polycategory)
Poly-bicategories (Cockett-Koslowski-Seely)

- 0-cells $x, y, \ldots$

  *Topology*: points; *Logic*: a unique 0-cell (polycategory)

- 1-cells $A, B, \ldots : x \rightarrow y$

  *Topology*: paths; *Logic*: formulae
Poly-bicategories (Cockett-Koslowski-Seely)

- 0-cells $x, y, \ldots$

  *Topology*: points; *Logic*: a unique 0-cell (polycategory)

- 1-cells $A, B, \ldots : x \rightarrow y$

  *Topology*: paths; *Logic*: formulae

- 2-cells $p, q, \ldots : (A_1, \ldots, A_n) \rightarrow (B_1, \ldots, B_m)$

  *Topology*: disks; *Logic*: sequents

![Diagram](image-url)
Composition (cut)
Composition (cut)

\[ \Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2 \]
\[ \frac{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{ CUT}_b \]

\[ \Gamma \vdash \Delta_1, A, \Delta_2 \quad A \vdash \Delta \]
\[ \frac{\Gamma \vdash \Delta_1, A, \Delta_2 \quad A \vdash \Delta}{\Gamma \vdash \Delta_1, \Delta, \Delta_2} \text{ CUT}_a \]

\[ \Gamma \vdash A \quad \Gamma_1, A, \Gamma_2 \vdash \Delta \]
\[ \frac{\Gamma \vdash A \quad \Gamma_1, A, \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta} \text{ CUT}_c \]

\[ \Gamma_2 \vdash A, \Delta_2 \quad \Gamma_1, A \vdash \Delta_1 \]
\[ \frac{\Gamma_2 \vdash A, \Delta_2 \quad \Gamma_1, A \vdash \Delta_1}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{ CUT}_d \]
Divisible 2-cells

Given $p : (A_1, \ldots, A_n) \rightarrow (B_1, \ldots, B_m)$, let $\partial_i^- p := A_i, \partial_j^+ p := B_j$. 
Divisible 2-cells

Given \( p : (A_1, \ldots, A_n) \to (B_1, \ldots, B_m) \), let \( \partial^- i \ p := A_i, \partial^+ j \ p := B_j \)

A 2-cell \( t : (A, B) \to (C) \) is \textbf{divisible} at \( \partial^+_1 \) if

\[
\forall \Gamma_1 A B \Gamma_2 \Delta \quad \exists \Gamma_1 \Delta C \Gamma_2 \quad \text{such that}
\]

[A diagram is shown illustrating the concept of divisibility of 2-cells at \( \partial^+_1 \).]
A 2-cell \( t : (A, B) \rightarrow (C) \) is **divisible** at \( \partial_2^- \) if

\[
\forall \ A \ C \Gamma \Delta p \exists! \ C \ A \ B \tilde{p}
\]
Divisible 2-cells produce rules of sequent calculus

\[ t : (A, B) \rightarrow (A \otimes B) \text{ divisible at } \partial_1^+ : \]

\[ \forall \Delta \quad \exists! \Delta \]

\[ \Gamma_1, A, B, \Gamma_2 \vdash \Delta \]

\[ \Gamma_1, A \otimes B, \Gamma_2 \vdash \Delta \]

\[ \otimes L \]
Divisible 2-cells produce rules of sequent calculus

\[ t : (A, B) \rightarrow (A \otimes B) \] divisible at \( \partial_1^+ \):

\[ \Gamma_1 \vdash \Delta_1, A \quad \Gamma_2 \vdash B, \Delta_2 \]

\[ \Gamma_1, \Gamma_2 \vdash \Delta_1, A \otimes B, \Delta_2 \]

\[ \otimes R \]
Units: the usual approach

2-cells $(A_1, \ldots, A_n) \rightarrow (A)$, with $n \geq 2$, divisible at $\partial_1^+$, model composition of paths in topology, and $n$-ary tensors (or conjunctions) in logic
Units: the usual approach

2-cells \((A_1, \ldots, A_n) \rightarrow (A)\), with \(n \geq 2\), divisible at \(\partial_1^+\), model **composition of paths** in topology, and **n-ary tensors** (or conjunctions) in logic

- Dually (self-dually in topology), \((B) \rightarrow (B_1, \ldots, B_n)\) divisible at \(\partial_1^-\) model \(n\)-ary pars or disjunctions
Units: the usual approach

2-cells \((A_1, \ldots, A_n) \rightarrow (A)\), with \(n \geq 2\), divisible at \(\partial_1^+\), model composition of paths in topology, and \(n\)-ary tensors (or conjunctions) in logic

- Dually (self-dually in topology), \((B) \rightarrow (B_1, \ldots, B_n)\) divisible at \(\partial_1^-\) model \(n\)-ary pars or disjunctions

**Units**/constant paths (in Cockett-Seely and Hermida)

\(\leadsto\) divisible 2-cells with a degenerate boundary (0-ary tensors/pars)
Coherence via universality

### Multicategory

A polycategory where all 2-cells have a single output.

(⇝ intuitionistic sequent calculi)

### Representable multicategory

For all composable \((A_1, \ldots, A_n)\), \(n \geq 0\), there exists an “n-ary tensor” 2-cell \((A_1, \ldots, A_n) \to (\otimes_{i=1}^{n} A_i)\) divisible at \(\partial_1^+\).

Hermida, 2000

Monoidal categories and strong monoidal functors are equivalent to representable multicategories (with a choice of divisible 2-cells) and morphisms that preserve divisibility at \(\partial_1^+\).
Coherence via universality

**Multicategory**

A polycategory where all 2-cells have a single output.

(⇒ intuitionistic sequent calculi)

**Representable multicategory**

For all composable \( (A_1, \ldots, A_n) \), \( n \geq 0 \), there exists an “\( n \)-ary tensor” 2-cell \( (A_1, \ldots, A_n) \rightarrow (\bigotimes_{i=1}^{n} A_i) \) divisible at \( \partial_1^+ \).

**Hermida, 2000**

Monoidal categories and strong monoidal functors are equivalent to representable multicategories (with a choice of divisible 2-cells) and morphisms that preserve divisibility at \( \partial_1^+ \).
Coherence via universality

Representable polycategory

For all composable \((A_1, \ldots, A_n)\), \(n \geq 0\), there exists an “\(n\)-ary tensor” 2-cell \((A_1, \ldots, A_n) \rightarrow (\otimes^n_{i=1} A_i)\) divisible at \(\partial^+_1\), and an “\(n\)-ary par” 2-cell \((\exists^n_{i=1} A_i) \rightarrow (A_1, \ldots, A_n)\) divisible at \(\partial^-_1\).

Linearly distributive categories and strong linear functors are equivalent to representable polycategories (with a choice of divisible 2-cells) and morphisms that preserve divisibility at \(\partial^+_1\) and \(\partial^-_1\).
So, all’s good up to dimension 2...

But:

- If we allow 2-cells with degenerate input or output boundary, we must allow 2-cells with overall 0-dimensional boundary.

(Although in most examples these are unnatural.)
But:

- If we allow 2-cells with degenerate input or output boundary, we must allow 2-cells with overall 0-dimensional boundary. (Although in most examples these are unnatural.)

- If we want (in topology) to model higher-dimensional homotopy types, or (in logic) the dynamics of reduction/cut elimination, we need **higher-dimensional cells**.
So, all’s good up to dimension 2...

But:

- If we allow 2-cells with degenerate input or output boundary, we must allow 2-cells with overall 0-dimensional boundary. (Although in most examples these are unnatural.)
- If we want (in topology) to model higher-dimensional homotopy types, or (in logic) the dynamics of reduction/cut elimination, we need **higher-dimensional cells**.
- Put these two together $\Rightarrow$ problems, problems, problems!
So, all’s good up to dimension 2...

But:

- If we allow 2-cells with degenerate input or output boundary, we must allow 2-cells with overall 0-dimensional boundary. (Although in most examples these are unnatural.)
- If we want (in topology) to model higher-dimensional homotopy types, or (in logic) the dynamics of reduction/cut elimination, we need **higher-dimensional cells**.
- Put these two together $\Rightarrow$ problems, problems, problems!

**A solution: regularity**

Input and output boundaries of 2-cells are 1-dimensional (in general: $k$-boundaries of $n$-cells are $k$-dimensional)
We need a new definition for units

Idea: Saavedra unit (J. Kock, 2006), reformulated

**Tensor unit** $1_x : x \rightarrow x$

For all $A : x \rightarrow y$, $B : z \rightarrow x$, there exist

![Diagram showing the tensor unit and its properties](attachment:image.png)

respectively divisible at $\partial_1^+$ and $\partial_2^-$, and at $\partial_1^+$ and $\partial_1^-$. 

**Induces the correct coherent structure** (triangle equations, etc)
But we can do better

Tensor left divisible 1-cell $E : x \to x'$

For all $A : x \to y$, $A' : x' \to y$, there exist

\[
\begin{aligned}
&x & \xrightarrow{A} & y \\
&\downarrow & & \downarrow \\
&E & \xrightarrow{E \to A} & E \\
&x' & & x'
\end{aligned}
\]

\[
\begin{aligned}
&x & \xrightarrow{E \otimes A'} & y \\
&\downarrow & & \downarrow \\
&E & \xrightarrow{E \circ A'} & E \\
&x' & & x'
\end{aligned}
\]

divisible both at $\partial^+_1$ and $\partial^-_2$. 
But we can do better

Tensor right divisible 1-cell $E : x \to x'$

For all $B : z \to x$, $B' : z \to x'$, there exist

$$z \xrightarrow{B'} x'$$

$$B' \circ E \xrightarrow{e_{E,B'}^l} E$$

$$x \xrightarrow{B} E \xrightarrow{t_{B,E}} x'$$

divisible both at $\partial_1^+$ and $\partial_1^-$.

Tensor divisible 1-cell $E : x \to x'$

Tensor right and left divisible 1-cell.
Theorem

The following are equivalent in a regular poly-bicategory:

- for all 0-cells $x$, there exists a tensor unit $1_x : x \to x$;
- for all 0-cells $x$, there exist a 0-cell $\overline{x}$ and a tensor divisible 1-cell $e : x \to \overline{x}$;
- for all 0-cells $x$, there exist a 0-cell $\overline{x}$ and a tensor divisible 1-cell $e : \overline{x} \to x$. 

If enough equivalences exist, units exist! 

Representability: existence of enough divisible 2-cells and 1-cells.
From divisible cells to units

<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>The following are equivalent in a regular poly-bicategory:</td>
</tr>
<tr>
<td>- for all 0-cells $x$, there exists a tensor unit $1_x : x \to x$;</td>
</tr>
<tr>
<td>- for all 0-cells $x$, there exist a 0-cell $\overline{x}$ and a tensor divisible 1-cell $e : x \to \overline{x}$;</td>
</tr>
<tr>
<td>- for all 0-cells $x$, there exist a 0-cell $\overline{x}$ and a tensor divisible 1-cell $e : \overline{x} \to x$.</td>
</tr>
</tbody>
</table>

If enough equivalences exist, units exist!
From divisible cells to units

Theorem
The following are equivalent in a regular poly-bicategory:

- for all 0-cells $x$, there exists a tensor unit $1_x : x \to x$;
- for all 0-cells $x$, there exist a 0-cell $\bar{x}$ and a tensor divisible 1-cell $e : x \to \bar{x}$;
- for all 0-cells $x$, there exist a 0-cell $\bar{x}$ and a tensor divisible 1-cell $e : \bar{x} \to x$.

If enough equivalences exist, units exist!

Representability: existence of enough divisible 2-cells and 1-cells
Some of this is in my PhD thesis:

Equivalences and units

Some of this is in my PhD thesis:


A formulation of bicategory theory where “divisible cells” are the single fundamental notion (composition and units are derived):

- A.H., *Weak units, divisible cells, and coherence via universality for bicategories*. (Soon to be available)
Some of this is in my PhD thesis:


A formulation of bicategory theory where “divisible cells” are the single fundamental notion (composition and units are derived):

- A.H., *Weak units, divisible cells, and coherence via universality for bicategories*. (Soon to be available)

Scales to higher dimensions:

- A.H., *A combinatorial-topological shape category for polygraphs*. (Later this year)
An observation on the sequent calculus side

Tensor units as 0-ary tensors:

\[ \Gamma_1, \Gamma_2 \vdash \Delta \]

\[ \Gamma_1, 1, \Gamma_2 \vdash \Delta \]

\( \rightsquigarrow \text{ introduction of units is a “divisibility property” rule} \)
An observation on the sequent calculus side

Tensor units as divisible 1-cells:

\[
\begin{align*}
\Gamma_1, 1, \Gamma_2 \vdash \Delta \\
\Gamma_1, 1, \Gamma_2 \vdash \Delta
\end{align*}
\]

\[\leadsto \text{elimination of units is a "divisibility property" rule}\]

\[\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\end{array} \quad \begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\end{array}
\]

\[\begin{array}{c}
A \\
\downarrow \\
A \\
\end{array} \quad \begin{array}{c}
B \\
\downarrow \\
B \\
\end{array}
\]

\[\begin{array}{c}
1 \\
\uparrow \\
1 \\
\end{array}
\]

\[\begin{array}{c}
1 \\
\downarrow \\
1 \\
\end{array}
\]
An observation on the sequent calculus side

Tensor units as divisible 1-cells:

$$
\begin{array}{c}
\bullet & \xrightarrow{A} & \bullet \\
\downarrow & & \downarrow \\
1 & \xleftarrow{A} & 1
\end{array}
\quad
\begin{array}{c}
\bullet & \xrightarrow{B} & \bullet \\
\downarrow & & \downarrow \\
B & \xleftarrow{1} & 1
\end{array}
$$

$\rightsquigarrow$ elimination of units is a “divisibility property” rule

$$
\Gamma_1, 1, \Gamma_2 \vdash \Delta
$$

This difference is not captured by the induced structure

(monoidal categories, etc)
Regularity constraint: cannot empty either side of a sequent
Questions on the sequent calculus side (1)

Regularity constraint: cannot empty either side of a sequent

- Proofs in “regular MLL” are valid in MLL. In the other direction, we can obtain regular proofs by “introducing enough units”.
Regularity constraint: cannot empty either side of a sequent

- Proofs in “regular MLL” are valid in MLL. In the other direction, we can obtain regular proofs by “introducing enough units”.

\[
\begin{align*}
\text{AX} & \quad \text{AX} \\
A \vdash A & \quad 1 \vdash A \rightarrow \bot, A \\
A, 1 \vdash \bot, A & \quad 1 \vdash (A \rightarrow \bot), 1 \vdash \bot, A \\
\bot \vdash \bot & \quad 1 \vdash A \rightarrow \bot, A \\
\bot \vdash (A \rightarrow \bot), 1 \vdash \bot, A
\end{align*}
\]
Questions on the sequent calculus side (1)

Regularity constraint: cannot empty either side of a sequent

Proofs in “regular MLL” are valid in MLL. In the other direction, we can obtain regular proofs by “introducing enough units”.

\[
\begin{align*}
\text{AX} & \quad A \vdash A \\
\text{1}_L, \bot_R & \quad A, 1 \vdash \bot, A \\
\text{AX} & \quad \bot \vdash \bot \\
\text{1}_R & \quad 1 \vdash A \rightarrow \bot, A \\
\text{1}_L & \quad \bot \rightarrow (A \rightarrow \bot), 1 \vdash \bot, A
\end{align*}
\]

What does the number of “residual units” count?
Two-sided sequent calculi that fit this framework (this includes ones for full linear logic) can be seen as “calculi of divisible 2-cells”.

Questions on the sequent calculus side (2)
Two-sided sequent calculi that fit this framework (this includes ones for full linear logic) can be seen as “calculi of divisible 2-cells”.

What is the logical/computational significance of divisible 1-cells? (And 3-cells, etc.)
Two-sided sequent calculi that fit this framework (this includes ones for full linear logic) can be seen as “calculi of divisible 2-cells”.

What is the logical/computational significance of divisible 1-cells? (And 3-cells, etc.)

What could be a “calculus of divisible cells in all dimensions”?
Two-sided sequent calculi that fit this framework (this includes ones for full linear logic) can be seen as “calculi of divisible 2-cells”.

What is the logical/computational significance of divisible 1-cells? (And 3-cells, etc.)

What could be a “calculus of divisible cells in all dimensions”?

Thank you for your attention.